Secondary torsion moment deformation effect (STMDE) by Boundary Element Method in composite bars with variable cross sections

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Abstract. This paper proposes a formulation for solving torsion problems in composite bars with variable cross sections including secondary torsion moment deformation effect (STMDE). The Boundary Element Subregion-by-Subregion (BE-SBS) Technique is applied to model cross sections with general shape and constituted of any number of different materials. Four boundary value problems are solved. The primary and secondary warping functions are determined by applying the Boundary Element Method, while the primary and secondary twist angles, θ^p and θ^s, along the element axis are solved by a process based on the weighted residual method. It is worth mentioning that iterative Krylov's solvers are used, which results in saving memory and solver time. The efficiency, robustness and accuracy of the implemented algorithm are demonstrated in several numerical examples.

Keywords: Secondary torsion moment deformation effect, generic composite bars, subregioning technique, preconditioned iterative solvers

1 Introduction

In engineering practice, composite bars with variable cross sections are often used in 3d frameworks of curved bridges and viaducts. To obtain an accurate description of the behavior of these structural elements, we can use 3D formulations based on the Finite Element Method (FEM) or the Boundary Element Method (BEM). The results of these methods are usually very accurate, but great computational effort is necessary, and for many structures, due to the richness of details, modeling using these methods is impractical because of the processing time (CPU-time) and quantity of RAM memory required for analysis. An alternative to this difficulty is to have formulations based in beam theory, which are applicable when the solid presents the aspect of a bar.

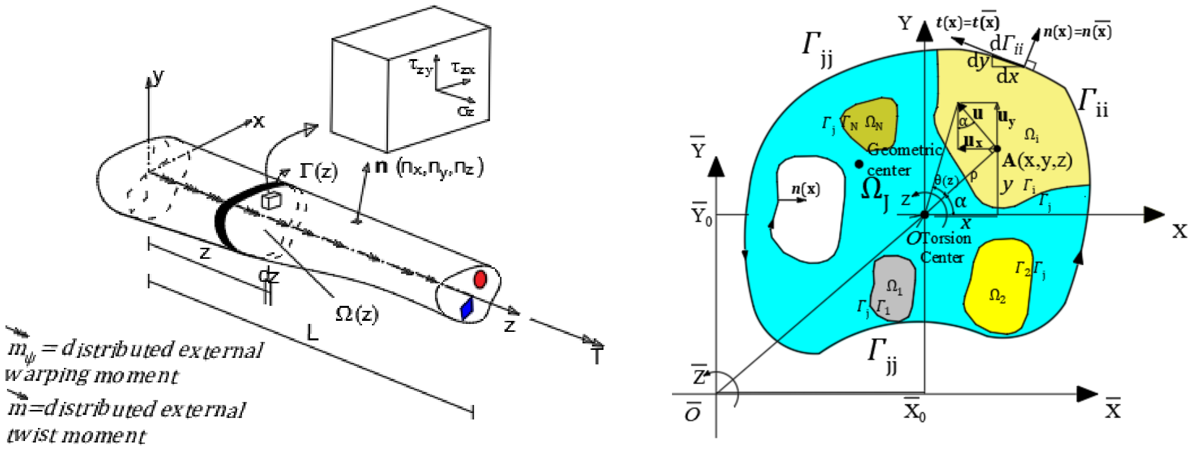
In beam theory, the very complex problem of torsion was satisfactorily solved in a work presented by Saint-Venant [1] and published in 1855. This formulation respects the relations of the theory of elasticity and is based on the semi-inverse method, where the warping function of the cross section is described by the partial differential equation of Laplace, which has an analytical solution only for simple cross sections, such as triangular and elliptical ones. For the other sections, the Laplace equation is solved using numerical methods such as two-dimensional formulation of the BEM, Athanasiadis [2], Sapountzakis [3]. An important aspect of the Siaint-Venant formulation is that it is developed for structural elements in which the warping is free at the supports, and in this case, the angular displacement rate is constant along the length of the member (Uniform torsion). But when there is restriction to warping at the end of bar, and/or distributed torsional moment load, nonuniform torsion occurs, and the angular displacement rate is no longer constant. In addition, there are normal stresses and secondary shear stresses arising to obtain equilibrium. The study of nonuniform torsion using beam theory and BEM have been intensively addressed in many research works, for example, Sapountzakis and

Mokos [4],[5],[6], Sapountzakis and Dikaros [7],[8] and [9]. The special feature of this paper is the application of the robust BE-SBS technique together with the weighting residues method, allowing the creation of a powerful tool to approach composite bars with variable cross section submitted to nonuniform torsion taking into account the secondary shear deformations associated with the secondary torsion moment.

