# An isogeometric Boundary Element formulation for solids containing trimmed surfaces

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**Abstract.** The Isogeometric Boundary Element Method (IGABEM) is an accurate and recent approach for solving boundary value problems. This approach is especially accurate in the representation of complex geometries and mechanical responses. Besides, it enables the direct application over Computer Aided Design (CAD) models, once they utilise same basis functions in the parametrization of geometric entities. In the context of elasticity problems, the Non-Uniform Rational B-Splines (NURBS) functions approximate both geometry and mechanical fields. Additionally, CAD models often represent complex solids with trimmed NURBS surfaces. The generation of these type of surfaces rely on the trimming operation, in which NURBS curves represent the edges of a cut. Consequently, a robust IGABEM analysis must account for trimmed surfaces, which is a challenging task nowadays. Hence, the present research deals with the mechanical analysis within IGABEM and trimmed geometries. For this purpose, an identification task based on the ray-casting algorithm detects regions and control points unaffected by trimming operation, trimmed or inactivated. Afterwards, the collocation points in surfaces' corners are moved into the surface, which follows the modified Greville Abscissae (GA) strategy. Besides, the integration process applies the transformations proposed by Kim et al (2009). The proposed formulation applies a non-singular strategy, which avoids additional singularities' treatment. One application demonstrates the accuracy of the proposed formulation, in which the responses have been compared against analytical solutions. A new extension of mechanical analysis within IGABEM has been reached herein, once trimmed surfaces are also incorporated in this numerical method.

**Keywords:** Isogeometric analysis, Boundary Element Method, Three-dimensional modelling, Trimmed analysis

## 1 Introduction

The computational mechanical modelling is a powerful approach to predict the solids' structural behaviour under their current engineering applications. Once it is possible to determine adequately the body's response, the need of experimental tests is reduced, which provides money saving in the design task. Because of the advantages of such tool, they have been widely applied for determining displacements and stress fields of simple and complex engineering components.

Between the existent numerical methods, the Boundary Element Method has been widely used to accurately determine the response of elasticity problems. This method builds an algebraic system of equations based on integral equations written solely by the boundary's displacements and tractions. This manner removes the domain mesh's requirement. Hence, the surface's description is sufficient for the structural analysis of three-dimensional problems. Additionally, Computer Aided Design (CAD) strategy, which uses Non-Uniform Rational B-Splines (NURBS) as basis functions, is a convenient procedure to represent complex geometries, since this enables the description of high curvature surfaces exactly. In this scenario emerges the Isogeometric Boundary Element Method (IGABEM), where NURBS functions interpolate both geometry and mechanical fields. The IGABEM have been applied in various mechanical analysis, such as elasticity Simpson et al. [1], acoustics Simpson et al. [2], and heat transfer An et al. [3], for example, demonstrating a remarkable performance when compared to analytical and numerical benchmarks.

However, the representation of holes or specific details in NURBS surfaces requires an additional task: the trimming operation. Since a tensor product operation between two univariate NURBS curves results in a NURBS patch, these particular geometric aspects become impossible to add directly. Then, the trimming operation uses trimming curves (usually NURBS curves) to define which portions of a NURBS patch remain or not. Therefore, a IGABEM formulation is complete only when it considers the presence of trimmed NURBS surfaces. Because of this, the aim of the present study is to consider trimmed patches in the IGABEM approach. In particular, there are two main aspects to consider: the identification and the integration of these patches. This formulation have been applied herein in one numerical application to analyze its preciseness.

