An improvement to the frequency response in non-homogeneous Helmholtz problem using the Double Fictitious Background Media Formulation

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Abstract. This article proposes a new formulation of the Boundary Elements Method to solve the response problem in non-homogeneous Helmholtz problems, called Double Fictitious Background Media (DFBMF). The DFBMF is based on a simple algebraic procedure, creating a fictitious medium, which consists of adding and subtracting the same term in the governing equation, which divides it into two parts: a homogeneous Helmholtz equation and a fictitious reactive term. This new governing differential equation is mathematically independent of the properties of the chosen fictional background medium. Still, considering the Weighted Residual Principles, the associated integral boundary equation is affected by the Green Function argument through the reactive term, which accounts for the variation in physical properties. To reduce the numerical error, two integral equations with different arguments are generated where the reactive term is considered a common source to the two equations. The numerical simulations show a significant increase in the precision of the results with the application of this strategy.

Keywords: Boundary Element Method, Helmholtz Equation, Inhomogeneous media.

1 Introduction

Better BEM results are achieved using formulations in which the fundamental solutions are closely related to the original differential operator. However, in important applications such as advective-diffusive problems with variable velocity fields or then stationary waves in inhomogeneous media, the most appropriate fundamental solutions are given respectively in terms of constant velocities or uniform properties, that is, the strictly adequate Green's functions are not available.

In order to get around this restriction, dealing with advective-diffusive problems with variable velocity fields, Wrobel and DeFigueiredo [1] combined the Dual reciprocity model [2] with the use of a fundamental solution corresponding to an advective-diffusive problem for a constant velocity profile. Itagaki [3] imposed a similar transformation to handle the variable coefficients of the modified Helmholtz Equation for inhomogeneous means, which results in an intelligent way to calculate the remaining domain integral. In this context, Dan and Mansur [4] also added and subtracted a term in the Helmholtz equation and considered a Green function correlated to a homogeneous fictitious background medium. The additional reactive term then generated is resolved using cells. This formulation was called Fictitious Background Media Inverse Formulation (FBMIF).

In this article, a new variant of FBMIF is applied to solve Helmholtz's direct problems for non-homogeneous media. The proposed approach is called Double Fictitious Background Media Formulation (DFBMF). DFBMF proposes two independent integral equations to model Helmholtz problems in inhomogeneous media, generating a double system of independent linear equations that promotes a considerable increase in the numerical accuracy of the numerical simulations performed. In addition, a proper mathematical interpretation of the integrals with appropriate fundamental solutions in combination with the additional fictitious source terms is presented. The procedures presented here are generic and can be extended to other mathematical models other than the Helmholtz equation, such as advective-diffusive problems with a variable velocity field.

2 Fictitious background medium for Helmholtz problem

Despite the Helmholtz equation with variable coefficients finds a wide field of practical applications, still persists a demand for BEM formulations capable of solving such problems efficiently [5,6,7]. The main idea presented here starts with modifying the original mathematical model by adding and subtracting a term in the Helmholtz equation. Thus, considering equation (1), one has:

$$\nabla^2 p + k^2(x,y)p=0 \tag{1}$$

In Eq. (1) $k(x,y)$ is related to the variable property of the physical medium. For acoustic problems, the coefficient $k(x,y)$ corresponds to the wavenumber, that is, the angular frequency divided by variable acoustic velocity profile. Thus, adding and subtracting the term $k_0^2 p$ in the equation (1) and rearranging the terms, the following equation is obtained:

$$\nabla^2 p + k_0^2 p= - [k^2(x,y) - k_0^2]p= - k_v p \tag{2}$$

The coefficient k_0 corresponds to a wavenumber of a hypothetical homogeneous medium. Equation (2) exactly corresponds to the equation (1) and can be interpreted as the response of a homogeneous fictitious background medium subjected to the reactive term $(-k_v p)$.