2 Nonuniform torsion in composite bars

To develop the beam theory under nonuniform torsion, considering the shear deformations related to the secondary torsion moment in the solution of the problem (STMDE), the displacement field of a point in cross section located at z coordinate (see Fig. 01) is given by equation 01 and is separated into a primary and a secondary part, as shown below:

$$\begin{aligned} u_x &= u_x^p + u_x^s, \\ u_y &= u_y^p + u_y^s, \\ u_z &= u_z^p + u_z^s. \end{aligned} \tag{1}$$



a) Composite bar with stress tensor b) Composite cross section

Figure 01: Composite bars with variable cross section subjected to a twist moment.

In beam theory, the bar is represented by a single-line axis, the z axis, with the twist angle measured in radians for the torsion problems. Twist angle is partitioned into two components: primary and secondary, according to equation 02:

$$\theta(z) = \theta^p(z) + \theta^s(z). \tag{2}$$

The displacement field of a point in cross section is a function of the twist angle $\theta(z)$ and is described by the expressions defined in equation 03:

$$\begin{aligned} u_x &= -(\theta^p + \theta^s)y, \\ u_y &= (\theta^p + \theta^s)x, \\ u_z &= \frac{d\theta^p}{dz}\psi^p(x,y) + \frac{d\theta^s}{dz}\psi^s(x,y). \end{aligned} \tag{3}$$

Submitting the displacement field defined in equation 03, for a sub-region of domain Ωi of the cross section, in the stress and strain relations and then in the equilibrium equations of the theory of elasticity, we obtain the two boundary value problems (BVPs): the first, equation 4, associated with the primary warping function $[\psi^p(x,y)]$, and the respective boundary conditions (Γ_{ii}) and interface (Γ_{ij}), defined by equations 5 and 6. The second, BVP associated with secondary torsion, which is defined by equation 7, and the respective boundary and interface conditions which are defined by equations 8 and 9.

$$\frac{\partial \tau_{xz}^p}{\partial x} + \frac{\partial \tau_{yz}^p}{\partial y} = G_i \frac{d\theta}{dz}\left(\frac{\partial^2 \psi^p}{\partial x^2} + \frac{\partial^2 \psi^p}{\partial y^2}\right) = \nabla^2 \psi^p(xy) = 0, \qquad se\ x \in \Omega_i, \tag{4}$$

$$p(x) = \nabla\psi^p(x) \cdot n(x) = \frac{\partial\psi^p}{\partial n} = x \cdot t(x),\ if\ x \in \Gamma_{ii}, \tag{5}$$

$$G_i\left[p_{ij} - \big(xt(x)\big)_{ij}\right] = G_j\left[p_{ji} - \big(xt(x)\big)_{ji}\right], \qquad if\ x \in \Gamma_{ij}. \tag{6}$$

It is worth explaining that the boundary value problem defined in equation 07 corresponds to the increment of the warping function $\Delta\psi(x,y)$, and must be added to the primary warping to obtain the secondary warping $\psi^s(x,y)$.