## 2 Isogeometric Boundary Element Method

The Boundary Element Method (BEM) is a numerical method in which integral equations considers solely the mechanical fields at the boundary to determine the body's structural response. For a given solid with boundary $\Gamma$, there are known displacements at the boundary $\Gamma_u$ and tractions at $\Gamma_p$ so that $\Gamma_u \cup \Gamma_p = \Gamma$ and $\Gamma_u \cap \Gamma_p = \varnothing$. In this context, the unknown values are the displacements at $\Gamma_p$ and the tractions at $\Gamma_u$. Considering the linear elasticity, the Somiglian Identity establishes an equation that relates the mechanical fields at the boundary with the solid's material properties. After a limit process, and assuming as nil the body forces, this equation is written as:

$$\delta_{\ell k}\left(\mathbf{x}^s\right) u_k\left(\mathbf{x}^s\right) + \int_\Gamma P^*_{\ell k}\, u_k\, d\Gamma = \int_\Gamma U^*_{\ell k}\, p_k\, d\Gamma \tag{1}$$

in which $u_k$ and $p_k$ are displacements and tractions at the boundary and $\mathbf{x}^s$ is the collocation point. Additionally, the terms $U^*_{\ell k}$ and $P^*_{\ell k}$ refer to the Kelvin's fundamental solution, written as:

$$\begin{aligned} U^*_{\ell k} &= \frac{1}{16\pi\mu\left(1-\nu\right)r}\left[\left(3-4\nu\right)\delta_{\ell k} + r_{,\ell} r_{,l}\right] \\ P^*_{\ell k} &= \frac{1}{8\pi\mu\left(1-\nu\right)r^2}\left\{\frac{\partial r}{\partial \mathbf{n}}\left[\left(1-2\nu\right)\delta_{\ell k} + 3 r_{,\ell} r_{,k}\right] - \left(1-2\nu\right)\left(r_{,\ell} n_k + r_{,k} n_\ell\right)\right\} \end{aligned} \tag{2}$$

where $r$ is the distance between the collocation point and the field point $\mathbf{x}^f$ at the boundary, $\mathbf{n}$ is the normal outward vector, $\delta_{\ell k}$ is the Kronecker delta, and $\mu$ and $\nu$ are the Young's Modulus and Poisson ratio, respectively.

In the present study, the Non-Uniform Rational B-Splines (NURBS) surfaces describe the solid's geometry and mechanical fields at the boundary. Cottrell et al. [4] and Piegl and Tiller [5] provide more details regarding their evaluation and geometrical properties. To obtain such surfaces, there is a tensor product between two univariate NURBS curves. The definition of NURBS curves considers the B-Splines curves with weights at their control points. Hence, a $p$th degree NURBS curve $R_{i,p}$ at the control point $i$ is:

$$\begin{aligned} N_{i,0}\left(\xi\right) &= \begin{cases} 1 & \xi_i \le \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \\ N_{i,p}\left(\xi\right) &= \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}\left(\xi\right) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}\left(\xi\right) \\ R_{i,p}\left(\xi\right) &= \frac{N_{i,p}\left(\xi\right) w_i}{\sum_{j=1}^{n} N_{j,p}\left(\xi\right) w_j} \end{aligned} \tag{3}$$

where the $w_i$ is the corresponding weight and $\mathbf{\Xi} = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$ is the knot-vector. Then, a internal segment $\xi_t \le \xi < \xi_{t+1}$ defines a knot-span. This characteristic enables the representation of an extensive segment with only one NURBS curve.

The tensor product between a $p$th degree NURBS curve $R_{i,p}$ with $n$ control points and knot-vector equals to $\mathbf{\Xi} = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$ and a $q$th degree NURBS curve $Q_{j,q}$ with $m$ control points and knot-vector $\mathbf{N} = \{\eta_1, \eta_2, ..., \eta_{m+q+1}\}$ results in a NURBS surface $S^{p,q}_{i,j}$ as follows:

$$S^{p,q}_{i,j}\left(\xi, \eta\right) = \frac{R_{i,p}\left(\xi\right) Q_{j,q}\left(\eta\right) w_{i,j}}{\sum_{k=1}^{n}\sum_{\ell=1}^{m} R_{k,p}\left(\xi\right) Q_{\ell,q}\left(\eta\right) w_{k,\ell}} \tag{4}$$