Searching for Green's function that allows the best approximation for non-homogeneous Helmholtz cases, a fundamental solution correlated with respect to the fictitious background medium described in equation (2) is proposed. Thus, considering an infinite medium characterized by a constant acoustic velocity profile and subject to a point domain source, the Helmholtz fundamental problem is defined as:

$$\nabla^2 \Psi^* + k_0^2 \Psi^* = - \delta(\xi; r) \tag{3}$$

In Eq. (3) k_0 corresponds to the wavenumber of the fictitious medium. As can be extensively found in the literature [8], equation (3) is satisfied by the following equation:

$$\Psi^* = \frac{-i}{4} H_0^2\left(k_0 \left|\vec{r} - \vec{r}_\xi\right|\right) \tag{4}$$

H_0^2 is the Hankel function of the second kind and zero order. Eq. (4) corresponds to the Green's function relative to the fictitious background medium defined in this work.

Considering Eq. (3) and following common procedure used to establish BEM's integral sentences, the following integral equation can obtained from the equation (4):

$$c(\xi)p(\xi) - \int_\Gamma q\Psi^* d\Gamma + \int_\Gamma p\, q^* d\Gamma = \int_\Omega k_v p \Psi^* d\Omega \tag{5}$$

In Eq. (5), $c(\xi)$ is a coefficient that depends on the location of the source point ξ [9], and q and q* are the normal derivatives of function p and Ψ^* respectively.

3 Double fictitious background technique

Equation (5) presents a BEM integral in which it was not possible to use a Green function that exactly fits the fundamental problem. Although the use of FBMIF has produced a homogeneous Helmholtz operator on the left side, there is an additional reactive term whose numerical effect must be analyzed. The best way to carry out this analysis is given by the Weighted Residual Method (WRM) [10]. WRM mathematically explains the reason for the better accuracy of integral formulations that use fundamental solutions related as auxiliary functions. It is assumed that when approaching the field of unknowns p in the governing equation, a residual ε results, since the exact value of p is not available. As this residual is multiplied by the auxiliary function and integrated into the domain, this operation can be interpreted as an internal product. The greater the orthogonality between the functional spaces of the residual in relation to the auxiliary function, the lower the value of the inner product and the better the accuracy of the approximation used. Thus, the most effective auxiliary functions are the most complete and closely related to problems, as is the case with Green's functions.

In this sense, consider the left side of Eq. (5). Based on WRM, the auxiliary function Ψ^* is a weighting function and the following expression for the domain can be written:

$$\int [\nabla^2 p + k_0^2 p]\Psi^* d\Omega = \int \varepsilon_{hom}\, \Psi^* d\Omega = R_{hom}(p) \tag{6}$$

The application of the well-known procedures of the BEM such as integration by parts will only generate the left side of Eq. (5) if the portions referring to residues on the boundaries are considered. These parcels were not considered for simplicity, since only the examination of residues in the domain is of interest. Therefore, examining the right hand side of Eq. (5), one has another residual, given by:

$$\int [-k_v p]\Psi^* d\Omega = \int \varepsilon_{desv}\, \Psi^* d\Omega = R_{desv}(p) \tag{7}$$

In a homogeneous problem, k_0 is the uniform property of the medium, and therefore R_{desv} is null. It can be stated, based on computational experiments, that the residual related to R_{hom} is minimal, given the effectiveness of the Green function as a weighting function under homogeneous conditions. However, in heterogeneous cases, R_{desv} is not zero. It is not possible to point out any particular quality regarding the minimization of residues given by the interaction of R_{hom} with R_{desv}. As will be shown later, there is an optimal value for k_0, producing minimal R_{desv} and R_{hom} residues, but this value cannot be identified a priori.

The DFBMF proposes the use of two independent integral sentences to model Helmholtz problems in non-homogeneous media. Therefore, two independent fictitious wavenumbers k_{01} and k_{02} are considered resulting in the following pair of integral equations:

$$\begin{aligned} c(\xi)p(\xi) - \int_\Gamma q\Psi^* d\Gamma + \int_\Gamma p\, q^* d\Gamma &= \int_\Omega [k^2(x,y) - k_{01}^2] p\Psi^* d\Omega = \int_\Omega k_{v1} p\Psi^* d\Omega \\ c(\xi)p(\xi) - \int_\Gamma q\bar{\Psi}^* d\Gamma + \int_\Gamma p\, \bar{q}^* d\Gamma &= \int_\Omega [k^2(x,y) - k_{02}^2] p\Psi^* d\Omega = \int_\Omega k_{v2} p\bar{\Psi}^* d\Omega \end{aligned} \tag{8}$$

