Recalculation of internal directional derivatives using the integral equation of the Boundary Element Method in Poisson problems previously solved by the Finite Element Method

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Abstract. The strategy of reusing the integral equation of the Boundary Element Method (BEM) to improve the accuracy of the values of variables previously calculated on the boundary have showed consistency, addressing scalar problems governed by the Laplace, Poisson, and Navier equations. In this work, the boundary nodal values obtained with the application of the Finite Element Method (FEM) in Poisson problems are substituted in the BEM integral equation for the calculation of the internal values of potential and its derivative. For the example showed, the results obtained were compared with the values found by the FEM and BEM, both in their classic form, and the performance was evaluated through the analytical results.

Keywords: Boundary Element Method, Recursive Scheme, Poisson's Problems, Finite Element Method, Radial Basis Functions.

1 Introduction

In a previous work [1], the boundary integral equations of the Boundary Element Method (BEM) were used to calculate the internal values of the directional derivatives in Laplace problems, using the nodal values of the potential determined by the Finite Element Method (FEM). The internal values of the potential were also recalculated in the same way, with favorable results, that is, with greater accuracy than the results obtained from the classic FEM.

The proposal of reusing the integral equation of the BEM in association with the FEM was based on the observation that the internal variables of the BEM are calculated by reapplication of the integral boundary equation, with the nodal values being previously calculated [2, 3]. These internal values always have more accuracy than the boundary; the reason for this behavior is that the inverse integral equation is equivalent to a global sentence of weighted residuals and its reuse implies a new minimization of numerical errors. So, initially, this procedure was applied to recalculate the boundary values at new nodal points, called recursive procedure (RBEM).

The RBEM was successfully used to solve numerical problems governed by the Laplace [4,5] and Poisson equations [6,7] and also linear elastic problems expressed by the Navier equation [8,9]. The RBEM results are especially effective for poor meshes since mesh refinement reduces the performance of the RBEM. The recursive procedure also features a very positive effect on problems where there is a discontinuity in the prescribed boundary conditions. The perceived discontinuities in the BEM solution are completely eliminated by the recursive procedure, indicating that the reuse of the integral equation of the BEM, in fact, minimizes the errors committed in the direct solution in accordance with the weighted residuals method.

With regard to the use of the RBEM scheme together with FEM, no new system of equations is solved, which makes the proposal computationally cheaper. However, it is necessary to have all potential values and potential normal derivatives at the boundary to implement the proposed scheme. The potential is obtained directly through the solution of the matrix system of the FEM, but the potential normal derivatives at the boundary need to be calculated by some approximation technique [10] they are obtained here in two ways. The first consists of the classical calculus of derivatives of polynomial functions used as approximation functions within finite elements. The second technique is given by a model based on the BEM. The objective of this work is to demonstrate that the BEM can be applied to improve the solution of the FEM since the medium is homogeneous; however, the calculation of potential normal derivatives with the FEM can also be advantageously calculated using the BEM.

However, there is nothing to stop us from adopting some of those proposals to improve the performance of other methods, especially with regard to internally calculated values, since the domain is homogeneous. In this work, the BEM has efficient calculation strategies to deal with the independent term of the Poisson equation.

2 The FEM Poisson Problems

Poisson's equation is a generalization of Laplace's equation, in which a stationary scalar field, called potential, is subject to sources, sinks, or external actions that act directly in the field. Henceforth it is called body forces. Denoting by u(X) the potential scalar variable which represents the field, considered a two-dimensional homogeneous isotropic body subjected to a thermal or mechanical field at a steady state, the Poisson equation in indicial notation is given by:

$$u_{,ii}(X) = P(X) \tag{1}$$

The Finite Element Method in its classical form, as well as the Boundary Element Method, are based on integral formulations. Thus, considering Eq. 1, choosing auxiliary functions $\emptyset(X)$ the weak integral form related to Laplace's problem can be found, according to Eq. 2:

$$\int_{\Omega(X)} u_{,i} \phi_{,i} d\Omega(X) - \int_{\Gamma(X)} u_{,i} \phi \, d\Gamma(X) = \int_{\Omega(X)} P\phi d\Omega(X) \tag{2}$$

