Analysis of centrifugal body force problems in anisotropic materials using the boundary element method

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Abstract. The main objective of this work is the development of a formulation of boundary elements for the anisotropic material problem evaluation under centrifugal loads. The fundamental anisotropic solutions are used and terms of inertia are considered as body forces. The domain integrals that result from the inertial terms are transformed into boundary integrals using the radial integration method. As a result, no internal points are needed to improve the accuracy of the solution. Discontinuous quadratic boundary elements are used whose degrees of freedom are written in a local reference system, where the directions of the coordinate axes coincide with the normal and tangent directions to the boundary at the collocation point. Problems with known analytical solutions are used in order to assess the accuracy of the proposed formulation. There is a good agreement between the numerical and exact solutions.
Keywords: Boundary Element Method, Anisotropic Materials, Centrifugal Loads.

1 Introduction

Monocrystalline alloys are the material used in aeronautic turbine blades because its high strength to high temperatures. They work in very high angular rotations which produce stresses in the blades that, due to oscillation, can nucleate smalls cracks that will propagate and break the blade due to fretting process. There are few works in literature that consider the centrifugal forces loading the blades. The majority of papers just consider that the blade is under tension by Papanikos et. al [1]. Because the monocrystal, these alloys are anisotropic.

The anisotropic behaviour of materials increases the number of variables in structural analysis. Due to this, analytical solutions are limited to simple problems. Numerical methods are necessary for the analysis of complex structures.

The use of boundary element method (BEM) for the solution of problems with anisotropic materials has become more common in recent years. As anisotropy increases the number of material elastic constants, difficulties in modelling arise in the development of the numerical formulations. Particularly, in the boundary element formulation, the larger number of variables means far more difficulty in deriving fundamental solutions. This aspect is evident in the literature. The number of references in which the boundary element method is applied to anisotropic structure is significantly smaller than the number for isotropic structures. However, in the last 10 years, important advances in the application of boundary element techniques to anisotropic materials have been published in the literature. For example, plane elasticity problems have been analyzed by Tyagnii [2], Cordeiro and Leonel [3], contact problems in elastic solids by Nguyen and Hwu [4], three-dimensional problems by Gu et al. [5], Shiah and Hematiyan [6], Shiah and Hematiyan [7].

In the general boundary element formulation with body forces, domain integrals arise in the formulation owing to the domain loads. In order to evaluate these integrals, a cell integration scheme can be used to give accurate results, as carried out by Shi and Bezine [8] for anisotropic plate bending problems. However, the discretization of the domain into cells reduces one of the main advantages of the BEM, that is, the discretization of only the boundary. An alternative to this procedure was presented by Rajamohan [9], which proposes the use of particular solutions to avoid domain discretization. However, the use of particular solutions requires us to find a suitable function which satisfies the governing equation. Depending on how complicated the governing equation is, this function may be quite difficult to find.

In the work described in this paper, domain integrals which arise from domain loads are transformed into boundary integrals by exact transformation using the radial integration method. This method was initially presented by Venturine [10] for isotropic plate bending problems. Later, Gao [11] has extended it to three-dimensional isotropic elastic problems. The most attractive feature of the method is its simplicity, since only the radial variable is integrated. For domain integrals which include unknown variables, the proposed procedure can be performed using a radial basis function as in the dual reciprocity method suggested by Gao [11]. In the case of centrifugal loads, as body forces do not depend on unknown variables, no radial basis function and, consequently, no internal points are necessary in the formulation.

In this paper, the BEM is applied to the analysis of stress and displacement fields in anisotropic problems under centrifugal loads. The radial integration method is used in order to transform domain into boundary integrals. No domain discretization and no internal points are necessary in the formulation. Unknown variables and boundary conditions are written in a local coordinate system that expand the range of problems that can be modeled by the formulation. Numerical results are compared with exact solutions and dependence on boundary discretization is accessed.

