A Comparison Between Cell-less Formulations for Domain Integrals Treatment in the Poisson type problem by the Boundary Element Method

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Abstract. By applying the Boundary Element Method (BEM) to solve Poisson problems, besides boundary integrals, the final equation also contains a domain integral involving the inhomogeneous term. This integral can be evaluated by discretizing the domain into cells. However, one of the main advantages of the BEM, which is the reduction of the problem's dimension by one order, is lost and several methods have been developed aiming to treat these integrals without the need to do this discretization. This paper makes a comparison among three formulations to do this: the Dual Reciprocity Method (DRM), the Multiple Reciprocity Method (MRM) and the Radial Integration Method (RIM). The formulations and features of each method are presented as well as numeric results obtained by their applications to some examples with known analytical solutions. The aim of this paper is to make a critical analysis in terms of the numeric efficiency and accuracy of each technique, primarily about the number of elements and internal points required for obtaining results inside an acceptable margin of error.

Keywords: Boundary Element Method, Dual Reciprocity Method, Multiple Reciprocity Method, Radial Integration Method, Direct Integration Method

1 Introduction

The Boundary Element Method (BEM) is currently a well-known and established numerical technique that has become an interesting and efficient alternative to other common and well-known methods, such as Finite Elements Methos and Finite Difference Method, to solve different types of ordinary problems in engineering [1–3]. One of its main features lies on the fact that it requires discretization of the boundary only, rather than its complete domain. It reduces dramatically the amount of data needed for the problem description and consequently the computational cost in solution processing, once it involves smaller order matrices.

However, in some cases, such as Poisson's problem, transient regime problem, physically nonlinear problems and others, some terms are incorporated to the final integral equation through a domain integral. They are called domain terms. An option to treat this integral is by internal cells, which basically is a discretization of (at least) part of the domain. However, it causes the loss of the main feature of the method, which is the discretization of the boundary only [4].

In order to preserve this important characteristic of the BEM, some alternative methods to cell discretization were developed. Some of them are the Galerkin vector approach [5], the Dual Reciprocity Method (DRM) [6, 7], the Multiple Reciprocity Method (MRM) [4, 8, 9], the Radial Integration Method (RIM) [10] and the Direct Integration Method (DIM) [11]. All of these methods have one thing in common: the goal to turn the non-desired domain integral into equivalent boundary integrals.

Problably, the most known and used cell-less method by the Boundary Element community is the DRM, which, however, requires the approximation of the previous known functions, a choice of a set of interpolation functions for this purpose and the adoption of a series of internal auxiliary points to guarantee a good efficy of such interpolation. Thus, this paperwork proposes the implementation and comparisson of the DRM with the MRM and the RIM. Some comparisons can be found in the literature, such as a comparison between DRM and MRM [12]. However, a more comprehensive analysis involving those methods and including the RIM is yet to be performed.

2 Integral Equations for Poisson's Problem

The Poisson's Equation is a very important equation in Engineering, once it describes a very large number of problems. By using index notation and assuming cartesian coordinates, it can be written as

$$u_{,ii}\left(\mathbf{x}\right) + b(\mathbf{x}) = 0 \tag{1}$$

where $u(\mathbf{x})$ is a scalar potential function, $b(\mathbf{x})$ is an known function and $\mathbf{a}$ are the coordinates of a generic point in the domain of the problem, Ω, whose boundary is denoted by Γ. In addition to this equation, two types of boundary conditions are considered: essential boundary conditions, given by $u = \bar{u}$ on Γ_1 and the natural boundary conditions, given by $q = \frac{\partial u}{\partial n} = \bar{q}$ on Γ_2. Here Γ_1 and Γ_2 are parts of the boundary such that $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. The outward-pointing normal vector to the boundary is defined by n_i.

