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On the MFS for potential problems in non-homogeneous media

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Abstract. This paper presents a simple formulation based on Method of Fundamental Solutions (MFS) for the numerical solution of non-homogeneous potential problems. This formulation makes use of Green's functions for dealing with the non-homogeneity of the problems. These Green's functions are here modified by the method of images, aiming decreasing the number of surfaces discretized by the MFS and thus reducing the computational cost of the proposed models. The MFS formulation is then evaluated for different Green's functions by comparing its results with the ones given by analytical solutions. Finally, an example is presented, illustrating the good performance of the MFS for such problems with embedded boundary conditions.

Keywords: MFS, Green's functions, non-homogeneous media

1 Introduction

Many methods have been used for the numerical solution of potential problems. Of these, the Boundary Element Method (BEM) allows an efficient analysis due to the use of suitable Green's functions which avoid the discretization of the whole domain and only the discretization of boundaries is required for the implementation of the numerical models [1–4].

In the last decades, the meshless methods have attracted great interest of the scientific community. The Method of Fundamental Solutions (MFS) is one of these methods and has its mathematical formulation quite simple and efficient. Like in the BEM, it is also based on the prior knowledge of fundamental solutions and/or Green's functions, but the numerical and analytical integration schemes that need to be performed in the BEM are not required. However, one disadvantage of the MFS is the determination of the localization of the virtual sources on which the singularities are located. In spite of this difficulty, the MFS has been applied successfully in several engineering problems. For example, Santos et al. [5] used the MFS with genetic algorithms for the simulation of cathodic protection systems with nonlinear boundary conditions. Fontes Jr et al. [6] applied the regularized MFS with numerical Green's functions to solve fracture mechanics problems. More recently, Costa et al. [7] developed a frequency-domain MFS formulation to simulate the sound field generated by a point source in the presence of a T-shaped thin noise barrier.

The purpose of this paper is to develop 2D MFS models for analysis of potential problems with embedded heterogeneity. This paper is organised as follows. In Section 2, basic concepts are reviewed for potential problems. The MFS is formulated for different Green's functions in Section 3. In Section 4, the proposed formulation is validated by comparing its results with those reported in the literature, and in Section 5, an application is carried out in order to illustrate the good performance of the MFS for problems with heterogeneity and embedded boundary conditions. Conclusions are drawn in Section 6.

2 Governing Equation

For potential problems in non-homogeneous media, the following partial differential equation can be written [1–3]:

$$\nabla \cdot (k(\mathbf{x})\nabla\phi(\mathbf{x})) = 0, \tag{1}$$

where $k(\mathbf{x})$ is the material function and $\phi(\mathbf{x})$ is the potential function.

The boundary conditions for described above equation are given by:

$$\phi(\mathbf{x}) = \bar{\phi}(\mathbf{x}) \quad \text{for Dirichlet condition} \tag{2}$$

$$q(\mathbf{x}) = -k(\mathbf{x})\frac{\partial\bar{\phi}(\mathbf{x})}{\partial n} = \bar{q}(x) \quad \text{for Neumann condition} \tag{3}$$

where n is the unit outward normal vector.

By considering a point source placed within this domain at $\mathbf{x}_0$, it is possible to establish Green's functions with embedded heterogeneity at a point $\mathbf{x}$, which can be written as follows:

$$G(\mathbf{x},\mathbf{x}_0) = -\frac{1}{2\pi}\frac{(c_1 + c_2 y_0)^{-1}}{(c_1 + c_2 y)}\ln(r) \quad \text{for } k(y) = k_0(c_1 + c_2 y)^2 \tag{4}$$

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{2\pi}\frac{(c_1 e^{\beta y_0} + c_2 e^{-\beta y_0})^{-1}}{(c_1 e^{\beta y} + c_2 e^{-\beta y})}K_0(\beta r) \quad \text{for } k(y) = k_0(c_1 e^{\beta y} + c_2 e^{-\beta y})^2 \tag{5}$$

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{2}\frac{(c_1 \cos\beta y_0 + c_2 \sin\beta y_0)^{-1}}{(c_1 \cos\beta y + c_2 \sin\beta y)}Y_0(\beta r) \quad \text{for } k(y) = k_0(c_1 \cos\beta y + c_2 \sin\beta y)^2 \tag{6}$$

where c_1 and c_2 are arbitrary constants, k_0 is a reference value for $k(y)$, Y_0 and K_0 are functions of Bessel and modified Bessel, respectively, of the second kind and of order zero. A more detailed description about these functions can be found in Sutradhar and Paulino [4].

