Performance Comparison between the Multiple Reciprocity and Direct Interpolation Boundary Element Method in Problems Governed by the Helmholtz Equation

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Abstract. In this work, response problems applied and governed by the Helmholtz equation are analyzed. The formulation Multiple Reciprocity Boundary Element Method (MRBEM) can be seen as an extension of the formulation with Dual Reciprocity (DRBEM) because the original problem as a whole is modeled by a sequence of fundamental solutions of a higher-order, while the formulation of the DRBEM uses a sequence of radial-based functions to approximate the kernel the domain integrals. Although both techniques apply the reciprocity theorem, the idea behind each method is essentially different. For the validation of this formulation, problems governed by the Helmholtz equation are solved, in which the MRBEM results were compared to a new formulation of the Boundary Element Method (BEM), denoted in this work as DIBEM-2 (Direct Interpolation Boundary Element Method without Regularization). DIBEM-2 makes use of radial basis functions to approximate domain integrals. Performance curves are generated by calculating the average percentage error for each mesh, demonstrating the convergence and precision of each method.

Keywords: Boundary Element Method, Radial Functions, Fundamental Higher-Order Solutions, Helmholtz Equation.

1 Introduction

Historically, the Boundary Element Method (BEM) began to stand out in engineering applications after the publication of the book “The Boundary Element Method for Engineers” in 1978 [1]. Thenceforward, several researchers have pointed out that the BEM is quite suitable in applications in which the operators that mathematically characterize the governing equation are self-adjoint [2]. In this context, with the help of an auxiliary function, known as fundamental solution [1], it is possible to transform the domain integrals into boundary integrals [3]; however, the problems addressed by the mathematical concept of BEM are not always capable of performing this procedure.

Based on this, the DIBEM [4] was proposed to offer an alternative to the use of the well-known DRBEM technique [5] in the approximation of domain integrals that normally represent the source, and inertia, among other field actions. The DIBEM showed good results in Poisson and Helmholtz problems [6, 7], an initiative derived from this technique that seeks to expand the study with the method without regularization, here called DIBEM-2. This technique approaches the same concept as DIBEM [4], however, it uses a well-elaborated auxiliary solution in the mathematical manipulations.

Another approach for solving these problems is the MRBEM technique, developed mainly by Nowak in 1989 [8] and extended to a series of applications by Nowak and Brebbia [9, 10, 11], including the problems by the Helmholtz equation. It is possible to view the MRBEM as a generalization of the Galerkin Tensor [5]. This method strategically adopts a sequence of higher-order fundamental solutions in terms of a primitive function related to Laplacian or a differential operator to the Helmholtz operation and thus applies the Green identity several times [12]. As a result, the method takes the domain integrals to the exact boundary with just the base of the problem.

The objective of this work is to add even more to the performance study carried out in the DIBEM-2 technique when compared to the MRBEM in response problems governed by the Helmholtz equation, thus increasing the precision and solution capacity of the BEM. In the simulations carried out, linear elements are used in both formulations, as well as in the numerical values compared with available analytical solutions.

2 Basic Equation

One can deduce the Helmholtz Equation as a particular case of the Acoustic Wave equation [13]. Thus, the following Eq. (1) is considered for a two-dimensional case in a homogeneous media:

$$\nabla^2 u + \lambda u = 0, \qquad \lambda = \frac{\omega^2}{k^2} \tag{1}$$

In Eq. (1), k is the propagation velocity of the wave in the medium; ω is considered vibration frequency. An inverse integral equation equivalent to this term can be easily found through resources well known by classical BEM theory [1], as follows:

$$c(\xi)u(\xi) + \int_\Gamma u(X)q^*(\xi;X)d\Gamma - \int_\Gamma q(X)u^*(\xi;X)d\Gamma = \lambda \int_\Omega u(X)u^*(\xi;X)d\Omega \tag{2}$$

In Eq. (2), $u(X)$ represents the scalar potential and $q(X)$ its normal derivative; mutually, an auxiliary function $u^*(\xi;X)$, called fundamental solution, and $q^*(\xi;X)$ is its normal derivative, were used. Both functions are dependent on the Euclidean distance $r(\xi;X)$ between the source point ξ and any field point of domain X, which can be found in specialized literature. The coefficient $c(\xi)$ is defined by the smoothness on the boundary $\Gamma(X)$ and depends on the position of the point ξ concerning the physical domain $\Omega(X)$ [14].