Therefore, a $n(P)$th $\times$ $m(P)$th degree $S^{p,q}_{i,j}$ NURBS surface over a solid's boundary discrete portion $P$ approximates the geometry and mechanical fields as:

$$
\begin{aligned}
x_k\left(\xi,\eta\right) &= \sum_{i=1}^{n(P)} \sum_{j=1}^{m(P)} S_{i,j}^{p,q}\left(\xi,\eta\right) x_k^{con(i,j,P)} \\
u_k\left(\xi,\eta\right) &= \sum_{i=1}^{n(P)} \sum_{j=1}^{m(P)} S_{i,j}^{p,q}\left(\xi,\eta\right) d_k^{con(i,j,P)} \\
p_k\left(\xi,\eta\right) &= \sum_{i=1}^{n(P)} \sum_{j=1}^{m(P)} S_{i,j}^{p,q}\left(\xi,\eta\right) t_k^{con(i,j,P)}
\end{aligned}
\tag{5}
$$

in which $d_k$ and $t_k$ do not have physical meaning, being responsible for approximating displacements and tractions, respectively. In addition, $con(i,j,P)$ refers to a connectivity function that associates the control points local indexes $i$ and $j$ of the surface $P$ to the global index.

Additionally, it is inherent to a method such as IGABEM the definition of a collocation strategy. In this research, a modified Greville Abscissae (Greville [6]) defines these points' position on the NURBS parametric space as proposed by Cordeiro and Leonel [7]. Afterwards, there is a repositioning step that moves the collocation point out of the boundary, where the moving distance is equal to a knot-span area's percentage in physical space to guarantee the non-singular IGABEM formulation's use.

The integration in IGABEM considers initially the Gaussian space $\{[\hat{\xi},\hat{\eta}] \in \Lambda \,|\, \Lambda = [-1;1] \times [-1;1]\}$. This task occurs for each knot-span $[\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]$ over the NURBS surface parametric space. In this scenario, $s_\xi$ and $s_\eta$ indicate the total amount of the univariate knot-span at each direction. Hence, two Jacobians , $J_1$ and $J_2$, arise to establish the transformation $d\Gamma = J_1 J_2 d\Lambda$. While $J_1$ refers to a transformation between the physical space and the NURBS parametric space, $J_2$ is the Jacobian between parametric and Gaussian spaces.

Thus, substituting the eq. (5) in the eq. (1) results in the discrete non-singular boundary integral equation:

$$
\begin{aligned}
&\sum_{i=1}^{n(P(\mathbf{x}^s))} \sum_{j=1}^{m(P(\mathbf{x}^s))} S_{i,j}^{p,q}\left(\xi\left(\mathbf{x}^s\right),\eta\left(\mathbf{x}^s\right)\right) d_\ell^{con(i,j,P(\mathbf{x}^s))} \\
&+ \sum_{P=1}^{\mathrm{NP}} \left[ \sum_{u=1}^{s_\xi} \sum_{v=1}^{s_\eta} \sum_{i=\mathrm{inc}_\xi(u)}^{\mathrm{inc}_\xi(u)+p} \sum_{j=\mathrm{inc}_\eta(v)}^{\mathrm{inc}_\eta(v)+q} \int_\Lambda P_{\ell k}^* S_{i,j}^{p,q}\, J_1 J_2 \, d\Lambda \, d_k^{con(i,j,P)} \right] \\
&= \sum_{P=1}^{\mathrm{NP}} \left[ \sum_{u=1}^{s_\xi} \sum_{v=1}^{s_\eta} \sum_{i=\mathrm{inc}_\xi(u)}^{\mathrm{inc}_\xi(u)+p} \sum_{j=\mathrm{inc}_\eta(v)}^{\mathrm{inc}_\eta(v)+q} \int_\Lambda U_{\ell k}^* S_{i,j}^{p,q}\, J_1 J_2 \, d\Lambda \, t_k^{con(i,j,P)} \right]
\end{aligned}
\tag{6}
$$