2 Boundary element formulation for 2D anisotropic elasticity

The boundary integral equation for 2D anisotropic problem is given by Sollero and Aliabadi [12]:

$$c_{ij}(\mathbf{z_o})u_i(\mathbf{z_o}) + \int_\Gamma T_{ij}(\mathbf{z},\mathbf{z_o})u_j(\mathbf{z})d\Gamma(\mathbf{z}) = \int_\Gamma U_{ij}(\mathbf{z},\mathbf{z_o})t_j(\mathbf{z})d\Gamma(\mathbf{z})\forall z\epsilon\Gamma \tag{1}$$

where the coefficient $c_{ij}(\mathbf{z_o})$ is given by $\delta_{ij} + A_{ij}(\mathbf{z_0})$ in which δ_{ij} is the Kronecker's delta. At a smooth boundary point, $c_{ij}(z_o) = \delta_{ij}/2$, at an internal point, $c_{ij}(z_o) = 1$. Fundamental solutions for displacement $U_{ij}(\mathbf{z},\mathbf{z_o})$ and traction $T_{ij}(\mathbf{z},\mathbf{z_o})$ are:

$$U_{ij}(\mathbf{z},\mathbf{z_o}) = 2\Re[q_{i1}A_{j1}\log(z_{o_1} - z_1) + q_{i2}A_{j2}\log(z_{o_2} - z_2)] \tag{2}$$

$$T_{ij}(\mathbf{z},\mathbf{z_o}) = 2\Re\left[\frac{g_{j1}(\mu_1 n_1 - n_2)A_{i1}}{(z_{o_1} - z_1)} + \frac{g_{i2}(\mu_2 n_1 - n_2)A_{j2}}{(z_{o_2} - z_2)}\right] \tag{3}$$

where the terms q_{ij}, g_{ji} and A_{ij} are complex material constants, $\Re$ stands for the real part of a complex number and $\log$ is the natural logarithm. Constants μ_k are complex numbers that are the roots of a characteristic polynomial as given by Lekhnitskii [13], Sollero and Aliabadi [12]. The field point $\mathbf{z}$ and the source point $\mathbf{z_o}$ are written in complex form as:

$$\mathbf{z} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} x_1 + \mu_1 x_2 \\ x_1 + \mu_2 x_2 \end{Bmatrix} \tag{4}$$

$$\mathbf{z}_o = \begin{Bmatrix} z_{o_1} \\ z_{o_2} \end{Bmatrix} = \begin{Bmatrix} x_{o_1} + \mu_1 x_{o_2} \\ x_{o_1} + \mu_2 x_{o_2} \end{Bmatrix} \tag{5}$$

3 Treatment of body forces

When there are body forces in the formulation, the boundary integral equations are given by:

$$c_{ki}u_i + \int_\Gamma T_{ik}u_i d\Gamma = \int_\Gamma U_{ik}t_i d\Gamma - \int_\Omega p_i U_{ik} d\Omega \tag{6}$$

In this work, the radial integration method is used in order to transform the domain integral of Equation (6) into a boundary integral.

Writing a domain integral in a polar form:

$$\int_\Omega p_i U_{ik} d\Omega = \int_{\theta_1}^{\theta_2} \underbrace{\int_0^r U_{ik} p_i[x(\rho,\theta), y(\rho,\theta)]\rho d\rho\, d\theta}_{F_i} \tag{7}$$

making some changes, it is possible to get:

$$u_l + \int_\Gamma T_{li} u_i d\Gamma = \int_\Gamma U_{li} t_i d\Gamma + \int_\Gamma F_i \frac{\vec{n}.\vec{r}}{r} d\Gamma \tag{8}$$

where $\vec{n}$ and $\vec{r}$ are unit vectors. The Eq. (8) will be used as the basic equation for the boundary element method to treat anisotropic problems under centrifugal loads. The absence of domain integrals can be noted.

