A VON MISES STRESS-BASED TOPOLOGY OPTIMIZATION OF CONTINUUM ELASTIC STRUCTURES THROUGH THE PROGRESSIVE DIRECTIONAL SELECTION METHOD

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Abstract. This work presents a study applying the von Mises equivalent stress as a performance parameter for topological optimization of two-dimensional continuous elastic structures employing the Progressive Directional Selection (PDS) method. A typical objective to achieve the ideal topology of a structure is to define the best material distribution of the design domain, considering an objective function and mechanical constraints. In general, most studies deal with minimizing the compliance of structures. Numerical methods for optimizing the topology of continuous structures have been widely investigated. Most of these methods are based on finite element analysis, where the design domain is discretized into a fine mesh of elements. Evolutionary Structural Optimization (ESO) is one of these methods based on the simple concept of gradually removing inefficient finite elements from a structure. This method was formulated from the engineering point of view that the topology of the structures is naturally conservative for safety reasons and contains an excess of material. In such a context, the optimization consists of finding the optimal topology of a structure and determining whether there should be a solid or void element for each point in the design domain. ESO's algorithms are easy to understand and implement. The stress level of each element is determined by comparing the von Mises stress of the element and the maximum von Mises stress of the entire structure. After each finite element analysis, elements that present a stress level below the defined rejection ratio are excluded from the model. However, the ESO is a heuristic method, and there is no mathematical proof that an optimal solution can be achieved by eliminating elements. In addition, the original approach is inefficient because it needs to find the optimal topology comparing several solutions generated intuitively, adjusting the rejection ratio and evolutionary rate. To avoid this problem, but taking advantage of the simplicity of applying ESO, a new approach using the PDS method is proposed, inspired by the natural directional selection observed in biology. In the first work using PDS, the optimization problem was the minimization of the strain energy of a structure analyzed through the Finite-Volume Theory (FVT). This investigation discusses a scheme to minimize the von Mises equivalent stress of a discretized domain with a volume constraint. One example of topological optimization of 2D continuous elastic structure inspired by a classic literature problem is investigated.

Keywords: topology optimization, progressive directional selection method, continuum elastic structures.

1 Introduction

Véras and Araujo [1] presented a new approach for Topology Optimization (TO) of two-dimensional continuum elastic structures through the Progressive Directional Selection (PDS) method, taking advantage of the simplicity of applying ESO and inspired by the natural directional selection observed in biology. From a certain point of view, TO methods are a process of evolution of a structure, where only the elements that contribute effectively are kept in the structural set. As a response to this evolution, it is expected to arrive at a structure that presents

characteristics that most interest the designer. Numerical methods for topology optimization of continuum structures have been investigated extensively since Bendsøe and Kikuchi [2]. There are two main fields in Structural Topology Optimization: gradient-based and non-gradient-based. The first one is a set of mathematical models derived from calculating the design variables' sensitivities. On the other hand, in non-gradient-based methods the material is removed or added using a sensitivity function. Both fields have been investigated in detail over the last two decades, and there are already real-world structures designed using topology optimization, Steven and Xie [3]. However, unlike gradient-based methods, which have more complexity for computational implementation, as the Solid Isotropic Material with Penalization (SIMP), heuristic methods (not based on gradients) are a good alternative because of their simplicity, with results like those found by gradient-based methods, Munk et al. [4].
The SIMP algorithm has been the first to become efficient, robust, and widely used. As a result, SIMP optimizers have recently been introduced in some of the main finite element packages worldwide, Bendsøe and Sigmund [5]. The material properties are assumed to be constant within each element of the discretized domain of analysis, and the design variables are the relative densities of the elements. Thus, the elastic properties are modeled from the relative density of the material raised to a given power to penalize the intermediate values for the relative densities of the material, Bendsøe [6], Zhou and Rozvany [7] and Mlejnek [8].
The ESO method initially proposed by Xie and Steven [9] is built on a pure heuristic principle that removes inefficient materials, and the structure evolves towards an optimum. Initially, ESO was implemented solely as a material removal method, which meant that removed parts could not be restored afterward. However, this led to convergence issues and mesh dependence. The latter problems were overcome by extending to the Bidirectional Evolutionary Structural Optimization (BESO) method that allowed both material addition and removal, Huang and Xie [10]. However, the solution may worsen in the objective function if the ESO/BESO technique continues with no stop or reaches a local optimum, Rozvany [11]. Because the initial development of ESO methods is based on a heuristic concept and lacks theoretical rigor, most of the early work on ESO/BESO neglected significant numerical problems in TO, such as the existence of a solution, checker-board, mesh-dependency, and local optimum, Xia et al. [12]. Most applications in structural topology optimization use the finite element method (FEM), but other numerical methods are also used, for example, boundary element methods (BEM) [13-14] and finite-volume theory (FVT) [15].
This work presents a new approach for TO of two-dimensional continuum elastic structures through the Progressive Directional Selection method. The PDS is demonstrated on a von mises stress-based topology optimization problem. A classic bidimensional cantilever beam example is analyzed applying finite-volume theory [16, 17] to define the optimum design by the PDS method.