Both DIBEM-2 and MRBEM use the fundamental solution $u^*(\xi;X)$ corresponding to the solution of a stationary diffusive problem as an auxiliary function in the information of the boundary integral equation, both by the Poisson equation in an infinite medium. The difference between the DIBEM-2 and the MRBEM method occurs in the domain integrals approach on the right side of Eq. (2), that is, the inertia of the system. In addition, DIBEM-2 information uses an artifice, applying a more elaborated auxiliary function, so that one can integrate Eq. (1) and take it to the boundary problem, as will be discussed in the scope of this work.

3 Direct Integration without Regularization

The first step is to propose a new auxiliary function $b^*(\xi;X)$ so that one can integrate Eq. (1) and take it to a boundary integral in the inverse form, according to the typical mechanisms of the BEM.

$$b^*(\xi;X) = u^*(\xi;X) - \lambda G^*(\xi;X) \tag{3}$$

In Eq. (3) the auxiliary function $b^*(\xi;X)$ consists of the fundamental solution $u^*(\xi;X)$ correlated to the problems governed by the Laplace Equation and G^* is the Galerkin tensor [5]. In the same way that we approached the concepts by the classical theory of BEM in Eq. (2), it is plausible to rewrite Eq. (1) with the auxiliary function of Eq. (3) as follows:

$$\begin{aligned} c(\xi)u(\xi) + \int_\Gamma uq^*d\Gamma - \int_\Gamma qu^*d\Gamma + \lambda \int_\Gamma (u_{,i}\, G^*)\eta_i d\Gamma - \lambda \int_\Gamma (uG^*{}_{,i})\eta_i d\Gamma \\ = -\lambda^2 \int_\Omega uG^*d\Omega \end{aligned} \tag{4}$$

The integral term on the right side of Eq. (4) can be approximated using a sequence of radial basis functions, in which the entire kernel of the domain integral is interpolated. The direct integration technique is similar to Dual reciprocity, as shown by Eq. (5).

$$u(X)G^*(\xi;X) \cong {}^{\xi}\alpha^j\, F^j\big(X^j;X\big) \tag{5}$$

In Eq. (5), the $F^j(X^j;X)$ represents a set of radial interpolation functions and the coefficient ${}^{\xi}\alpha^j$ is the constant that corresponds to the interpolation function, which depends on the source point ξ and arbitrary base points X^j. It is worth noting that the number of arbitrary base points X^j must be identical to the number of nodal points. To improve the proposed interpolation within the domain, these points should also be centered internally. As with Dual reciprocity, the DIBEM-2 technique also uses a primitive interpolation function ψ^j [5]. In this way, it's acceptable to write the domain integral of the inertia term, as follows:

$$\begin{aligned} \int_{\Omega} {}^{\xi}\alpha^j\, F^j\big(X^j;X\big)d\Omega &= \int_{\Omega} {}^{\xi}\alpha^j\, \Psi_{,ii}\big(X^j;X\big)d\Omega = \int_{\Gamma} {}^{\xi}\alpha^j\, \Psi_{,i}^{\,j}\,\eta_i\big(X^j;X\big)d\Gamma \\ &= {}^{\xi}\alpha^j \int_{\Gamma} \eta^j\big(X^j;X\big)d\Gamma \end{aligned} \tag{6}$$

In Eq. (6), the functions $\eta^j(X^j;X)$ and $\psi^j(X^j;X)$ are known, derived from the functions $F^j(X^j;X)$, which was chosen in this study as the thin-plate radial basis function, that is, the argument of the function is composed of the Euclidean distance $r(X^j;X)$ between the arbitrary base points X^j and the interpolation points X. It is worth emphasizing that the results were satisfactory when this transformation was carried out in previous problems [15]. Due to limited space, the matrix treatment of this equation won't be discussed, however, it resembles and can be acquired from previous works [4, 6]. Thus, the final system can be written as follows:

$$Hu - Gq - \lambda Wu + \lambda Sq = -\,\lambda^2 Mu \tag{7}$$

4 Formulation with Multiple Reciprocity

Considering again Eq. (2), to simplify the terms subscripted here, denoted as $u_0^*(\xi;X)$, known as the fundamental solution, these were used to distinguish themselves from other similar solutions that will be generated during mathematical manipulation.