$$\frac{\partial\tau_{xz}^s}{\partial x} + \frac{\partial\tau_{yz}^s}{\partial y} = \frac{\sigma_z}{dz} = G_i\frac{d\theta^s}{dz}\left(\frac{\partial^2\Delta\psi}{\partial x^2} + \frac{\partial^2\Delta\psi}{\partial y^2}\right) = G\frac{d\theta^s}{dz}\nabla^2\Delta\psi\ \ (x,y) = -\frac{d}{dz}E\frac{d^2\theta^p}{dz^2}\psi^p(x,y). \tag{7}$$

$$\tau_{xz}^s n_x + \tau_{yz}^s n_y = G\frac{d\theta^s}{dz}\left(\frac{\partial\Delta\psi}{\partial x}\right)n_x + G\frac{d\theta^s}{dz}\left(\frac{\partial\Delta\psi}{\partial y}\right)n_y = \frac{d\theta^s}{dz}\left(\frac{\partial\Delta\psi}{\partial n}\right) = -\sigma_z n_z \cong 0, \qquad if\ x \in \Gamma_{ii} \tag{8}$$

$$G_i\left[p_{ij} - \big(xt(x)\big)_{ij}\right] = G_j\left[p_{ji} - \big(xt(x)\big)_{ji}\right], \qquad if\ x \in \Gamma_{ij}. \tag{9}$$

In nonuniform torsion, the primary shear stresses are linked to the local equilibrium of the cross section (Saint-Venant Torsion), which, in order to obtain equilibrium, is necessary to warp, due to the intensity of the shear stresses and shear strain variation along the cross section. When there is a restriction to warping at the ends of the bars, normal stresses arise. Moreover, the normal stress varies along z axis, and secondary longitudinal shear stresses arise to the bar to obtain equilibrium. In this context, the primary shear stresses are linked with the primary torsion moment M_t^p and the secondary shear stresses are linked with the secondary torsion moment M_t^s. The total torsion moment is given by the sum of the primary and secondary parts, according to equation 10:

$$M_t\ = M_t^p + M_t^s, \tag{10}$$

where the primary torsion moment is defined by equation 11:

$$M_t^p = \sum_{i=1}^{nsub}\int x\tau_{yz}^p + y\tau_{xz}^p d\Omega_i = k_t^p\frac{d\theta}{dz}, \tag{11}$$

where, k_t^p is the primary torsion rigidity. The secondary torsion moment is defined by equation 12:

$$M_t^s = \sum_{i=1}^{nsub}\int x\tau_{yz}^s + y\tau_{xz}^s\, d\Omega_i = k_t^S\frac{d\theta^s}{dz}, \tag{12}$$

where, secondary torsion rigidity, given according expression 13,

$$k_t^s = \frac{k_\psi^2}{\sum_{i=1}^{nsub} G\int\nabla\Delta\bar{\psi}\ \ \cdot\nabla\Delta\bar{\psi}\ \ d\Omega_i}. \tag{13}$$

The function $\Delta\bar{\psi}^S$ is obtained through the BVP defined by equation 7 by suppressing the variable $d^3\theta/dz^3$.

The warping stiffness k_ψ, in equation 13 depends on the primary warping function $\psi^p(x,y)$ and is defined by equation 14,

$$k_\psi = \sum_{i=1}^{nsub} E_i\int[\psi^p(x,y)]^2 d\Omega_i\,. \tag{14}$$

Warping stiffness is associated with warping moment or bimoment and is a result of normal stresses σ_z that arise in nonuniform torsion according with equation 15,

$$M_\psi = \sum_{i=1}^{nsub} E_i \int [\psi^p(x,y)]^2 d\Omega_i \cdot \frac{d^2\theta^p(z)}{dz^2} = k_\psi(z)\frac{d^2\theta^p(z)}{dz^2}. \tag{15}$$

2.1 Global torsion BVP for the beam

Adding the twist moment (primary torsion moment and secondary torsion moment) and the bimoment, two differential equations associated with the primary and secondary parts of twist angle are obtained. Those equations are defined by equations 16 and 17, and the respective boundary conditions, given in equation 18 for z = 0 and z =L.

$$\sum M_t = \frac{dM_t}{dz} + m(z) = \frac{d}{dz}\left[k_t^p(z)\frac{d\theta(z)}{dz} + k_t^s(z)\frac{d\theta^s(z)}{dz}\right] + m(z) = 0 \tag{16}$$

$$\sum \frac{dM_\psi}{dz} = M_t^s(z) + m\psi(z) = \frac{d}{dz}k_\psi(z)\cdot\frac{d^2}{dz^2}\theta^p(z) + k_\psi(z)\cdot\frac{d^3}{dz^3}\theta^p(z) + m\psi(z) = 0. \tag{17}$$

$$\begin{cases} \alpha_1(z)\theta(z) + \alpha_2(z)M_t(z) + \alpha_3 \\ \beta_1(z)\dfrac{d\theta^p(z)}{dz} + \beta_2(z)M_\psi(z) + \beta_3. \end{cases} \tag{18}$$