2 Locally-Applied Average Stress Theorem to the Finite Volume Theory

A proposal for evaluating the square of the average equivalent von Mises stress for each subvolume is presented below based on the average stress theorem of micromechanics. It is a much more efficient way to carry out this analysis, which is possible for the finite-volume theory because of the satisfaction of the differential equilibrium equations in the subvolumes. The average stress theorem is applied as presented below for the correct selection of the subvolumes that discretize a two-dimensional domain.

$$\frac{1}{V}\int_S t_i \cdot x_j \cdot dS = \frac{1}{V}\int_S \sigma_{ki} \cdot n_k \cdot x_j \cdot dS = \frac{1}{V}\int_V \sigma_{ki} \cdot \frac{\partial x_j}{\partial x_k} \cdot dV + \frac{1}{V}\int_V \frac{\partial \sigma_{ki}}{\partial x_k} \cdot x_j \cdot dV = \frac{1}{V}\int_V \sigma_{ji} \cdot dV + \frac{1}{V}\int_V \frac{\partial \sigma_{ki}}{\partial x_k} \cdot x_j \cdot dV = \bar{\sigma}_{ij} + \frac{1}{V}\int_V \frac{\partial \sigma_{ki}}{\partial x_k} \cdot x_j \cdot dV \tag{1}$$

where $\bar{\sigma}_{ij} = \frac{1}{V}\int_V \sigma_{ji} \cdot dV$ is the volume-averaged stress.

Thus,

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_S t_i \cdot x_j \cdot dS - \frac{1}{V}\int_V \frac{\partial \sigma_{ki}}{\partial x_k} \cdot x_j \cdot dV$$

When the differential equilibrium equations are satisfied, in the absence of body forces:

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_S t_i \cdot x_j \cdot dS \tag{2}$$

The presented formulation has its roots in the finite-volume theory, developed by Bansal and Pindera [18], for bidimensional linear elastic structures. The adopted reference domain is rectangular in $x_1 - x_2$ plane with $0 \leq x_1 \leq L$ and $0 \leq x_2 \leq H$, which is discretized in N_β horizontal subvolumes and N_γ vertical subvolumes, Figure 1. The subvolume dimensions are designated by l_q and h_q for $q = 1, \dots, N_q$, where $N_q = N_\beta \cdot N_\gamma$ is the total number of subvolumes. In the present formulation, the components of the displacement field can be approximated by a Legendre polynomial expansion in the local coordinated system, Bansal and Pindera [18]:

$$u_i^{(q)} = W_{i(00)}^{(q)} + x_1^{(q)} W_{i(10)}^{(q)} + x_2^{(q)} W_{i(01)}^{(q)} + \frac{1}{2}\left(3{x_1^{(q)}}^2 - \frac{l_q^2}{4}\right) W_{i(20)}^{(q)} + \frac{1}{2}\left(3{x_2^{(q)}}^2 - \frac{h_q^2}{4}\right) W_{i(02)}^{(q)} \quad (3)$$

where $\mathrm{i} = 1{,}2$ and $W_{i(mn)}^{(q)}$ are unknown coefficients.

