Application of the Direct Interpolation Boundary Element Technique with Self-Adaptative Integration to Bidimensional Diffusive-Advective Problems

Loeffler, C.F.[1], Santos, A.J.[1], *Pinheiro, V.P.[1], Melo, L.D.V.[1]

[1]*Programa de Pós-graduação em Engenharia Mecânica - PPGEM, Universidade Federal de Espírito Santo*
Av. Fernando Ferrari, 514 - Goiabeiras, 29075-910 ES/Vitória, Brasil
loefflercarlos@gmail.com
aquiladossantos@gmail.com
*vitor.pinheiro1987@gmail.com**
lucca1000@gmail.com

Abstract.

The application of the Boundary Element Method (BEM) in the formulation of advective-diffusive models experiences a solid scientific revival and finds its main challenges in the numerical treatment of the transport term. Despite the classic formulation, which is based on the fundamental solution related to the problem, being able to adequately describe physical situations dominated by advection, it does not present flexibility when dealing with variable velocity fields. The recent Direct Interpolation technique (DIBEM), also based on the robustness of approximations by radial basis, handles in a more balanced way the challenges imposed to BEM, while able to represent hydrodynamic fields with spatial variation and maintaining numerical stability up to moderate Peclet numbers. Basically, the technique procedure transforms domain integrals into boundary integrals, approaching the entire kernel using radial basis functions. The DIBEM's mathematical structure is very similar to a classic interpolation, that in turn requires a significant number of interpolating points in the domain. In this context, the performance of the Direct Integration formulation as well as it is numerical stability depend, among other factors, on the quality of the approximation of ordinary, quasi-singular and singular integrals that are inherent to the discrete linear system. The current article proposes a comparison and analysis of the influence of the numerical integration scheme selection, testing the classic Gaussian quadrature against a self-adaptive scheme, aiming to evaluate possible accuracy gains of the Direct Interpolation Technique. Quantitative assessment of the impacts of integration schemes are evaluated using well-known analytical solutions.

Keywords: Boundary Element Method, Direct Interpolation Technique, Diffusive-Advective Model, Self-Adaptative Integration Scheme.

1 Introduction

The robustness of the Boundary Element Method (BEM) is strongly dependent on an accurate calculation of the boundary integrals that arise from the discretization procedure. It is possible to analytically determine these integrals in certain particular cases. However, unlike constant and linear elements, the coordinate transformation technique, numerical integration procedures and the treatment of singular integrals are not so simple for higher order elements, where the variation of the quantities of interest along the element is more elaborate. A significant challenging factor arises dealing with non-rectilinear elements, such as quadratic or superior isoparametric elements, in which the function that performs the transformation of global coordinates into a local strategic system, the Jacobian, is no longer a constant value, but now, a function of the element shape.

Such difficulties were the motivation for several research efforts that propose alternative approaches for the calculation of domain integrals of the BEM, once the analytical options becomes complicated. In general, well-established numerical integration techniques are used, such as Gaussian Quadrature, coupled with schemes that propose to increase their precision or make them viable in some way. Usual schemes are: the domain partition

[1]; the Telles integration scheme [2] or the adoption of the coordinate of the source point outside the boundary element [1]. Modernly, more advanced schemes are used to approach isogeometric problems, aiming higher levels of precision.

On the other hand, there is a significant research effort to offer an efficient numerical model, in terms of the BEM, for a wide range of engineering problems. In this context, the elimination of domain integrals is included, which can be problematic in certain cases, due to the peculiarities of the differential operator or even due the complexity of the related fundamental solution. It is possible to transform domain integrals into boundary integrals using strictly appropriate fundamental solutions - the Green functions [3] - which match exactly the differential operator; but such functions are not always available or easily adaptable, as in transport problems with variable velocity field or also in physically heterogeneous problems.

The most widespread solutions for dealing with domain integrals in these cases is to use approximation techniques that use sequences of radial basis functions. Several formulations of the BEM act like this: the Dual Reciprocity Technique (DRBEM), presented in detail in [4]; the Radial Integration Method (RIM) [5]; and the Direct Interpolation Technique (DIBEM), presented by [6] which has already been tested on several scalar field problems, consistently showing satisfactory results. [7], [8].

The DIBEM technique does not use the related fundamental solution, such as DRBEM, but a similar fundamental solution corresponding to a simpler problem. Despite a relative loss of precision, such simplicity allows free vibration problems, transport problems with variable velocity and physically inhomogeneous cases, to be solved more efficiently.