The main idea here is to treat any value of pressure p occurring inside de domain Ω as a new independent unknown $\hat{p}$. Hence, considering this new variable $\hat{p}$, integral equations (8) are rewritten as:

$$c(\xi)p(\xi) - \int_\Gamma q\Psi^* d\Gamma + \int_\Gamma p\, q^* d\Gamma = \int_\Omega k_{v1}\hat{p}\Psi^* d\Omega \tag{9}$$

$$c(\xi)p(\xi) - \int_\Gamma q\bar{\Psi}^* d\Gamma + \int_\Gamma p\, \bar{q}^* d\Gamma = \int_\Omega k_{v2}\hat{p}\bar{\Psi}^* d\Omega \tag{10}$$

Hence, equations (9) and (10) form a set of linearly independent integral equations that can be used to simulate physical problems modeled by the Helmholtz equation for both homogeneous and inhomogeneous media.

The reactive terms present on the right side of equations (8) are now treated as fictitious sources, so that their respective weighted residual sentences are given by:

$$\begin{aligned} R_{hom}^1(p) - R_{desv}^1(\hat{p}) &= 0 \\ R_{hom}^2(p) - R_{desv}^2(\hat{p}) &= 0 \end{aligned} \tag{11}$$

Written in terms of the fictitious pressure $\hat{p}$ the two residual equations involve different properties, but together the common value of the unknown pressure p can be minimized. However, as one more unknown has been introduced, it is necessary to work with two equations. The values of k_{01} and k_{02} can be arbitrary, but there are more suitable combinations that will be discussed opportunely. The consistency of this procedure will be confirmed with the numerical solution of some examples.

4 Matrix model

The matrix formulation for the DFBMF is obtained from the equations (9) and (10). The Eq. (9) is taken initially. The collocation method is applied at the boundary and the domain functional nodes with the domain Ω discretized using respectively **N** and **M** functional nodes. However, **M** independent equations should be introduced in order to make it a consistent system. A set of **M** extra independent equations can be obtained from the equation (10) by executing the collocation method in the domain functional nodes, resulting in the following linear system:

$$\begin{bmatrix} \mathbf{H_{01}} & -\mathbf{G_{01}} & \mathbf{0} \\ \overline{\mathbf{H}}_{\mathbf{01}} & -\overline{\mathbf{G}}_{\mathbf{01}} & \overline{\mathbf{I}} \\ \overline{\mathbf{H}}_{\mathbf{02}} & -\overline{\mathbf{G}}_{\mathbf{02}} & \overline{\mathbf{I}} \end{bmatrix} \begin{bmatrix} \mathbf{p_{Cont}} \\ \mathbf{q_{Cont}} \\ \mathbf{p_{Int}} \end{bmatrix} = \begin{bmatrix} \mathbf{FF_{01}} \\ \overline{\mathbf{FF}}_{\mathbf{01}} \\ \overline{\mathbf{FF}}_{\mathbf{02}} \end{bmatrix} \widehat{\mathbf{p}} \tag{12}$$

The sub matrices $\mathbf{H_{01}}$ and $\mathbf{G_{01}}$ comes from the boundary integrals in the equation (13), the matrix $\mathbf{FF_{01}}$ from the domain integral and $\overline{\mathbf{I}}$ is an identity matrix corresponding to the coefficient $c(\xi)$ from the internal colocation points. The notation $\overline{}$ is used to distinguish matrices obtained from domain **internal** colocations points to the ones obtained from boundary colocations points.

5 Numerical example

The chosen example corresponds to a simple case of a plane stationary wave on the x direction on a homogeneous and limited space region Ω. The physical domain and its acoustic properties are shown in the Fig 1. Null Neumann conditions are applied on horizontal lines.