$$\int_{\Omega} \nabla^2 u(X)u_0^*(\xi;X)d\Omega = -\lambda \int_{\Omega} u(X)\nabla^2 u_1^*(\xi;X)d\Omega \tag{8}$$

In Eq. (8), the term subscript as $u_0^*(\xi;X)$, on the left, represents the diffusive portion; and for the right side, the fundamental solutions $u_0^*(\xi;X)$ represents the Galerkin Tensor. Thus, the same strategy addressed by the classical theory of the BEM is used to treat the left side,

$$c(\xi)u(\xi) + \int_{\Gamma} \left(uq_0^* - qu_0^*\right) d\Gamma = \lambda \int_{\Omega} u\nabla^2 u_1^*\, d\Omega \tag{9}$$

where $q_0^*(\xi;X)$ is the normal derivative. The first step in approaching the domain integral with MRBEM is to adopt the fundamental higher-order solutions [12]:

$$\nabla^2 u_{j+1}^* = u_j^*, \qquad q_j^* = \left(\frac{\partial u_j^*}{\partial n}\right), \qquad j = 0,1,2,\dots, \tag{10}$$

Therefore, when we work on the fundamental solution for the first time, that is, the subscript term is equal to $u_1^*(\xi;X)$, the solution corresponds to the Galerkin Tensor. Thus, using the concept addressed in Eq. (10) and inputting in the domain integral of Eq. (9), it becomes:

$$\lambda \int_{\Omega} u\nabla^2 u_0^*\, d\Omega = \lambda \int_{\Omega} uu_1^*\, d\Omega = \lambda \int_{\Gamma} \left(uq_1^* - qu_1^*\right) d\Gamma - \lambda^2 \int_{\Omega} uu_2^*\, d\Omega \tag{11}$$

A similar procedure performed to N times on this domain integral converts as such:

$$c(\xi)u(\xi) + \sum_{j=0}^{N}(\lambda)^j \int_{\Gamma} uq_j^* \, d\Gamma - \sum_{j=0}^{N}(\lambda)^j \int_{\Gamma} uq_j^* \, d\Gamma = (-1)^N(\lambda)^{N+1} \int_{\Omega} uu_N^* \, d\Omega \tag{12}$$

The right side of Eq. (12) can be neglected in engineering practice, since it was proven that, for the two-dimensional and three-dimensional dimensions, the domain integral converges to zero when N is sufficiently large [16].

$$c(\xi)u(\xi) + \sum_{j=0}^{N}(\lambda)^j \int_{\Gamma} uq_j^* \, d\Gamma = \sum_{j=0}^{N}(\lambda)^j \int_{\Gamma} uq_j^* \, d\Gamma \tag{13}$$

Due to limited space, the manipulation of the Galerkin's Tensor won't be demonstrated either, however, it can be seen in previous works [5]. Finally, the complete discretization of Eq. (13) can be written as:

$$\begin{gathered} H = H_0 - \lambda H_1 + \cdots + (-\lambda)^n H_n \\ G = G_0 - \lambda G_1 + \cdots + (-\lambda)^n G_n \end{gathered} \tag{14}$$

5 Numerical Simulations

The simulations presented hereafter consist of response problems, that is, the system is resolved by a scan at different excitation frequencies. As a consequence, the potential equilibrium configurations are determined as a function of a set of known conditions. Two simple examples, with known analytical solutions, were chosen for an analysis of the results. Seeking to make a fair comparison, it was taken as a measure of error, being equal to the difference between the numerical and analytical values divided by the module of the highest analytical value.

5.1 Clamped sheet

This first example consists of consists of a one-dimensional harmonic problem governed by the Helmholtz equation. The geometric characteristics are shown in Fig. 1.

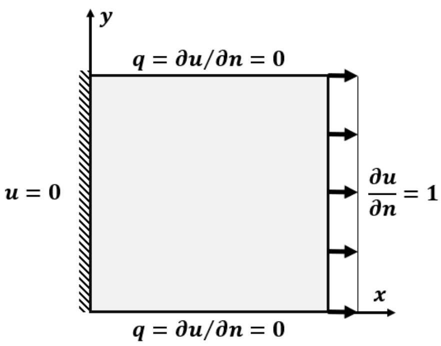

Figure 1. Clamped sheet and boundary conditions.