3.1 Discontinuous quadratic shape functions

The formulation of this work is isoparametric, that is, in addition to the physical variables in the boundary (displacements and tractions), the geometry is also approximated by discontinuous quadratic elements, as follows:

$$\mathbf{x} = \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left[\begin{array}{cccccc} \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} & 0 \\ 0 & \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} \end{array} \right] \left\{ \begin{array}{c} x_1^{(1)} \\ x_2^{(1)} \\ x_1^{(2)} \\ x_2^{(2)} \\ x_1^{(3)} \\ x_2^{(3)} \end{array} \right\} = \phi \mathbf{x}^{(n)} \tag{9}$$

where $u_i^{(n)}$ and $t_i^{(n)}$ are the nodal values of displacements and tractions, respectively, and $\phi^{(i)}$ are the quadratic discontinuous shape functions.

In this way, the boundary integrals can be written as:

$$H^{(j)} = \int_{\Gamma_j} T_{lk} \phi^{(j)} d\Gamma = \int_{-1}^{1} T_{lk} \phi^{(j)} |J| d\xi \tag{10}$$

$$G^{(j)} = \int_{\Gamma_j} U_{lk} \phi^{(j)} d\Gamma = \int_{-1}^{1} U_{lk} \phi^{(j)} |J| d\xi \tag{11}$$

where J represents the Jacobian module of the transformation, and is given by:

$$|J| = \frac{d\Gamma}{d\xi} = \left\{ \left(\frac{dx_1}{d\xi} \right)^2 + \left(\frac{dx_2}{d\xi} \right)^2 \right\}^{1/2} \tag{12}$$

where $dx_1/d\xi$ and $dx_2/d\xi$ are obtained by deriving the Eq. (9) with respect to ξ.

One of the contributions of this work is in the use of discontinuous quadratic shape functions for interpolation in space. It has been very common in the literature to use discontinuous quadratic functions for the physical variables of the problem, for example, temperature and flow in the problems of heat conduction and displacements and tractions in problems of structural analysis. However, the use of discontinuous functions to interpolate geometry was always disregarded. The justification for not using it was due to the nodes being located inside the element. Thus, in the case of curved elements, there is a risk of a discontinuity between the end of one element and the beginning of another.

This problem does not exist when the elements are continuous as there are nodes at the ends of the elements that are shared between neighboring elements.

In order to ensure that there is no overlap between two neighboring elements, in this work, the continuous functions are used to define the positions of the nodes of the discontinuous elements. Proceed as follows:

- Parabolic elements are generated using the continuous functions;
- The position of the nodes of the discontinuous elements are calculated by $\xi = -2/3$, $\xi = 0$ and $\xi = 2/3$.

Proceeding in this way, since 3 points always define a single parable, discontinuous elements can be used to interpolate the geometry without overlapping at the ends of the elements.

3.2 Local referral system

In this work, instead of the reference system (x_1, x_2), a local system with axes in the normal and tangent directions to the node, called system (n, t), is used to apply the boundary conditions. Thus, problems with boundary conditions of restricted displacements in the normal direction can be easily implemented, which significantly increases the number of problems that can be analyzed by the implemented formulation. In addition, the implementation of formulations for the analysis of contact mechanics is quite direct.

Thus, it is possible to obtain the following equation:

$$\hat{\mathbf{H}}\hat{\mathbf{u}} = \hat{\mathbf{G}}\hat{\mathbf{t}} + \mathbf{p} \tag{13}$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$ are vectors that contain displacements and tractions, respectively, in all nodes, written in the local reference system (n, t). The matrices $\hat{\mathbf{H}}$ and $\hat{\mathbf{G}}$ are the influence matrices $\mathbf{H}$ and $\mathbf{G}$, calculated taking into account that the shape functions were multiplied by the transformation matrices. Since the vector $\mathbf{p}$ does not depend on the values of displacements and tractions, it remains unchanged.

To calculate the unknown variables, a column exchange is made between the matrices $\hat{\mathbf{H}}$ and $\hat{\mathbf{G}}$, according to the boundary conditions, generating the matrices $\mathbf{A}$ and $\mathbf{B}$. Thus, all unknown variables in the boundary, be they displacements or tractions, pass to a vector $\mathbf{x}$ and the known variables pass to a vector $\mathbf{y}$, obtaining:

$$\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{y} + \mathbf{p} \tag{14}$$

which can be rewritten in the form of a linear system:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{15}$$

Once this linear system has been solved, unknown variables are reordered, which have now been calculated, and which are found in the vector $\mathbf{x}$ and the known variables, obtaining all values of the vectors $\mathbf{u}$ and $\mathbf{t}$.