The inverse integral form of this problem, on which the BEM is based on, is given by the following equation [2]:

$$c(\boldsymbol{\xi})u(\boldsymbol{\xi}) = \int_\Gamma u^*(\boldsymbol{\xi},\mathbf{x})q(\mathbf{x})\,d\Gamma - \int_\Gamma q^*(\boldsymbol{\xi},\mathbf{x})u(\mathbf{x})\,d\Gamma + \int_\Omega u^*(\boldsymbol{\xi},\mathbf{x})b(\mathbf{x})\,d\Omega \tag{2}$$

where $\boldsymbol{\xi}$ is the source point, $c(\boldsymbol{\xi})$ depends on the position of this point relative to the boundary and $u^*(\boldsymbol{\xi},\mathbf{x})$ and $q^*(\boldsymbol{\xi},\mathbf{x})$ are the fundamental solutions of the infinite domain correlated problem with $b(\mathbf{x}) = \delta(\mathbf{x}-\boldsymbol{\xi})$, i.e., when the domain term is the Dirac delta function on $\boldsymbol{\xi}$. Such solutions are given by

$$u^*(\boldsymbol{\xi},\mathbf{x}) = -\frac{1}{2\pi}\ln(r), \qquad q^*(\boldsymbol{\xi},\mathbf{x}) \equiv \frac{\partial u^*(\boldsymbol{\xi},\mathbf{x})}{\partial n(\mathbf{x})} = -\frac{1}{2\pi r^2}(r_i n_i), \qquad r_i = x_i - \xi_i \tag{3}$$

It is usually important to know the gradients of the potential function inside the domain.This can be done for every point by using the equation obtained directly from the derivation of the equation (2) with respect to the source point, i.e.,

$$q_i(\boldsymbol{\xi}) \equiv \frac{\partial u(\boldsymbol{\xi})}{\partial \xi_i} = \int_\Gamma \bar{u}_i^*(\boldsymbol{\xi},\mathbf{x})q(\mathbf{x})\,d\Gamma - \int_\Gamma \bar{q}_i^*(\boldsymbol{\xi},\mathbf{x})u(\mathbf{x})\,d\Gamma + \int_\Omega \bar{u}_i^*(\boldsymbol{\xi},\mathbf{x})b(\mathbf{x})\,d\Omega \tag{4}$$

where

$$\bar{u}_i^*(\boldsymbol{\xi},\mathbf{x}) \equiv \frac{\partial u^*(\boldsymbol{\xi},\mathbf{x})}{\partial \xi_i} = \frac{1}{2\pi r^2}r_i. \qquad \bar{q}_i^*(\boldsymbol{\xi},\mathbf{x}) \equiv \frac{\partial q^*(\boldsymbol{\xi},\mathbf{x})}{\partial \xi_i} = \frac{1}{2\pi r^2}\left[n_i - 2\frac{r_k n_k}{r^2}r_i\right] \tag{5}$$

2.1 Discrete Equations

The boundary of the domain can be completely discritized into elements, and each element will contain a specific number of source points. The boundary variables (potential and flux) are interpolated through these source points, which leads to the rewriting of the equation (2) such that

$$[H]\{u\} = [G]\{q\} + \{d\} \tag{6}$$

where the vectors $\{u\}$ and $\{q\}$ contain the boundary variable u^k and q^k ($k = 1,\ldots,N$), respectively, on the N source points. Both vectors may contain known and unknown variables, depending on the boundary conditions.

Each row of the matrices $[H]$ and $[G]$ corresponds to a specific source point of the boundary, where each coefficient on a same row is obtained from the integrations through the elements. For 2D case, such integrals are given respectively by

$$\int_{-1}^{+1} q^*(\boldsymbol{\xi}^i,\mathbf{x}(\eta))N_k(\eta)\mathcal{J}_e(\eta)\,d\eta, \qquad \int_{-1}^{+1} u^*(\boldsymbol{\xi}^i,\mathbf{x}(\eta))N_k(\eta)\mathcal{J}_e(\eta)\,d\eta \tag{7}$$

where η is a local (parametric) coordinate of the element, such that $\eta \in [-1,+1]$, N_k are interpolation functions of the boundary quantities, and $\mathcal{J}_e(\eta)$ represents the Jacobian of the coordinates transformation within the element. In this paper linear interpolation functions are considered.

It is also pointed out that free terms $c(\boldsymbol{\xi}^i)$ are added to the terms on the diagonal of the matrix $[H]$. Even so, in this work both, free terms and strongly singular integral, which is represented by the first integral of equation (7) when the source point corresponds to the k-th node of the element that is being integrated, are evaluated indirectly by application of a constant potential field.

On the other hand, the weakly singular integrals, which is represented by the second integral of equation (7), are treated analytically when the elements are continuous, or numerically through Gauss-Laguerre quadrature

when the elements are discontinuous. Discontinuous elements are used for treatment of corners or other flux discontinuities.