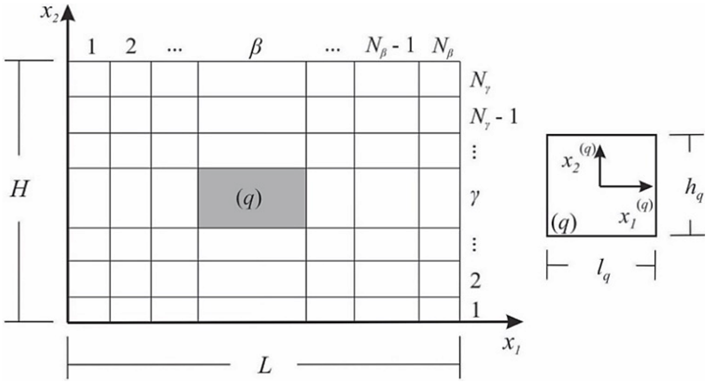


Figure 1. Discretized structure in rectangular subvolumes and local coordinate system of a generic subvolume

Strain field can be written as:

$$\begin{aligned} \varepsilon_{11}^{(q)} &= W_{1(10)}^{(q)} + 3x_1^{(q)} W_{1(20)}^{(q)} \\ \varepsilon_{22}^{(q)} &= W_{2(01)}^{(q)} + 3x_2^{(q)} W_{2(02)}^{(q)} \\ 2 \cdot \varepsilon_{12}^{(q)} &= W_{1(01)}^{(q)} + 3x_2^{(q)} W_{1(02)}^{(q)} + W_{2(10)}^{(q)} + 3x_1^{(q)} W_{2(20)}^{(q)} \end{aligned} \quad (4)$$

Then the stress field is defined as:

$$\begin{aligned} \sigma_{11}^{(q)} &= C_{11}^{(q)}\left(W_{1(10)}^{(q)} + 3x_1^{(q)} W_{1(20)}^{(q)}\right) + C_{12}^{(q)}\left(W_{2(01)}^{(q)} + 3x_2^{(q)} W_{2(02)}^{(q)}\right) \\ \sigma_{22}^{(q)} &= C_{12}^{(q)}\left(W_{1(10)}^{(q)} + 3x_1^{(q)} W_{1(20)}^{(q)}\right) + C_{11}^{(q)}\left(W_{2(01)}^{(q)} + 3x_2^{(q)} W_{2(02)}^{(q)}\right) \\ \sigma_{12}^{(q)} &= C_{44}^{(q)}\left(W_{1(01)}^{(q)} + 3x_2^{(q)} W_{1(02)}^{(q)} + W_{2(10)}^{(q)} + 3x_1^{(q)} W_{2(20)}^{(q)}\right) \end{aligned} \quad (5)$$

Using the equation (2), the volume-averaged stress components are evaluated as follows:

$$\overline{\boldsymbol{\sigma}}^{(q)} = \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} = \boldsymbol{T} \cdot \overline{\boldsymbol{t}}^{(\boldsymbol{q})} \quad (6)$$

where:

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$\overline{\boldsymbol{t}}^{(\boldsymbol{q})} = \left[\bar{t}_1^{(q,1)} \quad \bar{t}_2^{(q,1)} \quad \bar{t}_1^{(q,2)} \quad \bar{t}_2^{(q,2)} \quad \bar{t}_1^{(q,3)} \quad \bar{t}_2^{(q,3)} \quad \bar{t}_1^{(q,4)} \quad \bar{t}_2^{(q,4)}\right]^T$ is the surface-averaged traction vector of a subvolume $\boldsymbol{q}$.

The square of the volume-averaged von Mises stress can be evaluated as follow

$$\bar{\sigma}_{vM}^2 = {\overline{\boldsymbol{\sigma}}^{(q)}}^T \cdot \boldsymbol{P} \cdot \overline{\boldsymbol{\sigma}}^{(q)}$$

where:

$$\boldsymbol{P} = \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Thus,

$$\bar{\sigma}_{vM}^2 = \bar{\boldsymbol{t}}^{(q)^T} \cdot \boldsymbol{T}^T \cdot \boldsymbol{P} \cdot \boldsymbol{T} \cdot \bar{\boldsymbol{t}}^{(q)} \Rightarrow \bar{\sigma}_{vM} = \bar{\boldsymbol{u}}^{(q)^T} \cdot \boldsymbol{K}^{(q)^T} \cdot \bar{\boldsymbol{P}} \cdot \boldsymbol{K}^{(q)} \cdot \bar{\boldsymbol{u}}^{(q)} \tag{7}$$