Equation 2 was obtained using the basic BEM approach. Thereafter, the domain $\Omega(X)$ is divided into sub-domains $\Omega^e(X)$ - finite elements - whose boundaries are generally defined by the nodal points. The approximation is performed within each finite element, and so they are all connected by the continuity condition at the nodal points. FEM uses a finite linear combination of auxiliary functions $\emptyset(X)$ whose first-order derivatives belong to the space of integrable square functions $L^e(\Omega)$. Unlike BEM, the classical FEM idea does not approximate the behavior of the field variables $u(X)$ and $q(X)$ globally; the approximation is valid only within each finite element. They are all connected through the continuity conditions. The solution of the potential field is obtained using a first-order polynomial function $\varphi(X)$ commonly known as shape functions, with support located at the nodal points of the triangular finite elements. Thus, the primal variable $u^e(x_1, x_2)$ at any point of the finite element can be written by interpolating the nodal values u_j, as shown in Eq. 3:

$$u^e(x_1, x_2) = \varphi_1(x_1, x_2)u_1^e + \varphi_2(x_1, x_2)u_2^e + \varphi_3(x_1, x_2)u_3^e \tag{3}$$

Within the classical formulation of FEM presented here, the commonly used procedure is the Galerkin Method [11], to which the weighting functions $\emptyset(x_1, x_2)$ are chosen equal to the shape function $\varphi(X)$. Each finite element has a domain $\Omega^e(X)$ delimited by a boundary $\Gamma^e(X)$. Using Eq. (3), the complete weak integral equation [10] takes the following form:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u_j \int_{\Omega^e(X)} \left(\frac{\partial \varphi_j}{\partial x_1} \frac{\partial \varphi_i}{\partial x_1} + \frac{\partial \varphi_j}{\partial x_2} \frac{\partial \varphi_i}{\partial x_2} \right) d\Omega^e(X) = \sum_{i=1}^{n} \int_{\Gamma^e(X)} \left(\frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2 \right) \varphi_i d\Gamma^e(X) +$$
$$\sum_{i=1}^{n} \int_{\Omega^e(X)} [p_i(X)\varphi_i] \varphi_i d\Omega^e(X) \tag{4}$$

In Equation 4, u_j means the potential values at nodes j, which are calculated throughout the domain. However, the interpolation of the field is done at the level of each finite element, as shown in Fig. 1, subsequently, through the continuity of the potential at the nodal points, a system of equations comprising the entire domain is assembled.
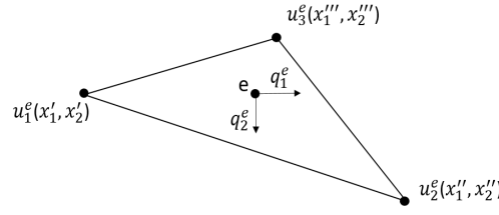
Figure. 1 Directional derivatives in the triangular finite element.

The Equation 5 presents interpolation process of force $P(X)$ in element finite generating the local vector $\mathbf{p}_L$. The h_{ii}^e terms results from de local integration of the $P(X)$:

$$\mathbf{p}_L = \begin{bmatrix} h_{11}^e & 0 & 0 \\ 0 & h_{22}^e & 0 \\ 0 & 0 & h_{33}^e \end{bmatrix} \begin{bmatrix} p_1^e \\ p_2^e \\ p_3^e \end{bmatrix} \tag{5}$$

The calculation of the directional derivatives, on the other hand, is generally performed by means of a potential field derivation, which is given by the interpolation functions, according to Eq. 6. Thus, the spatial derivatives of the potential must be assumed to be a valid approximation for the finite element centroid "e" (see Fig. 1). Thus, the numerical accuracy of q^e is also strongly dependent on the refinement of the mesh.

The determination of the spatial derivatives of the potential can be written in matrix form, as shown by Eq. 6:

$$\begin{bmatrix} q_1^e \\ q_2^e \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \varphi_1}{\partial x_1} & \dfrac{\partial \varphi_2}{\partial x_1} & \dfrac{\partial \varphi_3}{\partial x_1} \\ \dfrac{\partial \varphi_1}{\partial x_2} & \dfrac{\partial \varphi_2}{\partial x_2} & \dfrac{\partial \varphi_3}{\partial x_2} \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{bmatrix} \tag{6}$$

Regarding the spatial derivatives of the potential, the numerical robustness of the FEM is vastly different from the BEM approach. Using the FEM, the potential field given by Eq. 3 is submitted to a spatial derivative, which means that the interpolation functions have their order reduced. Accuracy is strongly affected by this procedure. The same problem occurs with respect to the determination of the derivatives at the boundary. Such values are approximated by those calculated at the centroid of the elements, which tangent the boundary through its edges. Problems can occur if the prescribed boundary conditions are different for each edge, in this case, certain more elaborate techniques must be used for an irregular geometrical shape.