DIBEM appears to be more robust than DRBEM, due it is a numerical procedure closer to a simple interpolation. Furthermore, it does not require the use of particular solutions that comply with the Principle of Reciprocity [4], which implies greater applicability to the technique, making it suitable for the application on plate models, for example. In contrast, DIBEM usually requires a higher quantity of internal points for generate an acceptable interpolation, however, inducing integration problems when the number of poles is too dense, and, for consequence, these points are positioned very close to the boundary.

This work aims to evaluate the accuracy improvement generated by using a self-adaptive integration scheme on a diffusive-advective model. Meshes with a larger number of internal poles improves the representation of physical fields, but also produces quasi-singular integrals that impair the accuracy of the technique. Thus, the use of the self-adaptive scheme can improve DIBEM performance.

Higher order elements are also implemented to verify the impact of a better representation of physical fields and geometry on these problems. A simple case is solved, with well-known analytical solution, but which allows a satisfactory evaluation of the effectiveness of the higher order elements and, especially, of the self-adaptive integration scheme with DIBEM technique .

2 Self-Adaptative Integration

The mathematical treatment of scalar field problems using DIBEM, uses fundamental solutions that generates singular or quasi-singular, depending on the distance of the so-called source point in relation to the boundary element to which the integration is carried out. According to the function to be integrated and the integration interval, the integrals can be classified as being regular, singular or quasi-singular [9]. The use of self-adaptive integration schemes allows to solve the three types of integrals previous mentioned more accurately. The advantage of self-adaptive integration schemes is that, with just a single integration procedure, it is possible to encompass all possible types of integrals to be evaluated.

Telles (1987) presented an efficient method to calculate singular or quasi-singular integrals that can be applied to calculate BEM integrals. It is a polynomial transformation of the third degree, which improves the approximation of the Gaussian Quadrature method in the vicinity of the singularity [2]. This procedure can be easily implemented in BEM using a variable that depends on the distance from the source point to the element. This self-adaptive scheme becomes inactive for large distances between the source point and the element to be integrated. This fact guarantees greater reliability and precision for a wide range of engineering applications.

Consider the integral presented below, where the function to be integrated is singular at the point $\bar{\eta}$,

$$I = \int_{-1}^{1} f(\eta) d\eta \tag{1}$$

being η the natural coordinate of the integral. A third-order coordinate transformation is then adopted, given according to the following relationship:

$$\eta(\gamma) = a\gamma^3 + b\gamma^2 + c\gamma + d \tag{2}$$

In eq. (2), γ is the coordinate given by the classic Gaussian Quadrature integration secheme and, after the transformation, and η will be associated with the weight of the Gaussian Quadrature corresponding to the original integration point γ ; furthermore, $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ are parameters that depends on the position of the singularity point.

Such coordinate transformation remains valid, eliminating the need of use a domain partition resource, to any singularity point, and automatically producing a higher concentration of integration points near to the singularity point [2]. The parameters $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ are defined by applying the following conditions:

$$\eta(1) = 0; \quad \eta(-1) = 0; \quad \left.\frac{d\eta}{d\gamma}\right|_{\bar{\eta}} = 0 \quad e \quad \left.\frac{d^2\eta}{d\gamma^2}\right|_{\bar{\eta}} = 0 \tag{3}$$

The first two boundary conditions exposed in eq. (3) impose that the range of the transformed function remains $[-1, 1]$, which coincides with the use of isoparametric elements in generalized coordinates in the BEM. The third condition allows the singularity to be properly eliminated and the fourth condition is introduced to produce a Jacobian minimum, point for point; a Jacobian maximum point can also be obtained. It is noteworthy that the second derivative presented in the fourth condition corresponds to the first derivative of the Jacobian, which behaves like a linear approximation of a transformation between two coordinate systems.