Applying the boundary conditions to the equation (6) one obtains

$$[A]\{x\} = [B]\{y\} + \{d\} = \{f\} \tag{8}$$

where $\{x\}$ is the vector which contains unknown variables and $\{y\}$ is the vector that contains the prescribed values.

The vector $\{d\}$ present on equation (6) is obtained from the domain integral of the equation (2), i.e.

$$\{d\} \rightarrow \int_{\Omega} u^*(\boldsymbol{\xi}, \mathbf{x}) b(\mathbf{x})\, d\Omega \tag{9}$$

The methods for obtaining it, which this paperwork intend to compare, are presented in the next section.

3 Methods for domain integrals treatment

The Dual Reciprocity Method, Multiple Reciprocity Method and Radial Integration Method are compared through their application on problems with known analytical solutions. Before that, their theoretical formulations are briefly presented.

3.1 Dual Reciprocity Method

The starting point, as described in [7], is to approximate the inhomogeneous term of Poisson's equation by a linear combination of $N + M$ pre-setted functions, $f_j(\mathbf{x})$, that have well-defined second-order primitives, i.e.,

$$b(\mathbf{x}) \approx \alpha_j f_j(\mathbf{x}) = \alpha_j \psi^{,j}_{,ii}(\mathbf{x}), \qquad j = 1, \ldots, N + M \tag{10}$$

where M is the amount of internal auxiliary points.

In this context, one can see that the functions $\psi^j(\mathbf{X})$ correspond to particular solutions of the original differential equation (1).

Substituting equation (10) into integral equation (9) and applying the Reciprocity Theorem, one obtains the following result:

$$\int_{\Omega} b(\mathbf{x}) u^*(\boldsymbol{\xi}, \mathbf{x}) d\Omega \approx \alpha_j \left\{ \int_{\Gamma} \phi^j(\mathbf{x}) u^*(\boldsymbol{\xi}, \mathbf{x}) d\Gamma - \int_{\Gamma} \psi^j(\mathbf{x}) q^*(\boldsymbol{\xi}, \mathbf{x}) d\Gamma - c(\boldsymbol{\xi}) \psi^j(\boldsymbol{\xi}) \right\} \tag{11}$$

where $\phi^j = \psi^j_{,i} n_i = \dfrac{\partial \psi^j}{\partial n}$ and the free term $c(\boldsymbol{\xi})$ is introduced to account for cases in which $\boldsymbol{\xi} \in \Gamma$.

Although the functions ψ^j and ϕ^j are known, when adopting the same boundary discretization to approximate these functions as performed for the quantities $u(\mathbf{x})$ and $q(\mathbf{x})$, the same integrals highlighted in equation (7) reappear in the calculation of the matrices that will result in the vector $\{d\}$ of equation (9). That is, writing explicitly,

$$\{d\} = \big([H][\Psi] - [G][\Phi]\big)\{\alpha\} \tag{12}$$

The matrices $[\Psi]$ and $[\Phi]$ can be constructed from the definition of the approximation functions f_j. For this, it is usual to adopt radial basis functions, i.e., functions that depend on the distance between a predefined point, $\mathbf{x}^j$, and the field point, $\mathbf{x}$. Mathematically, $f_j(\mathbf{x}) \equiv f_j(r(\mathbf{x}^j, \mathbf{x})) \equiv f_j(\mathbf{x}^j, \mathbf{x})$. Thus, inversely to $[H]$ and $[G]$, the matrices $[\Psi]$ and $[\Phi]$ have each column associated with one of these predefined points, these being the N boundary collocation points, added to the M internal auxiliary points. Each coefficient in a given column corresponds to the functions $\psi^j(\mathbf{x}^j, \mathbf{x})$ and $\phi^j(\mathbf{x}^j, \mathbf{x})$ calculated by taking into account the distance from the predefined point, $\mathbf{x}^j$, to the N boundary collocation point.

The vector $\{\alpha\}$, on the other hand, contains the $(N + M)$ scalar coefficients of the approximation defined in equation (10), initially unknown. However, from the pre-definition of f_j, one can write:

$$\{\alpha\} = [F]^{-1}\{b\} \tag{13}$$

where $[F]$ is a square matrix, involving the distances between internal and boundary points, and b is a vector containing the values of $b(\mathbf{x})$ at these same points, i.e., $F_{jk} = f_j(\mathbf{x}^j, \mathbf{x}^k)$ and $b_k = b(\mathbf{x}^k)$.