The fact is that the accuracy between the values of the basic variable and its derivative, both internal and on the boundary are effectively different on the FEM. This problem can be minimized through the refinement of the mesh or the choice of higher-order finite elements; however, the coupling with the BEM can be advantageous, as the numerical results will demonstrate.

3   Coupling procedure FEM-RBEM

According to the classical formulation of the FEM, an adequate arrangement of Eq. 4 and the application of the continuity condition between nodes belonging to finite elements that connect generate a well-known matrix system, given by:

$$\mathbf{Ku = f} \tag{7}$$

In scalar problems, the matrix $\mathbf{K}$ is called the global diffusivity matrix and the vector $\mathbf{f}$ represents the sum of the interpolation of the prescribed values of potential, potential derivatives and sources in both boundary elements and domain points. The vector $\mathbf{u}$ consists of the potential values in the domain $\Omega(X)$ and the boundary $\Gamma(X)$ that were not prescribed.

The coupling FEM-RBEM proposed here is very simple. It is considered that the fundamental solution has a major role as a weighting function and that the reuse of the boundary integral equation means a new minimization of residuals. Therefore, even if the nodal values used are not values calculated by the BEM, there will be an improvement in the numerical accuracy of the internal values determined using the integral equations

of the BEM.

As explained above, this approach is based on the typical BEM procedure for calculating internal variables, in which the integral boundary equation is used twice. However, the accuracy of the internal values is always greater than that obtained in the boundary. For this reason, it has a numerical precision superior to the classic FEM. Even the recalculation of the internal potential itself by the BEM presents values with better precision than the FEM.

The integral equation of the BEM for calculating the internal potential to Poisson problems is given by:

$$u^{int}(\xi) = \int_{\Gamma(X)} q(X)u^*(\xi;X)d\Gamma(X) - \int_{\Gamma(X)} u(X)q^*(\xi;X)d\Gamma(X) - \int_{\Omega(X)} P(X)u^*(\xi;X)d\Omega(X) \tag{8}$$

The directional derivatives of the potential $u^{int}(\xi)$ can be obtained, just being enough to derive it in relation to the coordinates x_1 and x_2 of the source point $\xi = \xi(x_1;x_2)$ according to Eq. 9:

$$\frac{du^{int}(\xi)}{dx_j} = \int_{\Gamma(X)} q(X)\frac{d[u^*(\xi;X)]}{dx_j}d\Gamma(X) - \int_{\Gamma(X)} u(X)\frac{d[q^*(\xi;X)]}{dx_j}d\Gamma(X) - \int_{\Omega(X)} P(X)\frac{d[u^*(\xi;X)]}{dx_j}d\Omega(X) \tag{9}$$

To solve Poisson problems with the BEM, there are several techniques able to deal with the domain integral and transforming it into a boundary integral: Gao's Direct Integration scheme [12], a Dual Reciprocity [13] and the Multiple Reciprocity Method [14,15] emphasizing that this last proposal contains the idea of Galerkin's Tensor and its primitives. It is important that the technique used by the BEM does not take any information regarding the domain. In this regard, Dual Reciprocity is not adequate in the association proposed here with the FEM.

The idea of Multiple Reciprocity Method (MRM) is the most powerful. Considering the integral equation for calculating internal potentials, and that a primitive function (also called harmonic) of the fundamental solution $T^*(\xi;X)$ and other associated primitives are known, the MRM assumes that:

$$T^*_{,ii}(\xi;X) = u^*(\xi;X); \quad K^*_{,ii}(\xi;X) = T^*(\xi;X); \quad Y^*_{,ii}(\xi;X) = K^*(\xi;X); \text{ (and so on)} \tag{10}$$