3.3 Calculation of stresses at the boundary

To calculate the stress tensor on a given boundary node, consider a node in which the directions of the tangent and normal vectors to the boundary do not coincide with the directions of the reference. In this node, a new $x_{1'}x_{2'}$ reference system is created, with directions that coincide with the tangent and normal vectors to the boundary in this node. Writing the displacements and tractions in this local system, we have:

$$\begin{aligned} u'_i &= l_{ij}u_j \\ t'_i &= l_{ij}t_j \end{aligned} \tag{16}$$

where l_{ij} are the cosine directors.

From the stress strain relationship, we have:

$$\begin{Bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{12} \end{Bmatrix} = \begin{bmatrix} Q'_{11} & Q'_{12} & Q'_{16} \\ Q'_{12} & Q'_{22} & Q'_{26} \\ Q'_{16} & Q'_{26} & 2Q'_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{12} \end{Bmatrix} \tag{17}$$

where Q'_{ij} are the components of the stiffness tensor written in the local frame.

In the Eq. (17) there are three unknowns σ'_{11}, ε'_{22}, ε'_{12}, which can now be calculated. Finally, the stress must be written in the global reference x_1x_2, that is:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \mathbf{T}^{-1} \begin{Bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{12} \end{Bmatrix} \tag{18}$$

where $\mathbf{T}$ is the coordinate transformation matrix.

4 Numerical application

In this section, a numerical application of the developed formulation will be presented. Consider a blade of an aeronautic turbine, fixed in the rotor motor by a dovetail joint as shown in Figure 1. The blade is of an hypothetical orthotropic material with properties given in Table 1. The friction between the rotor and the blade was not considered. The boundary conditions and the mesh used to analyze the problem are given in Figure 2 (a).

Table 1. Geometry, velocity, and material property - Aeronautic turbine blade.

Properties	**Symbol**	**Unity**
Constant Angular Speed	ω	100.000 RPM
Modulus of Elasticity	E_1	114 GPa
Modulus of Elasticity	E_2	228 GPa
Transverse Elasticity	G_{12}	11.4 GPa
Poisson Ratio	ν_{12}	0.3

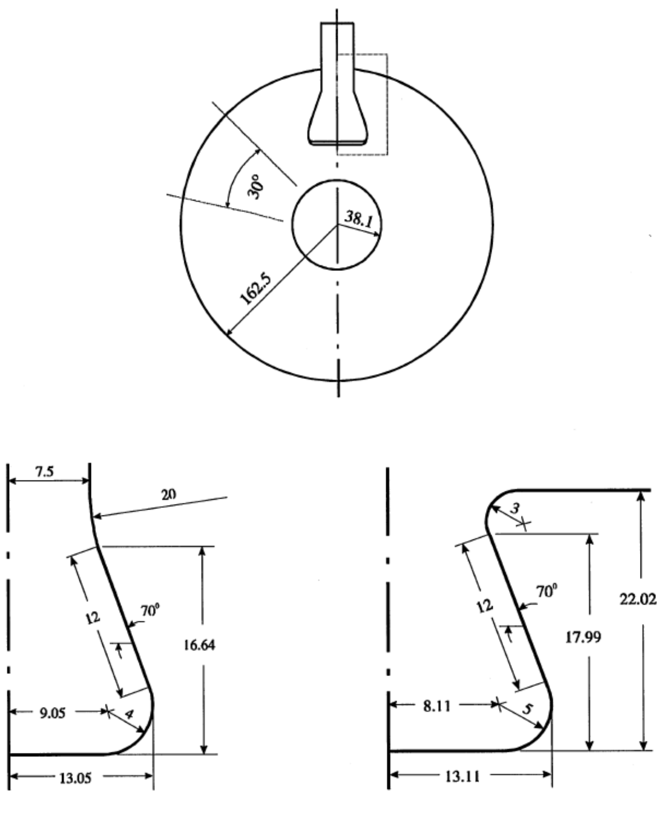


Figure 1. Turbine rotor and blade sketch by Papanikos [1].