Applying equation (13) to (12) yields a final expression for vector $\{d\}$ from the DRM:

$$\{d\} = \big([H][\Psi] - [G][\Phi]\big)[F]^{-1}\{b\} \tag{14}$$

Finally, it should be noted that internal values can be obtained in a step following the calculation of the boundary values, by changing the source point, $\boldsymbol{\xi}$, but taking advantage of the matrices $[\Psi]$ and $[\Phi]$, besides the vector $\{\alpha\}$, already calculated.

3.2 Multiple Reciprocity Method

In the Multiple Reciprocity Method, according to [8], the Reciprocity Theorem is applied multiple times to the domain integral of equation (9), unlike in the DRM, in which this theorem is applied only once and then the inhomogeneous term is approximated by the linear combination of radial basis functions.

In this procedure, each coefficient of the vector $\{d\}$ is obtained independently, which avoids the need for the introduction of auxiliary internal points. On the other hand, the MRM requires the knowledge and implementation of primitives of both the fundamental solutions and the inhomogeneous term [9].

Each coefficient of the vector $\{d\}$ is given by [8, 9]:

$$d(\boldsymbol{\xi}) = \sum_{j=0}^{\infty} \int_{\Gamma} \left[q^*_{j+1}(\boldsymbol{\xi}, \mathbf{x}) b_j(\mathbf{x}) - u^*_{j+1}(\boldsymbol{\xi}, \mathbf{x}) w_j(\mathbf{x}) \right] d\Gamma \tag{15}$$

where, $u^*_0(\boldsymbol{\xi}, \mathbf{x}) \equiv u^*(\boldsymbol{\xi}, \mathbf{x})$, $b_0(\mathbf{x}) \equiv b(\mathbf{x})$, and

$$u^*_{(j+1),ii}(\boldsymbol{\xi}, \mathbf{x}) = u^*_j(\boldsymbol{\xi}, \mathbf{x}) = \frac{1}{2\pi} r^{2j} \left[A_j \ln(r) - B_j \right] \tag{16}$$

$$q^*_j(\boldsymbol{\xi}, \mathbf{x}) = \frac{\partial u^*_j(\boldsymbol{\xi}, \mathbf{x})}{\partial n(\mathbf{x})} = \frac{1}{2\pi} r^{2(j-1)} \left\{ \left[2j \ln(r) + 1 \right] A_j - 2j B_j \right\} (r_i n_i) \tag{17}$$

$$b_j(\mathbf{x}) = b_{(j-1),ii}(\mathbf{x}), \qquad w_j(\mathbf{x}) = b_{(j),i}(\mathbf{x}) n_i(\mathbf{x}) \equiv \frac{\partial b_j(\mathbf{x})}{\partial n(\mathbf{x})} \tag{18}$$

with,

$$A_0 = -1, \qquad B_0 = 0, \qquad A_{j+1} = \frac{A_j}{4(j+1)^2}, \qquad B_{j+1} = \frac{1}{4(j+1)^2} \left[\frac{A_j}{j+1} + B_j \right] \tag{19}$$

Thus each component of vector $\{d\}$, is obtained by applying the equation (15) to a specific boundary collocation point, i.e., $d_i = d(\boldsymbol{\xi}^i)$. Evidently, one must choose a finite number of terms from the series presented in this equation. It should be pointed out, however, that in many cases, after a few applications of the Laplacian to the inhomogeneous term, b_j and w_j become zero and the series no longer contains infinite terms.

The calculation of internal quantities can be performed in a step subsequent to obtaining their boundary values, in a similar way to what is done in the usual solution of the Laplace Equation, by simply adding a new term $d(\boldsymbol{\xi})$, given in equation (15), where $\boldsymbol{\xi}$ is now an internal point.

3.3 Radial Integration Method

The Radial Integration Method was initially proposed and developed by [10] and is also based on the transformation of the domain integral into a boundary integral. For this purpose, a coordinate transformation is performed, taking the origin at the source point and adopting a polar coordinate system, (r, θ), for the two-dimensional case. Two of its main advantages over methods such as DRM and MRM refer to the fact that it does not require the use of particular solutions of the problem. Its main disadvantage, according to [13], is the high computational cost when compared to other techniques such as those already mentioned.