To solve the global boundary value problem associated with the twist angle, the primary and secondary θ^p(z) and θ^s(z) are approximated by polynomials, and the variable stiffnesses $k_t^p(z)$, $k_t^s(z)$ and $k_\psi(z)$ too. However, to obtain the rigidities, a simple interpolation of the polynomials from a generic sample of points obtained along the bar (z axis) is enough. To obtain the twist angle, a strategy based on the weighted residuals method is used, according to equation 19, as it is necessary to solve two BVPs, equations 16 and 17.

$$\begin{cases} \displaystyle\int_0^l \{\mathcal{L}_1[\theta^p(z),\theta^s(z)] + m(z)w_k\}\,dz = 0, & k=1,2...d-1, \\ \displaystyle\int_0^l \{\mathcal{L}_2[\theta^p(z),\theta^s(z)] + m_\psi(z)w_k\}\,dz = 0, & J=1,2...d-2. \end{cases} \tag{19}$$

Where the weighting function is defined by equation 20

$$w_k = z^{k=0} \cdots z^{k=d-4}, \tag{20}$$

To apply the weighted residuals method, explained in equations 19, the integrals of the process are involved, so the Gauss Legendre Standard numerical integration process is used. The application of the weighted residuals method results in a system of equations whose order depends on the degree (d) of the interpolating polynomial used to represent the twist angle.

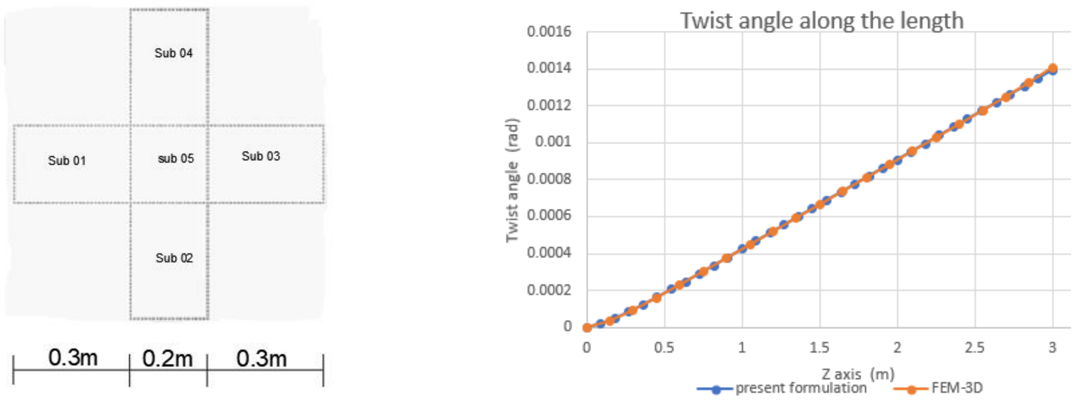
3 The Boundary Element Subregion-by-Subregion (BE-SBS) Technique

In the present paper, the boundary value problems described by equations 4 and 7 are solved using a two-dimensional formulation of the BEM, and because the present paper proposes to consider composite section bars, a sub-structuring strategy is needed. In this context, the BE-SBS incorporated in this work, whose development details can be seen in Araújo, at all [10],[11],[12],[13],[14], [15] and [16], allows coupling a generic number of subdomains, enabling the analysis of solid composites with a high complexity. Another relevant aspect of the technique is that the generation of each sub-region is independent and the search for coupled nodes is automated. One of its main advantages is the use of Krylov's Iterative solvers, which allows to dispense the explicit assembly of the global system of equations, and operations with large blocks of zeros that are common in coupled systems are avoided. These details bring efficiency and robustness to the algorithm. In the BE-SBS technique, several solvers are combined, being used in this paper, the BiCG-J solver, which combines the BiCG method with Jacobi's preconditioning. It is worth noting that for the nonuniform torsion, two BVPs are solved by BEM, and the pre-conditioned matrix for the first BVP (primary warping) does not change during the process. This means that the matrix of the first BVP (primary warping) can be used for the second BVP (warping increment), saving CPU-time and RAM memory.