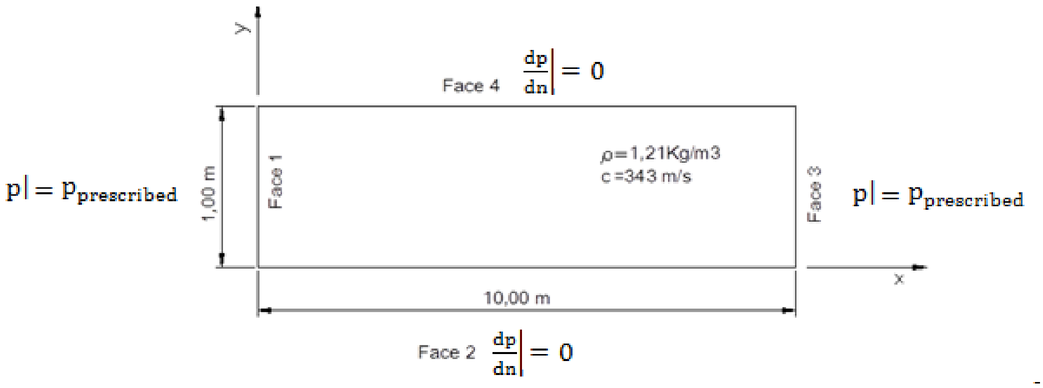

Figure 1. Acoustic domain and physical properties

Considering that no reflections occur in the faces 1 and 3, the analytical solution for this Helmholtz problem can be written as:

$$p = \rho c \, e^{-ikx} \tag{13}$$

Results corresponding to the FBMF were obtained assigning the followings values to k_0: 0.4, 0.6, 0.8, 1.0, 1.2 and 1.4 m^{-1}. On the other hand, for the DFBMF, it is necessary adjust two values of fictitious wavenumber. Thus, the value of k_{02} was fixed arbitrarily as $k_{02} = 0.3833 \ m^{-1}$ and the following values were assigned for the fictitious wavenumber k_{01}: 0.5, 0.7, 0.9, 1.1, 1.3 and 1.5 m^{-1}.

The use of the fictitious medium strategy in this homogeneous case is strategic, since it is possible to assign to the fictitious wavenumbers values different from the value pertinent to the proposed problem and verify its effects. Despite the correction imposed by the fictitious reactive term, there is a significant change in the results, as can be seen in Figure (2).

In the differential equation, the term concerning the pressure field that obeys equation (6) is independent of the fictitious background medium chosen and it has to be a null global effect. However, when an integral sentence is formulated, the results are affected by the choice of an auxiliary function. As provided by the Weighted Residual Method, since the Green's function is not the strictly appropriate fundamental solution, the minimum residuals is affected by the non-null reactive term introduced by the fictitious technique. It also means that the suitable argument of the correlate Green's function is determinant to the accuracy.

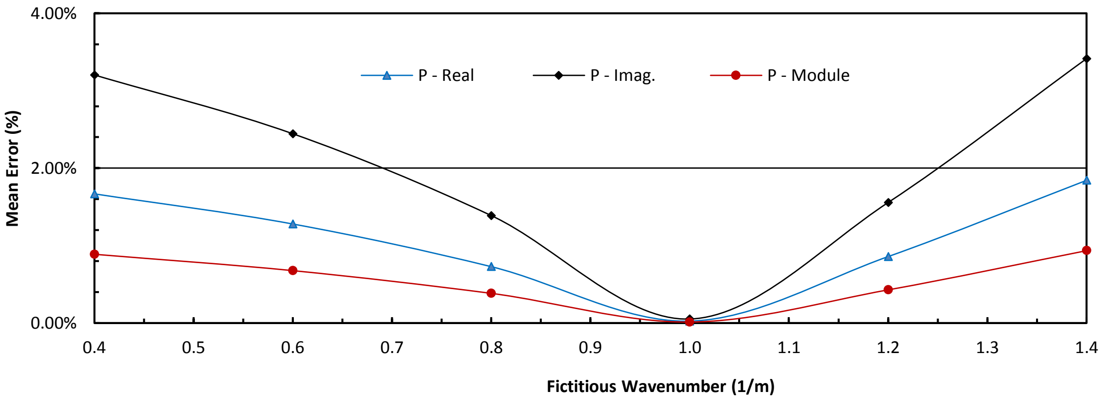

Figure 2. Mean error for the pressure. FBMF results for different values of fictitious wavenumbers k_0

The results shown in figure 3 are obtained using DFBMF. There is a strong reduction in errors for all ranges of values assigned to the fictitious wavenumbers k_{01}, confirming the consistency of the proposed strategy. On the other hand, analyzing figures (2) and (3) it is also possible to conclude that there is an optimal value for the fictitious wave number that leads to maximum accuracy. It is an expected result, since the FBMF is reduced to the original formulation of the boundary element for the Helmholtz equation when $k_0 = k_{Medium}$ so that the domain integral is null.