in which NP is the total amount of NURBS surfaces that discretizes the body's boundary. The variables $u$ and $v$ are each univariate knot-span index, varying from 1 to $s_\xi$ and $s_\eta$, respectively. In addition, $\mathrm{inc}_\xi$ and $\mathrm{inc}_\eta$ are indexing functions that return the control point local number in its corresponding knot-span. Thereby, the eq. (6) application for each collocation point leads to a algebraic system $\mathbf{Hd} = \mathbf{Gt}$. Then, the boundary conditions enforcing occurs by moving all unknown to the left-hand term and all known variables to the right-hand term of the algebraic system. Consequently, the solution of the final system $\mathbf{Ax} = \mathbf{b}$ obtains all the unknown parameters $\mathbf{d}$ and $\mathbf{t}$, which enables the complete representation of the solid's mechanical fields.

## 3 Trimmed surfaces procedures

Since NURBS surfaces are obtained by a tensor product between two univariate NURBS curves, they represent a complete patch in the space. However, this construction is unable to generate surfaces with holes or complex boundaries. Then, one manner to overcome such issue is the trimming operation. This task relies on the definition of a curve inside the NURBS surface parametric space to represent the removed or the remained portion. Commonly, NURBS functions parametrize these trimming curves. In this scenario, trimmed surfaces are present in almost every CAD model because they are mandatory to represent properly current applications' details. Therefore, a robust isogeometric numerical method such as IGABEM must incorporate this scheme in its analysis. Then, there are some specific procedures to consider trimmed patches, where the identification task and the integration assessment arise as two crucial aspects.

### 3.1 Identification task

Before the algebraic system's numerical assemble, it is necessary to classify each trimmed patch knot-span, in order to apply the suitable integration rule. Then, the present study uses the ray-cast algorithm to define whether specific points are internal or external to the trimming curve at the parametric space. This technique consists of counting the number of intersections of a segment and one closed boundary. This segment starts at the given point and must cross the closed curve. If there is an even number of crossings, then the point is outside of the boundary, and it is inside otherwise. To determine the crossing between the line and the trimming curve, this strategy proposes a $f$ function:

$$f = \eta(t) - \eta_L(\xi(t)) \Rightarrow f = \eta(t) - [m_L(\xi(t) - \xi_0) + \eta_0] \tag{7}$$

in which $(\xi(t),\eta(t))$ are the trimming curve's points for a $t$ univariate parametric coordinate, $(\xi_0,\eta_0)$ are the parametric coordinates of the given point and $m_L$ defines the line's slope. This slope considers the given point and the trimming curve's first control point coordinates. Then, there is a $f$ function evaluation for a discrete amount of $t$ that runs along the entire trimming curve. The crossings' number is equal to the number of changes in the $f$ signal, which means that there is a 0 between $t_i$ and $t_{i+1}$. By determining every line's parameters, it is possible to calculate the value of $f$ function for all the tested points with only one run over the $t$ parameter, which speeds the identification analysis.

The ray-casting algorithm application for each corner of a given knot-span allows its category's determination. When every corner is inside the trimming curve, the entire knot-span is internal to the trimming curve. On the other hand, if all corners are external to the trimming curve, the whole knot-span is external. Lastly, when there are internal and external corners simultaneously, the knot-span is a trimmed one, where specific integration techniques defines this region's influence properly. Furthermore, depending on the trimming curve's nature (inner or outer), there is a classification over the trimmed knot-spans corners, to inactivate those at the removed portion.