Similarly to the MRM case, each coefficient of the vector $\{d\}$ is obtained independently, which also avoids the need for use of auxiliary internal points. Such terms are determined from the following expression:

$$d_i = d(\boldsymbol{\xi}^i) = \int_{\Omega} u^*(\boldsymbol{\xi}^i, \mathbf{x}) b(\mathbf{x}) \, d\Omega = \int_{\Gamma} \frac{r_k n_k}{r^2} F(\boldsymbol{\xi}^i, \mathbf{x}) \, d\Gamma \tag{20}$$

where

$$F(\boldsymbol{\xi}^i, \mathbf{x}) = \int_0^{r(\Gamma)} u^*(\boldsymbol{\xi}^i, \mathbf{x}) b(\mathbf{x}(r)) r \, dr \tag{21}$$

where $b(\mathbf{x}(r))$ is obtained using the relation $x_i = \xi_i + r_i = \xi_i + r_{,i}\, r$.

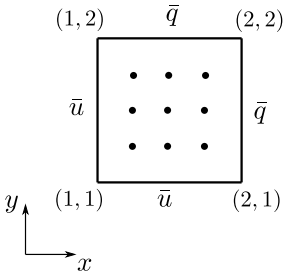
It should also be emphasized, that the integrals of equations (20) and (21) are regular, and can be solved numerically by conventional Gaussian quadrature. Moreover, the numerical integration of Eq. (21) requires the following parametrization:

$$r = \frac{r(\Gamma)}{2} \eta + \frac{r(\Gamma)}{2}, \qquad (-1 \leqslant \eta \leqslant +1) \tag{22}$$

As in MRM, the obtaining of internal quantities is done in a later step than the calculation of their values on the boundary, analogously to the usual procedure performed for the Laplace Equation, but now, adding a term d_i, given by the equation (20), where $\boldsymbol{\xi}^i$ refers, in this case, to the internal point where such quantities are required.

4 Numerical Results

Here, three different two-dimensional Poisson equations are solved numerically by the Boundary Element Method. The domain integrals are calculated using each of the methods described in the last section. The geometry (a square with unitary edge), as well as the boundary conditions of the examples are the same and are shown in Figure 1.



Internal points coordinates (x, y):

$(0.25, 0.75)$; $(0.50, 0.75)$; $(0.75, 0.75)$

$(0.25, 0.50)$; $(0.50, 0.50)$; $(0.75, 0.50)$

$(0.25, 0.25)$; $(0.50, 0.25)$; $(0.75, 0.25)$

$\bar{u} \rightarrow$ Potential prescribed

$\bar{q} \rightarrow$ Flux prescribed

Figure 1. Examples domain, boundary conditions and internal points.

Each example was analysed with different meshes of equal sized linear boundary elements. Nine internal points were introduced (see Figure 1). For the DRM analysis, such internal points are also used for the interpolation of the domain term. In addition, three approximating functions are used for the DRM: $f = 1 + r$, $f = r^3$ and $f = r^5$. The numerical results were compared with their respective analytical solutions. The following quantities were considered: (i) the average error for boundary calculated fields, (ii) the average error for potential at internal points and (iii) the average error for the potential gradient at internal points.

4.1 Example 1

In this example, the term of the domain integral is

$$b(x, y) = x + y + 2$$

whose analytic solution is

$$u(x, y) = 0.25(x^2y + xy^2 + x^2 + y^2)$$

Results are shown in Figure 2.

4.2 Example 2

In this example, the term of the domain integral is

$$b(x, y) = 11x^5 + 9x^4y + 8x^3y^2 + 8x^2y^3 + 9xy^4 + 11y^5$$

whose analytic solution is

$$u(x, y) = 0.25(x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7)$$

Results are shown in Figure 3.