Although the DFBMF is less sensitive to the variations imposed on the fictitious wavenumber it was also observed that its maximum accuracy occurs for a value slightly greater than 1. This small deviation is certainly produced by the use of two independent equations. However, the improvement given by DFBMF compensates for this effect.

In this homogeneous example, it is clear that DFBMF led to more accurate results than FBMF especially when the fictitious wavenumber values k_0 are further away from the prescribed or ideal. This superior performance is credited to the introduction of a new independent equation.

Considering ideal values of fictitious wavenumber k_0 and k_{01}, the figures 4 and 5 exhibit the nodal values of error for the module of pressure inside the domain obtained with the FBMF for $k_0 = 1\ m^{-1}$ and the DFBMF for $k_{01} = 1.07\ m^{-1}$.

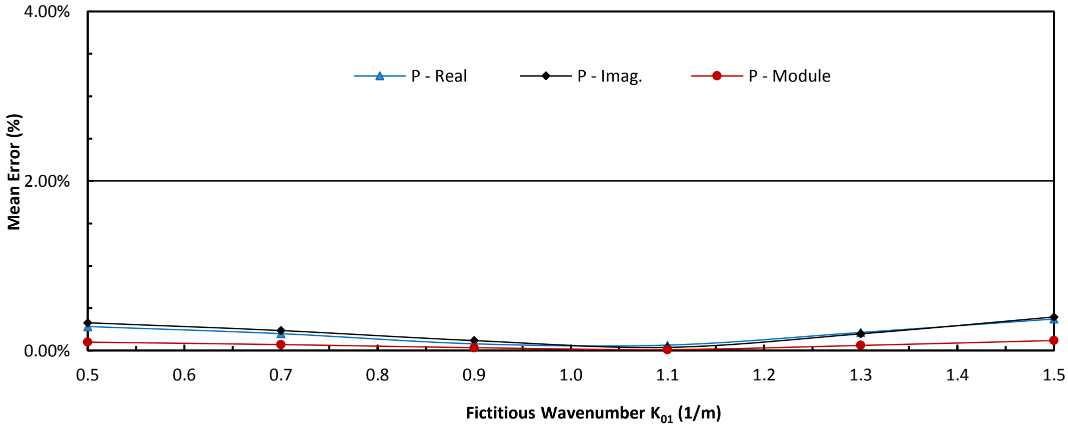

Figure 3. Mean error to the pressure. DFBMF for different values of wavenumbers k_{01} and fixed k_{02}

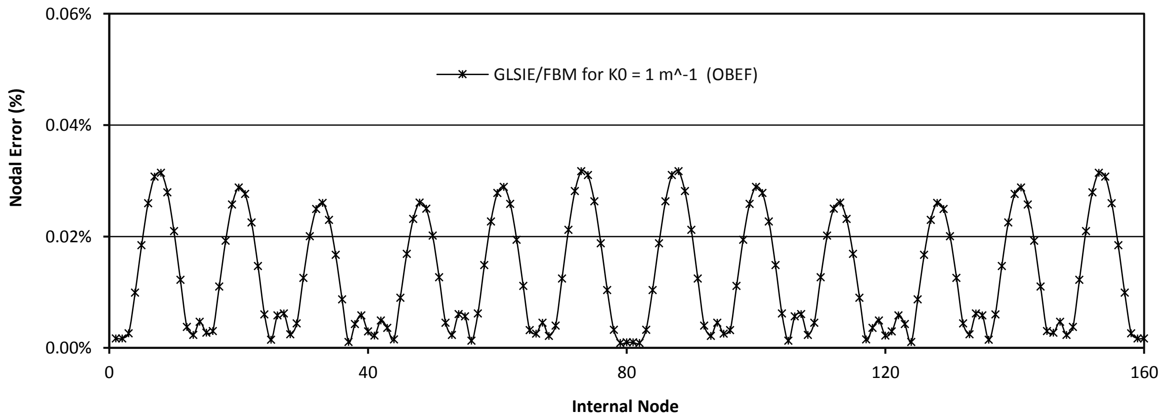

Figure 4. Nodal error of pressure (module) inside the domain Ω using the FBMF for $\mathrm{k_0 = k_{Medium} = 1\ m^{-1}}$