The value of k was considered as unitary. The analytical solution of this case is given by Eq. (15).

$$u(x) = \frac{sin(\omega x)}{\omega cos(\omega)} \tag{15}$$

For a better evaluation of the numerical behavior, simulations of DIBEM-2 and MRBEM were processed with the same number of nodal points and, for DIBEM-2, interpolating points were also used inside the mesh to provide reasonable results since all the kernel of the domain integral referring to inertia is interpolated. The excitation frequencies were varied from 1.0 to 20.0 in an interval of 0.5. The intention is to verify if the DIBEM-2 formulation presents satisfactory and faster results when compared to the MRBEM since it uses primitives and the refinement of the meshes to plot acceptable results. It is observed that in all computer simulations, the performance of DIBEM-2 offered satisfactory results, as well as that of MRBEM, as shown in Fig. 2.

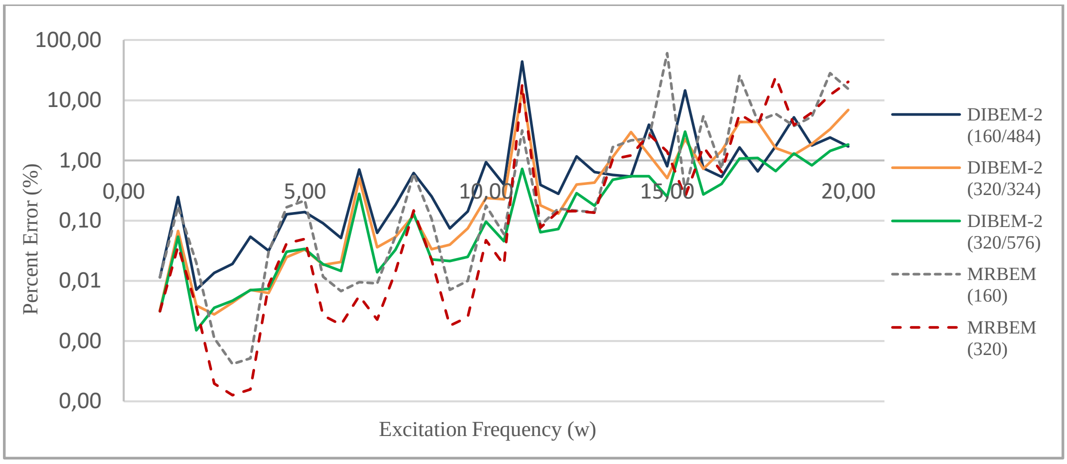

Figure 2. Weighted mean error curves, using linear elements, for the clamped sheet.

Also in Fig. 2, it can be noted that the error levels dropped significantly with the use of the boundary mesh refinement in DIBEM-2 and MRBEM, as the mesh with 320 nodes in the boundary (NBP) had a superior performance when compared to the 160 NBP, however, in DIBEM-2 it is not plausible to disregard the importance of the internal point cloud. It is possible to observe that, for DIBEM-2, the mesh with 320 NBP and 576 internal points (NIP) had a better performance when compared to the mesh of 320 NBP and 324 NIP, proving the importance of the internal points inside the mesh. It is noteworthy that the manifestation of the error peaks shown in Fig. 2 is because the calculated frequencies are very close to the natural frequency so that the analytical solution tends to infinity and an almost singular response from the numerical method is expected. As computational time is also an important point for the evaluation of DIBEM-2 with the MRBEM formulation, the same mesh was used for data collection as shown in Table 1.

Table 1. Relation of total computational time for each mesh.

BEM Mesh	Time (s)
DIBEM-2 (160/484)	291.73
DIBEM-2 (320/324)	422.56
DIBEM-2 (320/576)	941.44
MRBEM (160)	1531,29
MRBEM (320)	2473.42

It can also be seen in Fig. 2 and Tab. 1 that the precision of DIBEM-2 when scanning the excitation frequencies was similar to that of MRBEM, however, DIBEM-2 presented a drastic reduction in computational time concerning MRBEM, being possible to notice that the most refined mesh reached approximately 38%. Thus, it is plausible to say that the DIBEM-2 mathematical model is effectively more consistent.

5.2 Square membrane with three clamped edges

This second example consists of a two-dimensional harmonic problem, governed by the Helmholtz equation. The geometric characteristics, in this case, are presented in Fig. 3.