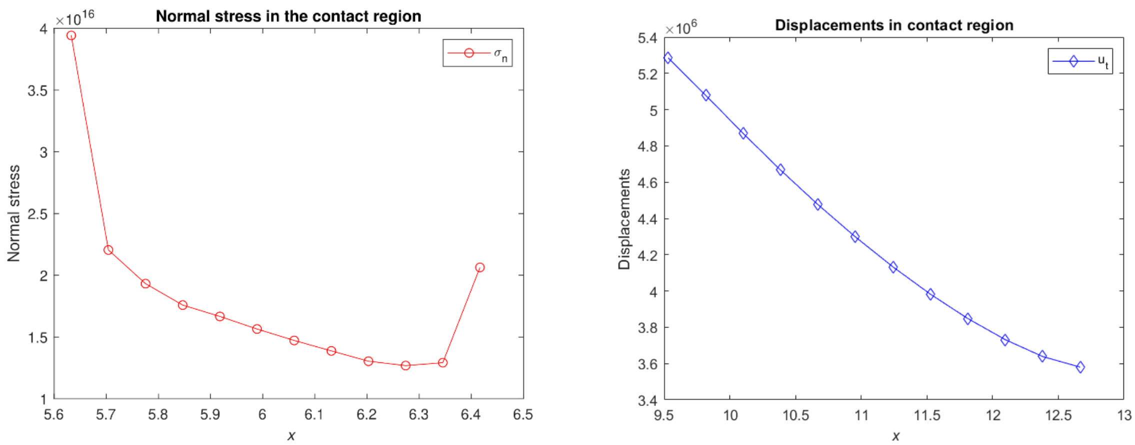
The dimensions of the rotor and the blade are available in Figure 1.

A mesh with 37 discontinuous quadratic boundary elements are used, distributed as shown in Figure 2 (a) and deformed the blade is shown in the Figure 2 (b).

(a) (b)

Figure 2. (a) Boundary conditions and (b) deformation blade.

The normal stresses in the contact region are shown in Figure 3 (a) and the normal tangential displacements are shown in the Figure 3 (b).



(a) (b)

Figure 3. Contact region (a) normal stress (b) displacements.

The results for normal stresses presents in the same shape of results shown by Sinclair and Cormier [14]. However, due to insufficient data provided in the cited article, it was not possible to reproduce their results. Note that the normal stress peaks at the ends of the contact region, falling in the central region of the contact. This behavior is the opposite of the normal stress that occurs in the contact of two cylinders without friction (Hertz problem), where the highest stress occurs at the central point of contact.

5 Conclusions

This work presents a formulation based on the boundary element method for the solution of problems with body force under centrifugal loads in anisotropic materials.

An analysis of an aeronautical turbine blade was used to demonstrate the possibility of a practical application of the developed formulation. The results had the expected behavior, although a quantitative analysis of the errors was not made, due to the insufficiency of the data of the work that presents the original problem.

The modeling of the problems was performed with discontinuous quadratic elements, which brings advantages and disadvantages with respect to the continuous elements. As advantages we can mention the non-sharing of nodes, which facilitates the implementation. As a disadvantage, we can mention a bigger number of degrees of freedom when compared to a continuous element mesh with the same number of elements, since in discontinuous element the nodes are not shared between neighboring elements.

Another peculiar feature of this work was the use of the local reference system to represent the boundary conditions and the unknown variables in the boundary. This facilitated the imposition of boundary conditions of restricted displacements in the normal direction to the boundary, which is a common type of boundary condition in engineering problems. For example, frictionless contact problems where a contact surface is flat can be implemented without any code changes. This was demonstrated in the numerical example shown in the results section.

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