3 Diffusive-Advective Model: DIBEM Approach

The diffusion-advection equation is widely used to model and describe physical phenomena where there is the transport of a certain physical quantity, such as mass or energy, together with the diffusion process, in a given volume of control. Considering a permanent regime, in the absence of sources or sinks, with constant and unitary diffusion coefficient and uniform velocity field, in indicial notation we have:

$$u_{,ii} = v_i u_{,i} \tag{4}$$

The term on the left side of eq. (4) represents the diffusion process while the term on the right side models advection effects. By multiplying both sides of the eq. (4) by an auxiliary function, called fundamental solution, using integration by partes and the divergent theorem, we can write an integral sentence of the problem as follows:

$$\begin{aligned} \textstyle\int_\Omega v_i(X)u_{,i}(X)u^*(\xi;X)d\Omega = \int_\Gamma n_i(X)v_i(X)u(X)u^*(\xi;X)d\Gamma.+ \\ -\textstyle\int_\Omega v_i(X)u^*_{,i}(\xi;X)u(X)d\Omega - \int_\Omega v_{i,i}(X)u^*(\xi;X)u(X)d\Omega \end{aligned} \tag{5}$$

The BEM classical formulation, which is based on the use of the fundamental solution related to Laplace's problem, is applied. The Green function associated with the advection-diffusion problem could be used; however, there are limitations to this formulation, especially for variable velocity fields [3]. The diffusive side, if viewed in isolation, corresponds to Laplace's differential equation, whose mathematical treatment is well known in the BEM literature [1].

Concerning the eq. (5), there are two domain integrals: one explicitly related to the velocity field and the other that contains the velocity field divergent, which is considered null, as a consequence of the incompressibility hypothesis considered in this work. So, the DIBEM procedure will only approximate the first of these two integrals, including a regularization scheme [10] to avoid singularity. Thus, one has:

$$\begin{aligned} \textstyle\int_\Gamma v_i(X)n_i(X)u(X)u^*(\xi,X)d\Gamma - \int_\Omega v_i(X)u(X)u^*_{,i}(\xi,X)d\Omega = \\ \textstyle\int_\Gamma v_i(X)n_i(X)u(X)u^*(\xi,X)d\Gamma - \int_\Omega v_i(X)[u(X)-u(\xi)]u^*_{,i}(\xi,X)d\Omega \\ \textstyle -u(\xi)\int_\Omega v_i(X)u^*_{,i}(\xi,X)d\Omega \end{aligned} \tag{6}$$

Using integration by parts and the divergence theorem in the last term on the right side of eq. (6), the surplus term can be properly rewritten in terms of boundary:

$$u(\xi)\int_\Omega v_i(X)u^*_{,i}(\xi,X)d\Omega = u(\xi)\int_\Gamma v_i(X)n_i(X)u^*(\xi,X)d\Gamma \tag{7}$$

So, only the regularized term persists, and it can be approximated by the DIBEM procedure, as shown below:

$$\int_\Omega [\mathrm{v_{i,i}(X)u^*(\xi;X) - v_i(X)u^*_{,i}(\xi;X)}]\,[\mathrm{u(X) - u(\xi)}]\mathrm{d}\Omega = \int_\Gamma \alpha^{\mathrm{j}^\xi}\eta^{\mathrm{j}}\left(\mathrm{X^j;X}\right)\mathrm{d}\Gamma \tag{8}$$

From this point on, the problem is discretized according to the typical strategy of the boundary element method. A system of equations is generated relating known and unknowns values of potentials and fluxes on the boundary. Once this procedure is performed, the governing integral equation is transformed into a linear system of algebraic equations, in which the unknowns in each boundary element are calculated, considering the known values. An overview of a row of the matrix system is presented below:

$$\{q_j G_{\xi j}\} = \{v_i^k n_i^k u_j G_{\xi j}\} + \left\{\alpha_{\xi j} N_j - u(\xi)\left\{(v_i n_i)_j G_{\xi j}\right\}\right\} \tag{9}$$

Expanding to all nodes of the numerical mesh, its possible to generate a matrix form for the final linear system, as follows:

$$[\mathbf{H}]\{\mathbf{u}\} - [\mathbf{G}]\{\mathbf{q}\} = [\mathbf{G'}]\{\mathbf{u}\} - [\alpha]\{\mathbf{N}\} - [\mathbf{D}]\{\mathbf{u}\} \tag{10}$$

4 Numerical Tests

Consider a two-dimensional diffusive-advective with an uniform velocity field, as shown in Figure 1. The boundary conditions imposed are prescribed flux on the horizontal edges and fixed potentials on the vertical edges of the domain.