where $\bar{\boldsymbol{u}}^{(q)} = \left[\bar{u}_1{}^{(q,1)} \quad \bar{u}_2{}^{(q,1)} \quad \bar{u}_1{}^{(q,2)} \quad \bar{u}_2{}^{(q,2)} \quad \bar{u}_1{}^{(q,3)} \quad \bar{u}_2{}^{(q,3)} \quad \bar{u}_1{}^{(q,4)} \quad \bar{u}_2{}^{(q,4)}\right]^T$ is the surface-averaged displacement vector, $\bar{\mathbf{P}} = \boldsymbol{T}^T \cdot \mathbf{P} \cdot \boldsymbol{T}$ is an auxiliar symmetric matrix and $\boldsymbol{K}^{(\mathrm{q})}$ is the local stiffness matrix of a generic subvolume $\boldsymbol{q}$.

3 Progressive Directional Selection Method

In general, topological optimization methods seek the best structure design that produces the stiffest response with a given volume of material. In the traditional ESO method, a structure can be optimized by removing elements and, if the correct parameters are provided, the solution can be achieved. Although it is difficult to define these parameters, several preliminary analyses must occur until an engineer decides which solution to adopt. To overcome this problem, the Progressive Directional Selection method, inspired by Darwin's natural selection theory, specifically the directional selection, takes a discrete problem as a discretized structure in a "population" of structural elements. The selection can then be made progressively by eliminating the individuals who least contribute to the structure's stiffness.

In nature, when directional selection acts on a population, a specific characteristic can guarantee these individuals' survival. Thus, the PDS method optimizes the structure by minimizing the objective function (compliance, strain energy, and von Mises stress) and defines which structural elements will remain at the end of the selection. The process is simple because, once the desired final volume of the structure is known, the main idea is to gradually remove the elements from an initial configuration, as many times as necessary, wherein each stage increases the number of removing and decreases the number of removed elements by removing, until verifying whether the process leads to the same solution.

3.1 Numerical implementation of PDS

Based on the performance criteria adopted for the problem, the selected population is reached through an iterative process that converges when the optimal topology does not evolve anymore, i.e., there is no change in the final set of selected elements. The proposed PDS technique applies a strategy to minimize the objective function.

The optimization problem in their standard form can be expressed mathematically as:

$$\begin{gathered} Minimize\ \mathrm{mean}(\boldsymbol{\sigma}_{\mathrm{vM}}^{\mathrm{population}}) \\ subject\ to\colon \boldsymbol{KU} = \boldsymbol{F} \\ \frac{V}{V_0} = f \end{gathered} \tag{8}$$

where $\boldsymbol{\sigma}_{\mathrm{vM}}^{\mathrm{population}}$ is the array with the von Mises average stress of the population, $\boldsymbol{U}$ is global displacement vector, $\boldsymbol{K}$ is global stiffness matrix, $\boldsymbol{F}$ is the global force vector, V and V_0 are the material volume and design domain volume, respectively, and f is the prescribed volume fraction.

The procedures for running PDS are the following:

1. Initialize an original model (assemble the stiffness matrix and initial parameters) and determine boundary and loading conditions.
2. Assemble an array that identifies the elements.
3. Start the stage of the selection loop.
4. For the actual stage, specify the number of steps and the number of removed elements for each step.
5. Solve the problem and specify the optimization criterion.
6. Ranking individuals according to the optimization criterion.
7. For each step, remove the elements that contribute least to the structure.
8. Save the identities of the selected individuals.
9. Repeat the procedure steps from 4 to 8 until the current stage's selected individuals are the same as one

or more previous stages.

However, in the case of continuous two-dimensional elastic structures, applying a penalty factor (PF) to the stiffness of the eliminated elements of the discretized analyzed domain is necessary to avoid remeshing and singularity of the global stiffness matrix. In practice, values in the order of magnitude of 10^{-3} are recommended. Particular attention should be given to the classification of the individuals. Convergence criteria are named depending on the number of repetitions desired for the selected individuals. For example, the C4 criterion refers to the selection of individuals from a population that is repeated in four consecutive stages. To achieve a good ranking that overcomes some numeric imprecisions, for each removal step, a selection tolerance (ST) is applied to the ranked array (Figure 2), which can add an individual with a value of the objective function significantly closer to the last selected element at the ranked array. This procedure directly interferes with the optimal topology, especially in analyzing two-dimensional continuous structures.