Another required task to integrate the trimmed knot-spans is defining the trimming curve's univariate parametric coordinate at each given knot-span's edge. To this assessment, a local Newton-Raphson procedure over an error $E(t) = \lambda_S - \lambda(t)$ determines the corresponding $t$ parameter. In this non-linear procedure, $\lambda$ denotes either $\xi$ or $\eta$, depending on the side's known parametric coordinate. For vertical edges, the known variable is $\eta$, while for horizontal edges $\lambda$ refers to $\xi$. Furthermore, $\lambda(t)$ is the corresponding parametric coordinate $\xi$ or $\eta$ calculated by the trimming curve's interpolation. In this scenario, truncating $E(t)$ in its Taylor series' first term leads to an iterative scheme in $t$:

$$\Delta t_k = \frac{E(t_k)}{\left.\dfrac{\partial E}{\partial t}\right|_k} \tag{8}$$

in which $\partial E/\partial t|_k$ is the first derivative of the error $E(t)$ towards $t$ parametric direction, which becomes a trimming curve's tangent component.

### 3.2 Integration of trimmed knot-spans

Integrating trimmed patches must account for the modifications inserted by the trimming operation over a NURBS surface. Then, the first task for this process is subdividing the trimmed knot-span in three-side cells. At least one cell has the trimming curve as one of its side (type A cells), while others are regular triangles in the parametric space (type B cells). According to Kim et al. [8], this scheme transforms a point in the Gaussian space to the physical space using four space transformations.

The first transformation $\mathbf{T}_1$ considers a point $(\hat{\xi},\hat{\eta})$ in the Gaussian space and returns a corresponding point $(t,\zeta)$ in a space $\Omega_1$:

$$\begin{aligned} &\mathbf{T}_1 : \{\hat{\xi},\hat{\eta}\} \rightarrow \{t,\zeta\} \\ &t = \frac{\hat{\xi}}{2}(t_2 - t_1) + \frac{1}{2}(t_2 + t_1) \\ &\zeta = \frac{\hat{\eta}}{2} + \frac{1}{2} \end{aligned} \tag{9}$$

in which $t_1$ and $t_2$ are the univariate trimming curve parametric values at the edges of the type A cell. It is worth mentioning that the space $\Omega_1$ is a rectangular space $[t_1,t_2] \times [0,1]$.

The second mapping $\mathbf{T}_2$ is responsible for degenerating the rectangular space $\Omega_1$ into a triangular space $\Omega_2$ with edges $(0,0)$, $(1,0)$ and $(0,1)$. Simultaneously, $\mathbf{T}_2$ incorporates the trimming curve's information by using $\phi = \mathbf{T}_3^{-1}\mathbf{C}(t)$, where $\mathbf{T}_3^{-1}$ is the inverse $\mathbf{T}_3$ transformation, which will be depicted in sequence, and $C(t)$ is the trimming curve at the NURBS surface parametric space. In this sense, $\mathbf{T}_2$ is as follows:

$$
\begin{aligned}
&\mathbf{T}_2 : \{t,\zeta\} \rightarrow \{X,Y\} \\
&X = \phi_X(t)(1-\zeta) \\
&Y = \phi_Y(t)(1-\zeta) + \zeta
\end{aligned}
\tag{10}
$$

The third transformation $\mathbf{T}_3$ relates the space $\Omega_2$ to the NURBS surface parametric space, where both trimming curves and triangular cells have been defined. This third mapping uses a simple triangular FEM shape functions. Hence, denoting the parametric space triangle vertexes as $(\xi_1,\eta_1)$, $(\xi_2,\eta_2)$ and $(\xi_3,\eta_3)$, $\mathbf{T}_3$ is:

$$
\begin{aligned}
&\mathbf{T}_3 : \{X,Y\} \rightarrow \{\xi,\eta\} \\
&\xi = Y\,\xi_1 + (1-X-Y)\,\xi_2 + X\,\xi_3 \\
&\eta = Y\,\eta_1 + (1-X-Y)\,\eta_2 + X\,\eta_3
\end{aligned}
\tag{11}
$$

The last transformation is the NURBS parametrization given by eq. (4). These mappings are also valid for type B cells by using $t_1 = 0$ and $t_2 = 1.0$ in eq. (9), and $\phi_X(t) = t$ and $\phi_Y(t) = 0$.