Due to the characteristic of primitive or harmonic functions being self-adjoint, the same integration procedures by parts and application of the divergence theorem can be applied repeatedly to domain integrals:

$$\int_{\Omega(X)} P(X)T^*_{,ii}(\xi;X)\,d\Omega(X) = \int_{\Gamma(X)} P(X)T^*_{,i}(\xi;X)n_i(X)\,d\Gamma(X) -$$
$$\int_{\Gamma(X)} P_{,i}(X)n_i(X)T^*(\xi;X)\,d\Gamma(X) + \int_{\Gamma(X)} P_{,ii}(X)K^*_{,k}(\xi;X)n_k(X)\,d\Gamma(X) + \tag{11}$$
$$+ \int_{\Gamma(X)} P_{,iik}(X)n_k(X)K^*(\xi;X)\,d\Gamma(X) + \int_{\Omega(X)} P_{,iikk}(X)Y^*_{,mm}(\xi;X)\,d\Omega(X)$$

The process is repeated until the successive derivatives of body force are considered negligible. In this work, for simplicity, the problems discussed have body forces such that their Laplacian is null and just the classic idea of the Galerkin tensor is applied. This does not invalidate the generality of the procedure and will be sufficient to assess its performance. Thus, the following integral equation to the potential internal is achieved:

$$u^{int}(\xi) = \int_{\Gamma(X)} q(X)u^*(\xi;X)d\Gamma(X) - \int_{\Gamma(X)} u(X)q^*(\xi;X)d\Gamma(X) +$$
$$\int_{\Gamma(X)} P_{,i}(X)n_i(X)T^*(\xi;X)\,d\Gamma(X) - \int_{\Gamma(X)} P(X)T^*_{,i}(\xi;X)n_i(X)\,d\Gamma(X) \tag{12}$$

Derivating Eq. 12 in relation to the coordinates x_1 and x_2 of the source point $\xi = \xi(x_1;x_2)$ the derivatives of the internal potential are defined according to Eq. 13:

$$\frac{du^{int}(\xi)}{dx_j(\xi)} = \int_{\Gamma(X)} q(X)\frac{d[u^*(\xi;X)]}{dx_j(\xi)}d\Gamma(X) - \int_{\Gamma(X)} u(X)\frac{d[q^*(\xi;X)]}{dx_j(\xi)}d\Gamma(X) +$$
$$\int_{\Gamma(X)} P_{,i}(X)n_i(X)\frac{d[G^*(\xi;X)]}{dx_j(\xi)}d\Gamma(X) - \int_{\Gamma(X)} P(X)\frac{d[G^*_{,i}(\xi;X)n_i(X)]}{dx_j(\xi)}d\Gamma(X) \tag{13}$$

The derivatives contained in the kernel of the Eq. 13 are given by:

$$\frac{d[G^*(\xi;X)]}{dx_j(\xi)} = \frac{r_j}{8\pi}\ \{2ln[r(\xi;X)] - 1\} \tag{14}$$

$$\frac{d[G^*_{,i}(\xi;X)n_i]}{dx_j(\xi)} = \frac{1}{4\pi}\left(-0,5n_j + \frac{r_j}{r^2}r_in_i + ln[r(\xi;X)]n_j\right) \tag{15}$$

4   Numerical simulation

This example consists of a square bar fixed at $x_2 = L$ subjected to a body force $P(X)$ and free to deform as shown in Figure 2. Note that the coordinate system is fixed at point (0;0). The governing equation for this case is given by:

$$u_{,ii}(X) = P(X) = -\frac{\rho g}{EL} x_2 \qquad (16)$$

This example was successfully solved by Ramos et al [7] using the boundary element recursive procedure. The values of E, L ρ and g were considered unitary.
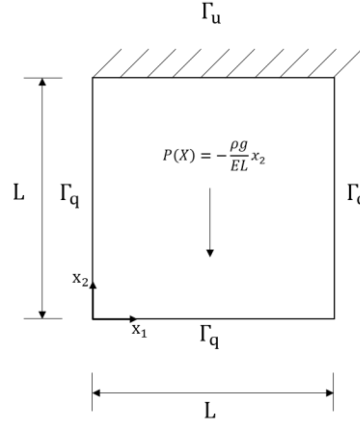

Figure 2: Clamped bar submitted to constant domain force.