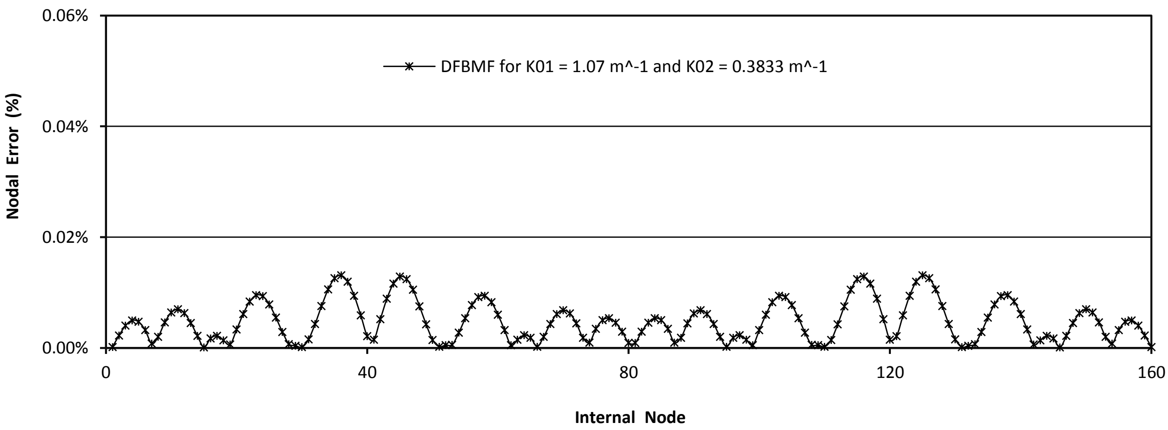

Figure 5. Nodal error in the numerical evaluation of pressure (module) inside the domain Ω using the DFBMF for $\mathrm{k_{01} = 1.07\ m^{-1}}$ and $\mathrm{k_{02} = 0.3833\ m^{-1}}$

In this homogeneous example, it is clear that the DFBMF led to more accurate results than the FBMF, mainly when values of fictitious wavenumbers are longer different from the prescribed or ideal ones are used. This superior performance is credited to the introduction of a new independent equation.

6 Conclusions

This article proposes a new BEM formulation to solve the non-homogeneous Helmholtz problem. DFBMF is based on the simple algebraic procedure of the Fictitious Background Media technique, but uses two integral equations that produces an improvement in accuracy. It occurs because the non-zero fictitious reactive term interacts negatively, as can be confirmed by the numerical results. This first important conclusion is that the Green's function used with average wavenumber as uniform constitutive property and Dual Reciprocity to solve the complementary term cannot be the best choice, since an optimal condition needs to be sought.

The second conclusion, however, is about the capacity of the proposed DFBMF to significantly reduce the numerical perturbations produced by the fictitious reactive term. Modeling the problem in terms of two independent integral equations, each related to a Green function for different values of the dummy background medium, it is possible to use empirical control parameters that improve the quality of the results. The theoretical substance for the use of two integral equations was confirmed by the numerical results obtained in the example presented here.

The convergence tests performed here can be replaced by a more adequate mathematical analysis. The ideal values of the fictitious wavenumbers are reached when the error curves reach minimum points. Since this minimum of the error curve corresponds to the point of the null derivative, this aspect can be used to establish a stopping criterion for an iterative algorithm. This algorithm should converge to optimal values for the fictitious wavenumbers when the minimum point on the error curve is reached. The derivative of one of the two integral equations of DFBMF can be used for this purpose. However, this is a subject to be explored in future works.

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