Two other relevant aspects of the technique should be mentioned, the first is the use of discontinuous elements to consider the discontinuities of flux in the corners of the interfaces; the second aspect is the use of special integration algorithms based on the cubic coordinate transformation of Telles [17], These algorithms of integration, whose details can be seen in Araújo at all [12] and [16], provide agility and accuracy to the code, as the coefficients of the matrices can be calculated with good precision, using few integration Gauss points.

4 Applications

Application 01 - It is a prismatic column with a homogeneous cross section, with the shape of the "+" symbol. The column is 3.0m high, with the base embedded and the upper end loaded with a twist moment of 25 kNm. The material used in the bar has (E=30Gpa, v = 0.2). The aspect of the boundary element mesh of the 5 sub-regions resulting from the section decomposition is shown in Fig. 02-a, and the comparison of twist angle obtained with 3D FE model by ANSYS [18] is shown in Figure 02-b. About the boundary element mesh, the sub1,2,3 and 4 contain 50 3-noded (quadratic) boundary elements, the sub 5 contains 40 3-noded (quadratic) boundary elements. In corners, discontinuous elements are used. To solve warping function, BVPs 4 and 7, BE matrices were calculated with 6 integration Gauss points. About 3D FE mesh, it has 458561 nodes and 105000 quadratic solid finite elements, shown in Fig. 03-a. For the global analysis of the bar, primary and secondary twist angles are represented by a 9-degree polynomial, and 9 integration Gauss points were also used to obtain coefficients of the equations system by weighted residuals method.



a) BE Mesh of cross section b) Twist angle (rad)

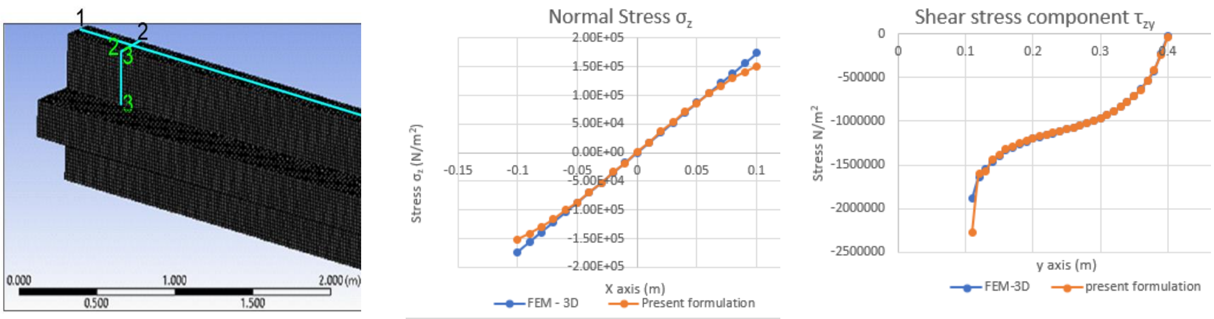
Figure 02- BE model of beam

The properties of the cross section are shown in table 01.

Table 01 – properties of the cross section for application 1

k_t^p(Nm^2)	k_t^s(Nm^2)	k_ψ (Nm^4)
51523466.070	136867052.468	1188559.542

For an even more consistent measurement, we compared the normal stress σ_z along the line (2-2) and the shear stress component τ_{zy} along the line (3-3), shown in Fig. 03 b and c respectively with 3D FEM. About the performance of the solvers used in this algorithm, with respect to BVPs solved with BE-SBS, the resulting BE model has 540 degrees of freedom with 73.4% sparsity. To solve the BVP associated with the primary warping function, 60 iterations were used and spent 0.29 seconds of CPU-time. To solve the BVP associated with the warping increment, 68 iterations were necessary and spent 0.32 seconds of CPU time. However, concerning the CPU time, it is worth explaining that the capacity of the computer used is appropriate for this kind of processing, since CPU time depends on the computer performance.



a) 3D FE models

b) Normal stress along the line (2-2)

c) Shear stress along the line (3-3)

Figure 03 – ANSYS 3D model, normal stress and shear stress

Application 02 – It is a composite bar with variable cross section studied by Sapountzakis and Mokos [5], which has a flange material of steel (E_s=2.1x10^{4}kN/cm^2 and G_s=8.05x10^{3}kN/cm^2) and web material of aluminum (E_a =7.0x10^{3}kN/cm^2 and G_a=2.6x10^{3}kN/cm^2). The variation of dimensions along the length and loads are given in Fig. 4 -a and b.