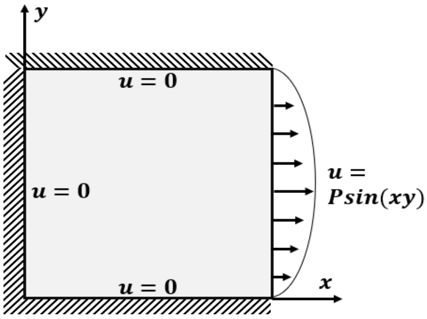

Figure 3. Square membrane and boundary conditions.

In the same way, as in the first problem, the value of k was considered unitary, as was that of P. Boundary conditions are all prescribed under essential conditions, making this problem more complex than the first. The analytical solution of this case is given by Eq. (16).

$$u(x,y) = \frac{sin\left(x\sqrt{\omega^2 - \pi^2}\right) sin(\pi y)}{sin\left(x\sqrt{\omega^2 - \pi^2}\right)} \tag{16}$$

To perform a performance comparison as in the previous example, regarding the behavior of DIBEM-2, the same context of analysis was approached, however, the excitation frequencies of 16.00 and 19.00 were changed respectively to 15.90 and 18.90 due to proximity with the natural frequencies calculated. As in the first example, the performances of DIBEM-2 and MRBEM were satisfactory, and the error levels dropped significantly as the number of nodes in the boundary (NBP) and interpolating points (NIP) inside of the mesh were refined, as can be seen below (Fig. 4).

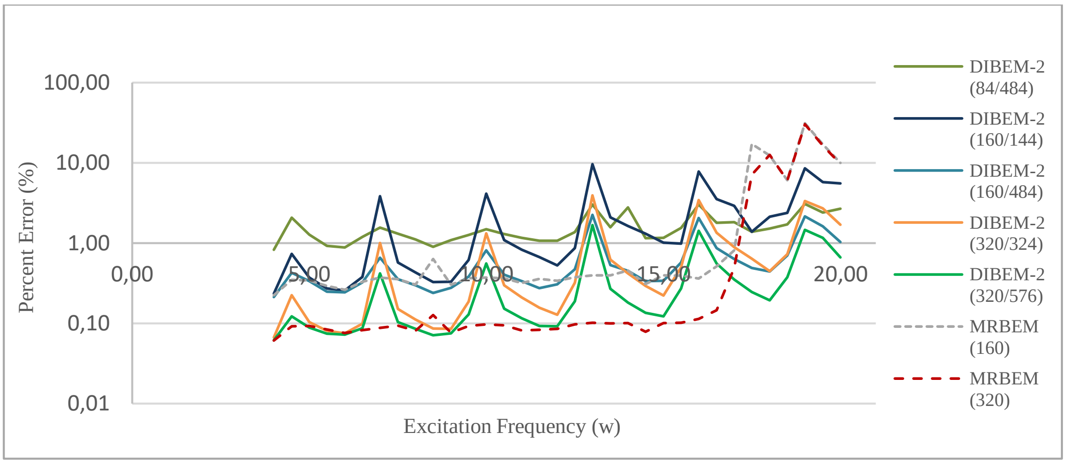

Figure 4. Weighted mean error curves, using linear elements, for the square membrane.

In Fig. 4, it is worth highlighting the importance of the internal point cloud present in DIBEM-2, in which some points have the mesh with lower NBP refinement and higher NIP. This generates better results when compared with a more refined mesh in the boundary, but the refinement of the mesh works favorably to gain precision. It is important to emphasize that, unlike DIBEM-2, the refinement of the boundary mesh in the MRBEM has a strong influence on the reduction of the error percentage, however, the number of primitives performed in the formulation also has an influence, causing some instability and requiring more computational time.

Finally, in Tab.2 the computational time results for the refinement of DIBEM-2 and MRBEM meshes are presented. The objective is to reaffirm how effective the DIBEM-2 mathematical model is in concerning the MRBEM. It can also be seen in this table that the refinement of the mesh of internal points automatically interferes with computational expenditure.

Table 2. Relation of total computational time for each mesh.