3 Method of Fundamental Solutions

Using the MFS, the response inside a two-dimensional space can be computed as a linear combination of fundamental solutions and/or Green's functions for a set of NVS virtual sources, which are placed outside the domain of interest to avoid singularities, as shown in Fig. 1.

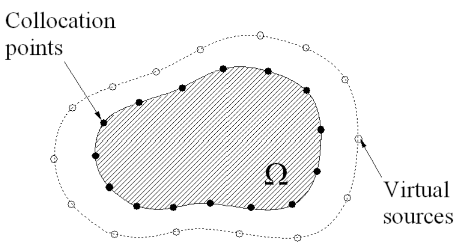


Figure 1. Schematic representation of the MFS model.

Thus, the potential function can be computed as:

$$\phi(\mathbf{x}) = \sum_{l=1}^{\text{NVS}} A_l G(\mathbf{x},\mathbf{x}_l) \tag{7}$$

where the coefficients A_l are the unknown amplitudes which are computed by imposing the boundary conditions for a set of NCP collocation points placed along the surface of the domain of interest and $G(\mathbf{x}, \mathbf{x}_l)$ is the Green's function at a point $\mathbf{x}$ for a virtual source placed at $\mathbf{x}_l$. In the MFS model here developed, the Green's functions presented in Sec. 2 are used. In this work, an equal number of collocation points and virtual sources is assumed, which allows obtaining a NCP $\times$ NVS system of equations. This system is built by prescribing at each collocation point $\mathbf{x}$, the boundary conditions defined in Eqs. 2 and 3.

4 Validation

To validate the results obtained by the MFS formulation, a square domain $[0, 1]^2$ with embedded heterogeneity was considered. In this analysis, the distance of virtual sources is equal to two times the distance between the collocation points. Here, a total of 128 collocation points was always used. Distribution of collocation and source points can be seen in Fig. 2. The responses provided by the MFS model were validated with one-dimensional analytical solutions provided by the Ref. [4] for three different profiles, $k(y) = 5(1 + 2y)^2$ (quadratic), $k(y) = 5e^{2y}$ (exponential) and $k(y) = 5(\cos y + 2\sin y)^2$ (trigonometric). The boundary conditions are prescribed as $\phi(x, 0) = 0$, $\phi(x, 1) = 100$, $\frac{\partial \phi(0,y)}{\partial x} = 0$ and $\frac{\partial \phi(1,y)}{\partial x} = 0$.

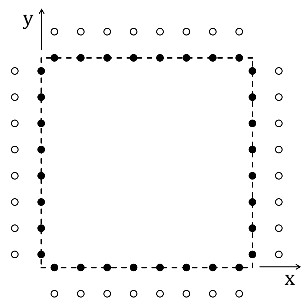


Figure 2. Distribution of collocation points (•) and virtual sources (○) for the MFS model.

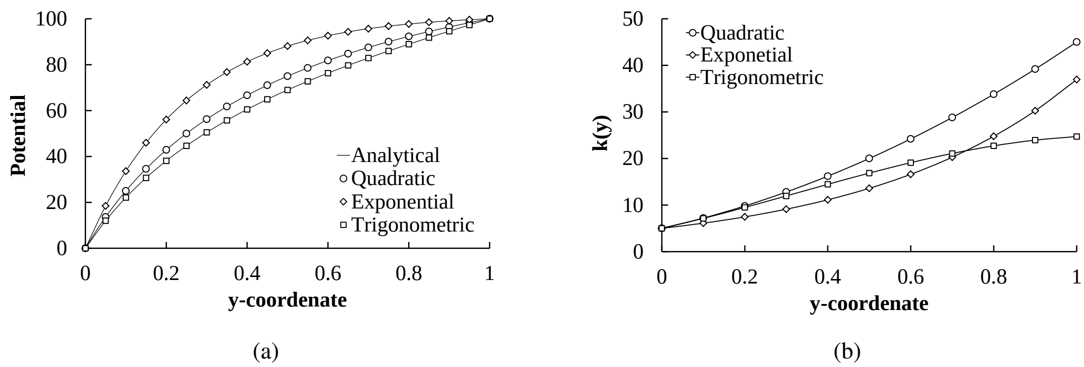
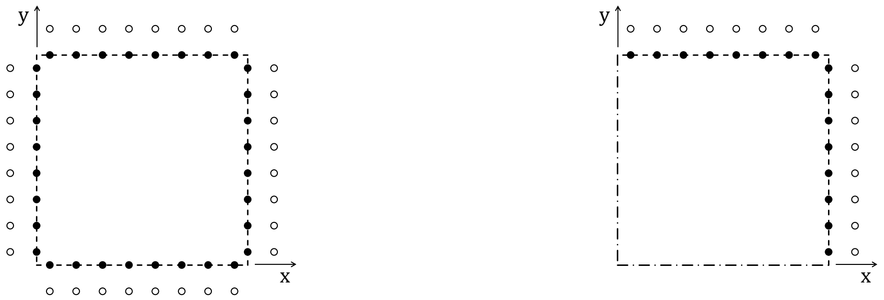