4.3 Example 3

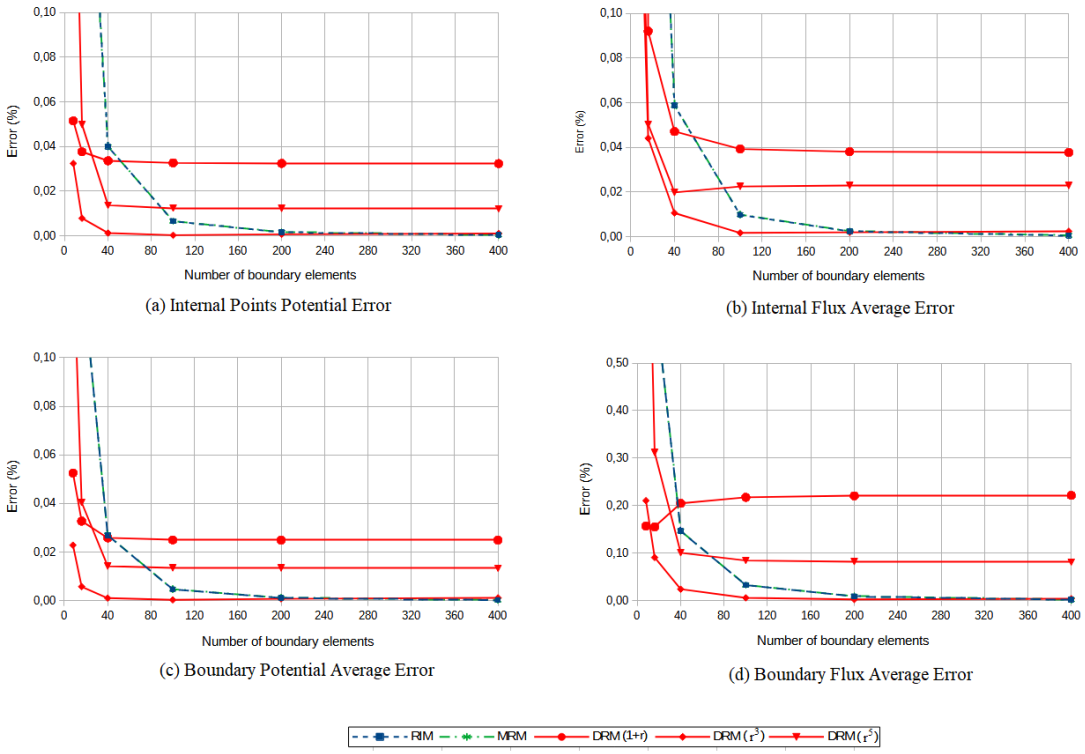
In this example, the term of the domain integral is

$$b(x, y) = (4x^2 - 4x - 1)e^{-(x-0.5)^2} + (4y^2 - 4y - 1)e^{-(y-0.5)^2}$$

whose analytic solution is

$$u(x, y) = e^{-(x-0.5)^2} + e^{-(y-0.5)^2}$$

Results are shown in Figure 4.