BEM Mesh	Time (s)
DIBEM-2 (84/484)	192.39
DIBEM-2 (160/144)	161.04
DIBEM-2 (160/484)	352.77
DIBEM-2 (320/324)	519.42
DIBEM-2 (320/576)	1067.22
MRBEM (160)	965.21
MRBEM (320)	3942.18

6 Conclusions

The Multiple Reciprocity technique is still the simplest alternative to overcome the mathematical difficulties that arise when applying the BEM to problems with operators that characterize governing equations that are not self-adjoint. However, the DIBEM-2 technique, through a scanning procedure of the excitation frequencies, demonstrated flexibility and robustness when used in problems governed by the Helmholtz equation. It's important to emphasize that the results showed less computational expenditure and more precision with the DIBEM-2 technique. In view of the observed aspects, in future works, it would be plausible to better examine the DIBEM-2 system, so that dynamic responses can be solved, that is, with advances in time and eigenvalue calculation problems and thus expanding its scale of applications.

References

[1] BREBBIA, C. A. The Boundary Element method for Engineers. London: Pentech Press, 1978.
[2] BREBBIA, C.A.; TELLES J. C. F., Wrobel L.C. Boundary Element Techniques. First Ed., Springer-Verlag, Berlin Heidelberg, 1984.
[3] BREBBIA, C. A.; WALKER, M. A. Boundary Element Techniques in Engineering. London: Ed. Newnes-Butterworths, 1980.
[4] LOEFFLER, C.F.; BARCELOS H.M.; MANSUR W.J.; BULCÃO A. Solving Helmholtz problems with the boundary element method using direct radial basis function interpolation. Eng Anal with Bound Elem 2015; 61:218-225.
[5] PATRIDGE, P.W.; BREBBIA, C.A.; WROBEL, L.C. The Dual Reciprocity Boundary Element Method. Computational Mechanics Publications, Southampton 1992.
[6] Loeffler, C. F., Cruz, A. L., & Bulcão, A. Direct Use of Radial Basis Interpolation Functions for Modelling Source Terms with the Boundary Element Method. Engineering Analysis with Boundary Elements, 2015c, vol. 50, pp. 97-108.
[7] Barcelos, H. M., Comparação de Desempenho entre a formulação Direta do Método dos Elementos de Contorno com Funções Radiais e o Método dos Elementos Finitos em Problemas de Poisson e Helmholtz. Vitória-ES: Dissertação de Mestrado em Engenharia Mecânica UFES, 2014.
[8] NOWAK, A. J. The Multiple Reciprocity Method of Solving Transient Heat Conduction Problems, in Boundary Elements. XI, Vol. 2, Computational Mechanics Publications, Southampton and Springer-Verlag, Berlin and New York, 1989.
[9] BREBBIA, C. A.; NOWAK, A. J. A New Approach for Transforming Domain Integrals to the Boundary, in Numerical Methods in Engineering. Vol. 1, Computational Mechanics Publications, Southampton and Springer-Verlag, Berlin and New York, 1989.
[10] NOWAK, A.J.; BREBBIA, C.A.; Solving Helmholtz Equation by Boundary Element Method Using the Multiple Reciprocity Method. Proc. Comp. Eper. Fluid Flow, ed. G.M. Carlomagno e C.A. Brebbia, Comp. Mech. Pub./Spinger-Verlag, 1989, p. 156-270.
[11] NOWAK, A. J.; BREBBIA, C. A. The Multiple Reciprocity Method: A New Approach for Transforming BEM Domain Integrals to the Boundary. Engineering Analysis, Vol. 6, No.3, p. 164-167,1989.
[12] KAMIYA, N.; ANDOH, E. Helmholtz Eigenvalue Analysis by Boundary Element Method. Journal of Sound and Vibration, p. 279-287. 1993.
[13] GAUL, L., M. KÖGL, e M. WAGNER. Boundary element methods for engineers and scientists: an introductory course with advanced topics. Springer Science & Business Media, 2013.
[14] Wrobel, L. C., Aliabadi, M. H., 2002. The Boundary Element Method, Wiley, Chichester.
[15] Loeffler, C. F., Cruz, A. L., & Bulcão, A. Aproximação direta de integrais de domínio usando funções de base radial com método dos elementos de contorno. Proceedings of the XXXIV Iberian Latin-American Congress on Computational Method in Engineering, Pirenópolis, GO, Brazil.
[16] KAMIYA, N.; ANDOH, E. A Note on Multiple Reciprocity Boundary Element Method: Neglect of Domain Term. Eng. Analysis with Boundary Elements, p. 273-275, 1991.