Figure 3. Comparison between the analytical and numerical solutions for three different profiles $k(y)$.

In Figure 3a, the analytical and numerical solutions are compared for three different profiles plotted in Fig. 3b. From the presented curves in Fig. 3a, it becomes clear that an excellent fit is found. Different distances of virtual sources were also tested and evaluated for an increasing number of collocation and source points, and an excellent convergence of the MFS solution was as well obtained (not presented). Further details on the optimal position of the virtual sources can be found in Costa et al. [7].

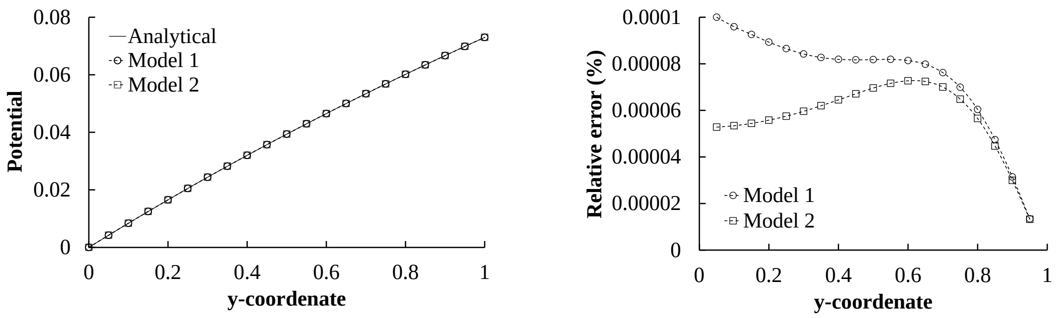
5 Application

To show the applicability of the present formulation, two numerical models were proposed. In the Model 1, all four surfaces were discretized, and in the Model 2, a Green's function that takes into account the presence of two plane surfaces was employed, and only two surfaces were discretized. In this analysis, it was used 128 collocation points for the model 1, and 64 collocation points for the model 2. Distribution of the collocation and source points for both MFS models can be seen in Fig. 4. As in the previous example, the virtual sources were computed as two times the distance between collocation points. The analytical solution for this problem is $\phi(x, y) = xy(5.6 + 0.5x + 0.6y + 0.8xy)^{-1}$ for a variable profile equal to $(5.6 + 0.5x + 0.6y + 0.8xy)^2$. The boundary conditions are prescribed as $\phi(x, 0) = 0$, $\phi(0, y) = 0$, $\phi(x, 1) = x(6.2 + 0.5x + 0.8x)^{-1}$ and $\phi(1, y) = y(6.1 + 0.6y + 0.8y)^{-1}$.



(a) Model 1. (b) Model 2.

Figure 4. Distribution of collocation points (•) and virtual sources (○) for the MFS models.



(a) (b)

Figure 5. Comparison between the analytical and numerical solutions for a non-homogeneous media.

In Figure 5, the numerical and analytical solutions are compared for a variable profile. Analysis of the results clearly reveals a good convergence of the MFS models (see Fig. 5a). It is important to note that the error provided by the Model 2 is lower than the Model 1 (see Fig. 5b), indicating that the MFS can reach an excellent accuracy with computational efficiency due to the use of suitable Green's functions.

6 Conclusion

In this paper, the MFS was applied to the numerical solution of non-homogeneous potential problems. The proposed formulation was validated by comparing its results with the ones reported in the literature, and an excellent agreement was always found. The performance of the proposed models was also evaluated, showing that the MFS can provide more accurate results making use of smaller-sized equation systems.

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