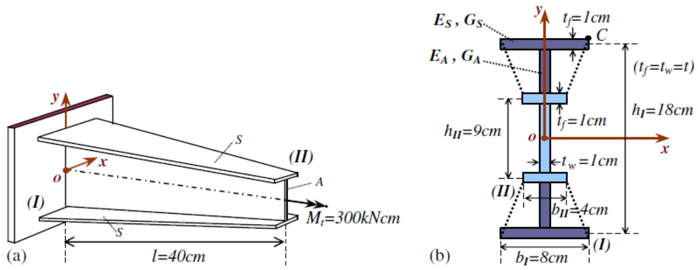
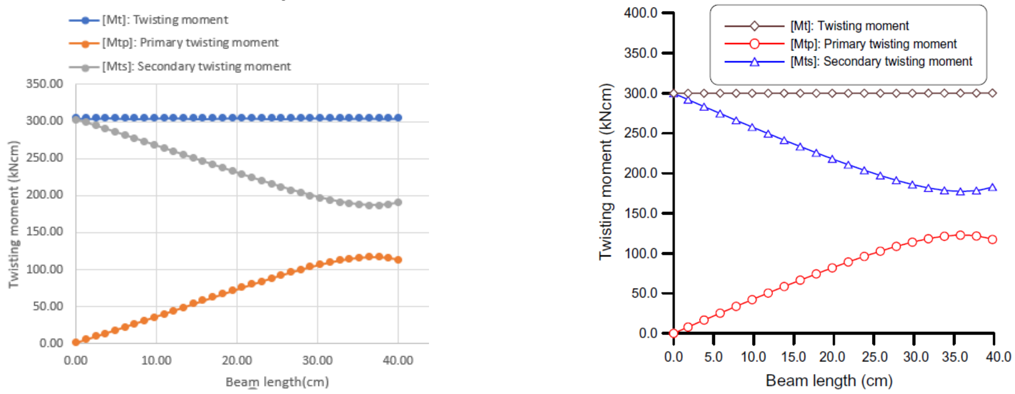


Figure 4 – Longitudinal (a) and transverse (b) sections of the variable I

Regarding the results, the torsion moments along z axis are shown in Fig. 05. The twist angle $\theta(l)$ obtained was also similar to those found in the literature, [5] obtained 7.44 x 10^{-2} rad and in the present formulation, it is 7.99 x 10-^{2} rad. The results are very similar to those in the literature.



Present formulation

Sapountzakis e Mokos [5]

Figure 05 – Torsion moments, M_t , M_t^p and M_t^s

It is important to note that the process used in the study [5] does not separate the primary and secondary parts of the twist angle, but the formulations give very similar results in structural elements with open cross section.

5 Conclusion

In application 1, we can see that the formulation developed based on beam theory was capable of reproducing the precise response of the analysis based on the 3D analysis of the theory of elasticity with FEM.

The similarity between the answers is surprising.

Special integration algorithms based on the Telles cubic transformation are implemented in the code developed in this paper, demonstrating that BEM is a viable alternative to solve the torsion problem. This can be seen by the good results observed in the two applications explored in this paper.

The developed process is robust and generic to solve the complex problem of nonuniform torsion, and the bars can be prismatic or with variable cross section, and can contain homogeneous or composite cross sections, whether thin or thick walled.

Another important detail is the simplicity of considering the most diverse boundary conditions in bars, implying a plainness in obtaining the spatial frame stiffness matrix containing degree freedom associated with the bimoment. Thus, it is easily incorporated into a space frame program that considers nonuniform torsion.

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