## 4 Numerical application

The present study's application consists of a solid cube of length $1.0$ with $r = 0.15$ radius cylindrical hole. Figure 1 illustrates its geometry and loading conditions. There are three displacements boundary conditions: $u_1 = 0$ over face $x_1 = 0$, $u_2 = 0$ along face $x_2 = 0$ and $u_3 = 0$ where $x_3 = 0$. Additionally, a uniform traction $p_3 = 1.0$ induces an uniform tensile response. The material properties are $E = 1000.0$ and $\nu = 0.3$. This application has an exact solution, as follows:

$$
\begin{aligned}
u_1(x_1,x_2,x_3) &= \frac{-\nu x_1}{E} \\
u_2(x_1,x_2,x_3) &= \frac{-\nu x_2}{E} \\
u_3(x_1,x_2,x_3) &= \frac{x_3}{E}
\end{aligned}
\tag{12}
$$

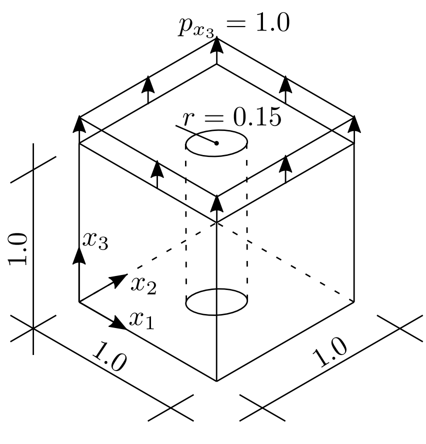


Figure 1. Application's geometry and boundary conditions.

The norm of error $L_2$ in displacements can evaluate the responses' quality once this application has an analytical solution, as follows:

$$||e||_{L2} = \frac{\int_\Gamma (\mathbf{u}_{app} - \mathbf{u}_{ex})(\mathbf{u}_{app} - \mathbf{u}_{ex})^T \, d\Gamma}{\int_\Gamma (\mathbf{u}_{ex}\mathbf{u}_{ex}^T) \, d\Gamma} \tag{13}$$

in which $\mathbf{u}_{app}$ refers to the approximate displacement vector obtained by the trimmed IGABEM and $\mathbf{u}_{ex}$ is the displacement vector calculated via eq. (12). Equation 13 integrals' evaluation utilises the same numerical integration method that have been enforced to determine the $\mathbf{H}$ and $\mathbf{G}$ coefficients. Therefore, the error measurement takes place at each integration point.

The trimmed isogeometric mesh of this application has a total of 10 NURBS surfaces, 6 for the cube and 4 for the cylindrical hole. The degree of all faces is 1 in both directions, except the cylinder's curved face, where a degree 2 over the radial direction guarantees an arc's exact geometric representation. There is a mid-point knot-insertion for each face, where $\xi = \eta = \{0, 0, 0.5, 1, 1\}$ are the knot-vectors for the cubic surfaces. For cylindrical surfaces, these values are $\xi = \{0, 0, 0, 0.5, 1, 1, 1\}$ and $\eta = \{0, 0, 0.5, 1, 1\}$. In addition, the trimmed surfaces are at $x_3 = 0$ and $x_3 = 1.0$. The faces' corresponding trimming curves have degree 2, knot-vector $\xi = \{0, 0, 0, 0.25, 0.5, , 0.75, 1, 1, 1\}$ and control points at parametric space in the format $P_i = (u_i, v_i, w_i)$ as follows: $P_1 = (0.65, 0.5, 1.0)$, $P_2 = (0.65, 0.35, \frac{\sqrt{2}}{2})$, $P_3 = (0.5, 0.35, 1.0)$, $P_4 = (0.35, 0.35, \frac{\sqrt{2}}{2})$, $P_5 = (0.35, 0.5, 1.0)$, $P_6 = (0.35, 0.65, \frac{\sqrt{2}}{2})$, $P_7 = (0.5, 0.65, 1.0)$, $P_8 = (0.65, 0.65, \frac{\sqrt{2}}{2})$ and $P_9 = (0.65, 0.5, 1.0)$.