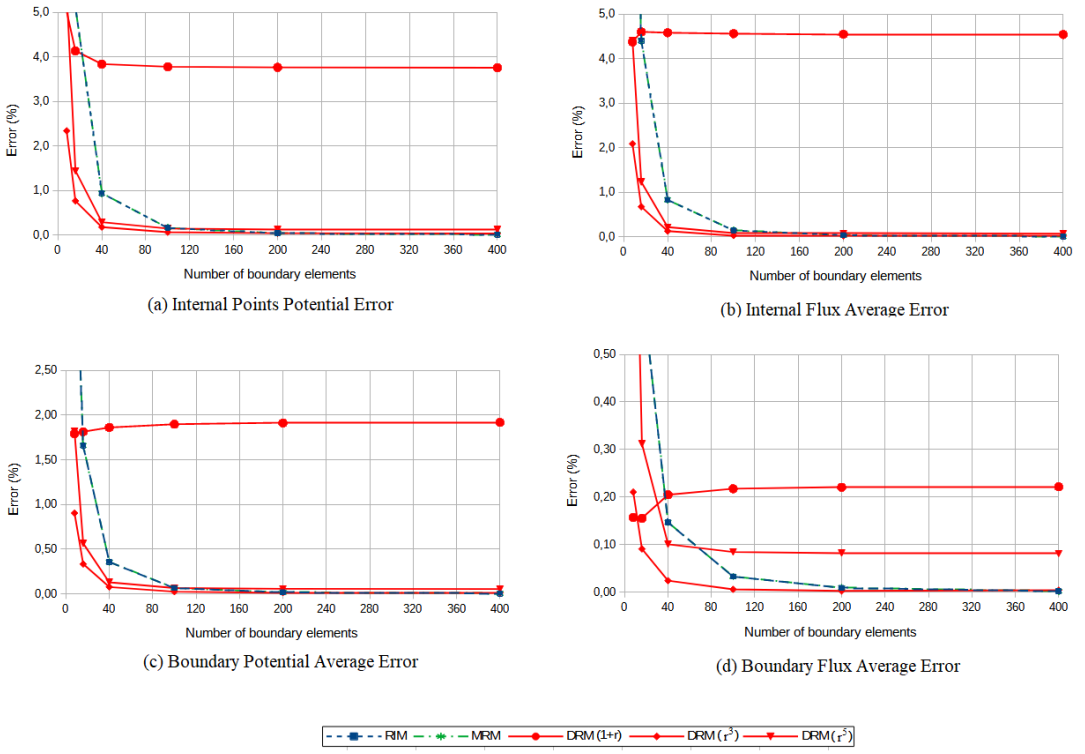
(a) Internal Points Potential Error

(b) Internal Flux Average Error

(c) Boundary Potential Average Error

(d) Boundary Flux Average Error

Figure 2. Percentage errors



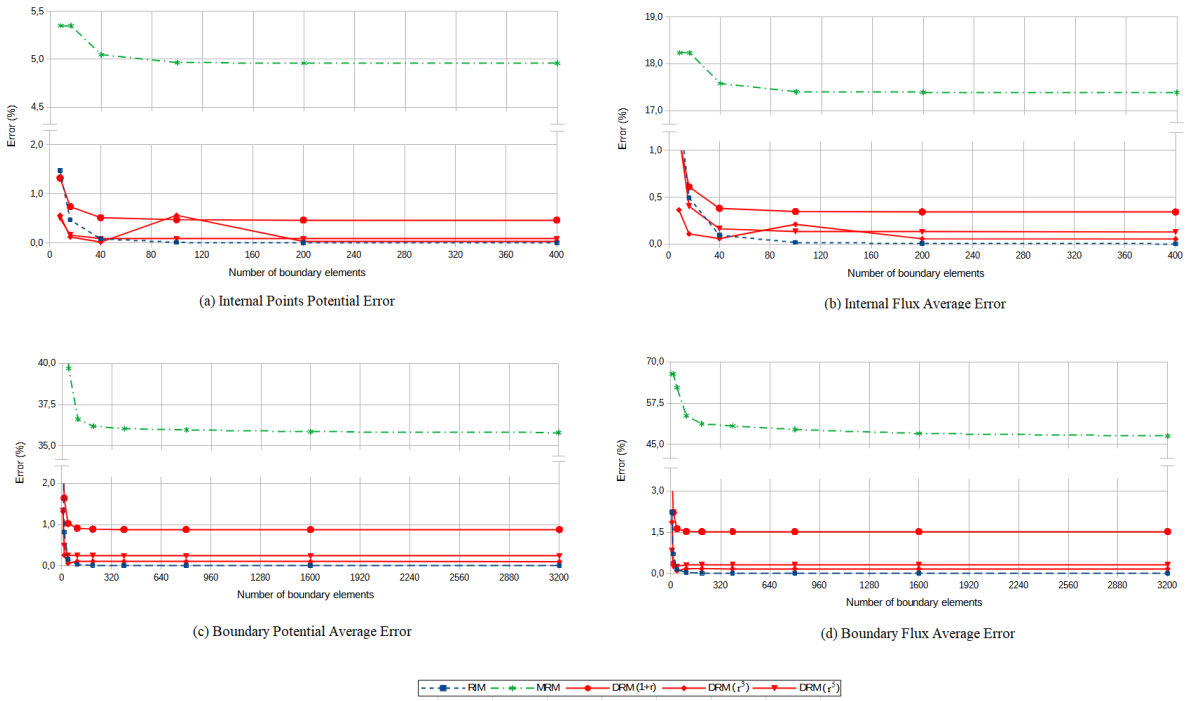
(a) Internal Points Potential Error

(b) Internal Flux Average Error

(c) Boundary Potential Average Error

(d) Boundary Flux Average Error

Figure 3. Percentage errors



(a) Internal Points Potential Error

(b) Internal Flux Average Error

(c) Boundary Potential Average Error

(d) Boundary Flux Average Error

Figure 4. Percentage errors

5 Conclusions

DRM results depend on the interpolation function and the use of a reasonable number of internal poles. On the other hand, RIM and MRM results don't.

When the complete series is used in MRM, which is common when the domain functions are polinomial, the result coincides with RIM. But this is not always possible, because some domain functions produce infinite series, in particular series that don't converge.

When MRM requires an infinite series, truncating it into a few terms often leads to inappropriate results, as shown in the last example.

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