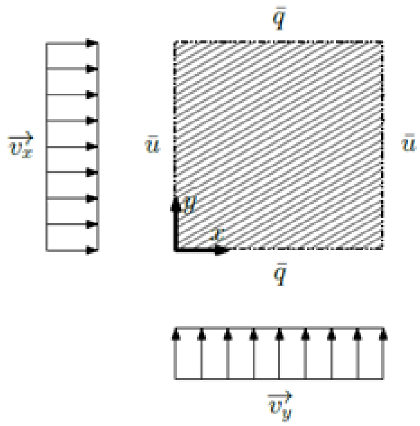

Figure 1. Computational Domain Scheme

The flow velocity field is given by v=1.0i+1.0j. The analytical solution for potentials and flows is shown below:

$$u(x) = e^x + e^y \qquad q(x) = e^x n_x + e^y n_y \tag{11}$$

Concerning the numerical tests, the boundary mesh was composed by 20 isoparametric elements and the numerical integration performed with 8 Gauss points (NGP), in order to observe the influence of the number of internal points. The impact is expected to be relevant, since these internal poles are used in interpolations that approximate the domain integral in the DIBEM technique.

The example presents geometry with unitary significant length and mechanical properties were also taken as unitary. Linear, quadratic and cubic isoparametric linear elements were used. The evaluation of numerical efficiency was made using the following nodal mean error criterion below:

$$E_m = 100\% \frac{1}{n} \sum_{j=1}^{n} \left| \frac{\hat{u}_j - u_j}{\hat{u}_j} \right| \tag{12}$$

In eq. (12), $\hat{u}_j$ is the value of the analytic solution at the *jth* point, u_j the value of the numerical solution at the *jth* point and n the number of internal poles of the mesh.

In formulations such as DIBEM, the mesh has two relevant parameters: boundary elements (BE) and internal poles (IP). Commonly, a minimum distance between the boundary and the rows of internal poles is required to avoid quasi-single integrations. This problem occurs because the interpolating points also act as source points and a high density of them results in points very close to the boundary.

In Figure 2 illustrates the mesh structure. There, l_e dimension represents the size of the boundary elements, d is the distance between double nodes, L is the characteristic length of the geometry, l is the distance between the poles not adjacent to the boundary and l_0 is the distance between the poles closer to the boundary. In this study, the l_0 offset is considered $0.5l_e$.

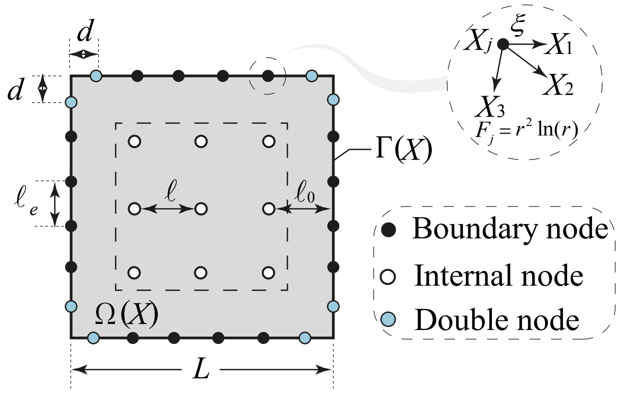

Figure 2. Mesh Structure

Figure 3 shows the average percentage errors obtained with a 20 boundary element mesh with different numbers of internal poles, using the special first-row spacing of $l_0 = 0.5l_e$.

Figure 4 shows the mean percentage errors obtained with the same mesh with 20 boundary elements with different numbers of internal poles without using offset, with homogeneous distribution of the nodes, that is, $l_0 = l_e$.