The Figure 2 illustrates the total displacements obtained by the trimmed IGABEM framework proposed herein. This response considers the collocation points at a $20\%$ distance outside the boundary and with 20x20 Gaussian points. Moreover, there is an evaluation of the L2 norm of error in displacement considering the variation of the collocation point distance factor from the boundary (using 5%, 10% and 20%) and the total of Gaussian points (from 5 to 30, every 5) in both directions. Hence, Figure 3 presents this comparison, where it is noticeable that the $20\%$ distance parameter leads to the lowest L2 norm of error when compared to other distances. Since the BEM has a singular nature in its fundamental solutions (O(1/r)), the collocation points' proximity to the boundary may trigger some numerical instabilities related to this singular behaviour. Thus, a singularity treatment must be enforced to solve this issue. Furthermore, it it worth noting that the increase in the Gaussian points' number provides more accurate results. This behaviour is expected once that an increment in the number of integration points induces a preciser integral's assessment. Lastly, the lowest L2 norm of errors are below $10^{-7}$, which demonstrates the accuracy of the proposed formulation.

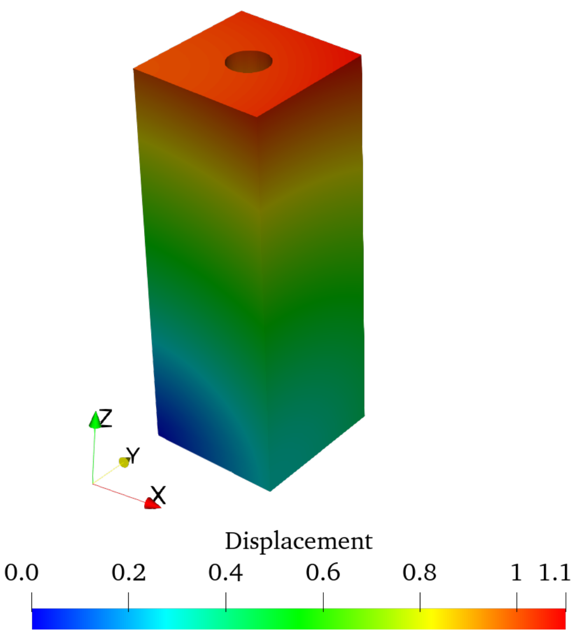


Figure 2. Application's total displacements.

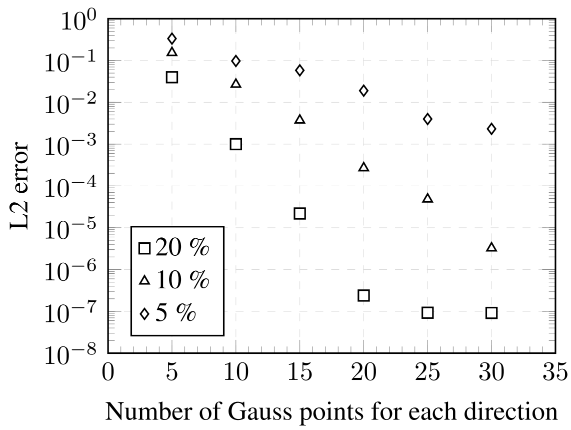


Figure 3. L2 norm of error.

## 5 Final remarks

This study deals with the mechanical analysis of three-dimensional solids considering non-singular IGABEM and models containing trimmed surfaces. One numerical application demonstrates the accuracy and robustness of the strategy proposed herein. However, the distance between the collocation point and the boundary affects the quality of the final response. Thus, a singular IGABEM formulation with trimmed surfaces is an ongoing development.

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