The analytical solution for the potential is:

$$u(x_1, x_2) = \frac{\rho g}{6EL} (L^3 - x_2^3) \qquad (17)$$

Consequently, one can obtain the spatial derivatives:

$$\frac{\partial u}{\partial x_1} (x_1, x_2) = 0 \qquad (18)$$

$$\frac{\partial u}{\partial x_2} (x_1, x_2) = -\frac{\rho g}{2EL} x_2^2 \qquad (19)$$

The performance of numerical solutions was evaluated through the Mean Relative Error (MRE%) curve for each method, using the analytical solution as well as the available meshes. (see Tab. 1). For the error measurement MRE% (see Eq. 20), the terms v_i^{analit} and v_i^{num} represents the analytical and numerical scanning values respectively; the denominator v_{analit}^* is given by the biggest analytic value over the domain and n is the total number of nodal points. Table 1 presents the FEM meshes used for this problem and Fig. 3 presents an example of a structured image of mesh 1, with the others following the same pattern.

$$MRE\% = \frac{100}{n} \sum_{i=1}^{n} \left| \frac{v_i^{analit} - v_i^{num}}{v_{analit}^*} \right|. \qquad (20)$$

Table 1. Features of the Finite Element meshes

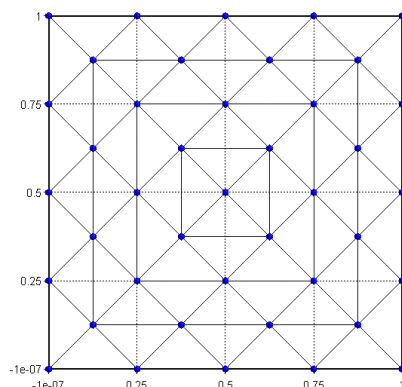| Nomenclature	Total number of nodal points	Total number of boundary nodes	Total number of finite elements	Total number of internal nodal points
Mesh 1	41	16	64	48
Mesh 2	145	32	256	224
Mesh 3	2113	128	4096	3968
Mesh 4	8321	256	16384	16128

Figure 3: Mesh 1

Figure 4 presents the error results for the calculation of the internal flows in the vertical direction, which include the error curves for the classic BEM, as it allows a better evaluation of the performance of the proposed technique compared to the others. As expected, classic BEM performed best. The error values of the FEM were higher than those of the BEM in all meshes, but these error levels were quite satisfactory, even considering the approximation of the calculation of the derivatives in the centroid of the finite elements (see Eq. 6) in addition to the reduction in the order of the primal variable itself (see Eq. 04.) It is noteworthy that when using the 5 mesh during the application of FEM, a matrix of order 10^4 is generated, increasing the computational cost during the solution of the linear system, this makes the use of RBEM advantageous during the resolution of large systems.

However, the proposed FEM-RBEM results were only better than FEM for the less refined mesh. This is particularly due to the simplicity of the proposed problem, confirming that the FEM-RBEM coupling should only be applied in more elaborate problems, in which simpler meshes are used with the FEM.
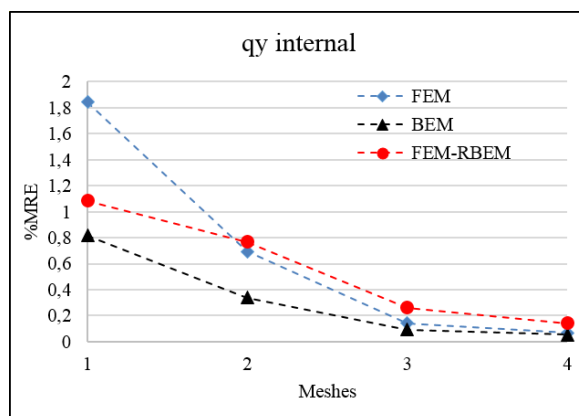

Figure 4: Mean Relative Error for the analyzed meshes

5   Conclusions

This work continues previous research in which the FEM-RBEM association was successfully applied to solve Laplace's problems. Now, by solving Poisson's problems, a wider range of problems can be analyzed. In this article, only one simple problem was solved, but the results confirm the previous conclusions: the FEM-RBEM association is very effective, especially if coarse FEM meshes are used. In general, it can be concluded that the use of RBEM integral equations contributed satisfactorily to obtain more accurate internal directional derivatives using FEM contour data. FEM-RBEM also guarantees a high gain in terms of computational cost, since for each internal point two equations for fluxes are solved using boundary values, eliminating the need to deal with numerous internal data.

As pointed out above, the good results resulting from this association between FEM and BEM are due to the way in which the values of potentials and spatial derivatives are calculated. Using BEM, an exact integral equation is used, whose interpretation towards a minimization of residues was proven through the results. Future work will be developed solving more complex Poisson problems to confirm the performance of the proposed technique.

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