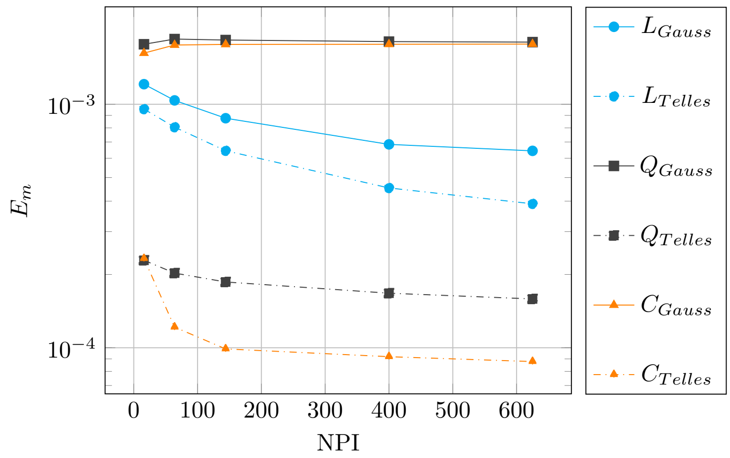

Figure 3. Potential Average Error - 20BE/ 8 NGP - Special Offset of First Row

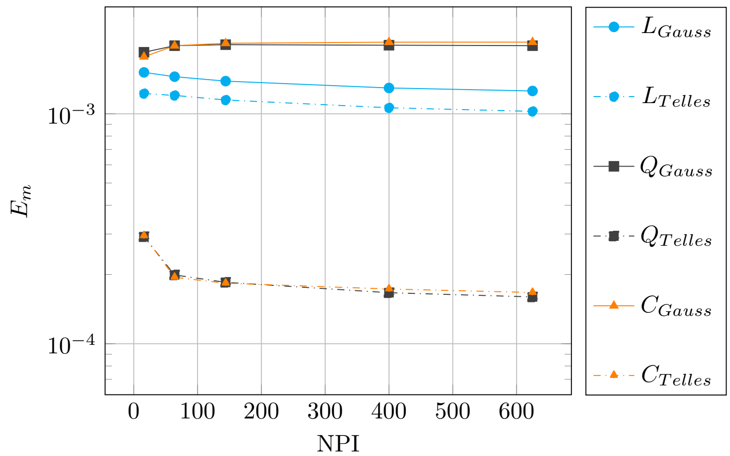

Figure 4. Potential Average Error - 20BE/ 8 NGP - No Offset - Structured Mesh

Comparing the graphs of Figures 3 and 4, with and without the offset, it is first noticed that, in general, the errors were reduced, approaching the first row of internal poles. This confirms that the uniform distribution of poles within the domain is important for better representation of the advective property.

The distancing of the nodal points of the boundary from a densely populated cloud of poles, with the purpose of avoiding quasi-singular integration problems, produces a constitutive irregularity around the boundary that mitigates the positive effect of their insertion. However, concerning quadratic and cubic elements without the adaptive scheme, the results practically did not change with or without distancing. This is probably because the nodal points on the boundary are also interpolating points; thus, a greater number of nodes in higher order elements implies a greater density of poles on the boundary, reducing the effect of the poor approximation of the advective field in the interval between the boundary and the cloud of internal poles. Thus, as the poles are closer to the domain boundary, integration errors increase and cancel out the gain with a best approximation of the advective field.

This explanation is confirmed when using the adaptive scheme with offset: the error values are drastically reduced for the higher order elements. Certainly, much of the error was reduced by better integration in the boundary; however, the moderate – but constant – accuracy gain using higher order elements with the insertion of a greater number of poles clearly contrasts with the stable behavior of the error curve without the integration scheme. This contrast is due to the fact that the increase in the number of interpolating points inside is not introducing integration errors, which occur when internal source points lie close to the boundary elements.

Using the uniform distribution of internal points and, consequently, without the problems caused by the distance, the positive effects of using the adaptive scheme are clearly visible, for all types of elements. The gain in precision is significant in higher order elements, to the point that they require few poles for good results. However, it is noteworthy that this problem consists of a case where the advective phenomenon is not predominant; cases in which the velocity are much higher, in its turn, it is not possible to approach without the insertion of a greater number of poles and the refinement of the boundary mesh.

5 Conclusions

The Boundary Element Method is a robust discrete method, but it requires numerical special treatment when dealing with its boundary integrals, since they have singular functions at their kernels, which are improper convergent integrals. Thus, the proper treatment of singular or quasi-singular integrals is essential for its good performance.

In the diffusive-advective cases solved by DIBEM, the importance of a self-adaptive scheme takes on a new dimension, although still related to numerical integration. Such problems are characterized by generating domain integrals that are transformed into boundary integrals by approximating the kernel of these domain integrals through radial basis functions, and, at the same time, using a simpler fundamental solution. However, techniques based on the use of radial basis functions require the introduction of interpolating poles on the domain, which are also source points and perform integration along the boundary elements. If the mesh of interpolating internal points is denser, integration problems may appear, due to the proximity between field and source points. The simulations presented here showed that the self-adaptive scheme greatly improved the quality of the numerical results. Certainly, the advantage of high-order elements would be even more pronounced if problems with curvilinear boundaries were considered.

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