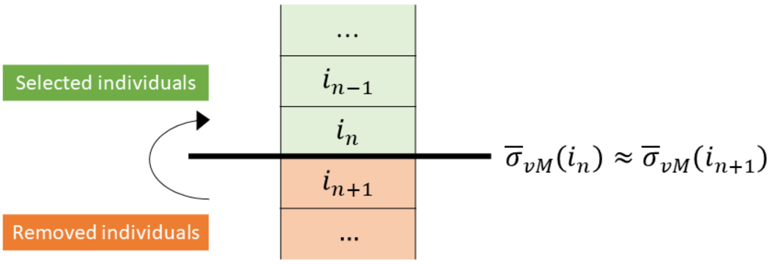


Figure 2. Selection tolerance scheme applied to the ranked array.

Another aspect of the procedure is to define the number of removed individuals (NR^{step}) in each removal step. The *i-th* selection stage is initially proportional to the final number of removed individuals (NS), which refers to the final volume desired for the structure in continuous problems. Thus, $NR^{step} = NS/i$, which must be an integer to access an array by index. When this does not occur, the NR^{step} must be adjusted, redistributing the decimal part among the other steps. The problem is treated as essentially it is, a discrete problem, differently of the approaches based on the concept of material density and penalization method.

3.2 Filtering scheme

A simple filtering technique can be adopted using a scheme based on the weight factors for subvolumes in the neighborhood. Depending on the position of the subvolume in the mesh, the contribution of the neighboring subvolumes must be adjusted, as illustrated in Figure 3. Only three cases are considered: internal, edge, and corners subvolumes, and the dark blue ones are the main subvolumes for each case.

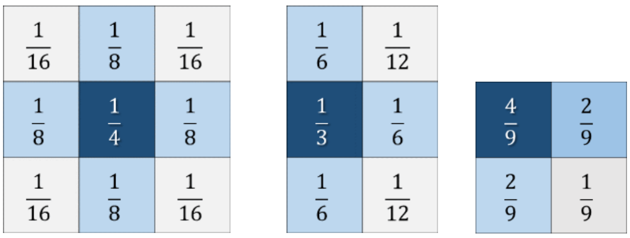


Figure 3. Filtering scheme: weight factors for internal, edge and corners subvolumes, respectively.

4 Cantilever beam Results and Discussion

The cantilever beam is fixed on the left border, and a concentrated load is applied in the middle of the right border, as shown in Figure 4. The proposed optimization problem minimizes the population's mean of the average von Mises stress of each subvolume. Also, this problem's objective is to find the stiffest structure with a given volume fraction of 40% of the design domain volume. The dimensions for the design domain are $H = 0.4\,m$ and $L = 0.8\,m$, and thickness $t = 10\,mm$. It is assumed the Young's modulus $E = 2 \times 10^{11}\,N/m^2$ and Poisson's ratio $\upsilon = 0.3$. The penalty factor is $PF = 10^{-3}$ and the adopted selection tolerance is $ST = 10^{-6}$. The convergence criterium employed is implemented to repeat the set of selected subvolumes in five cases: C3, C4, C5, C6 and C7. The structure is analyzed employing six different meshes: 22x11, 42x21, 62x31, 82x41, 102x51 and 122x61. Figure 5 shows the optimal topologies obtained from the PDS considering the von Mises stress-based topology optimization.

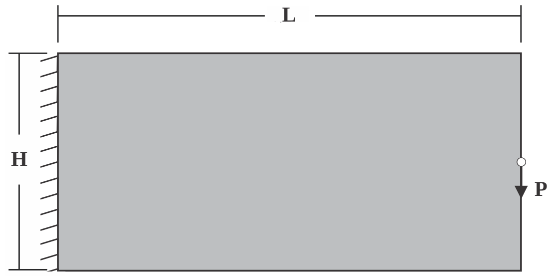


Figure 4. Cantilever beam.

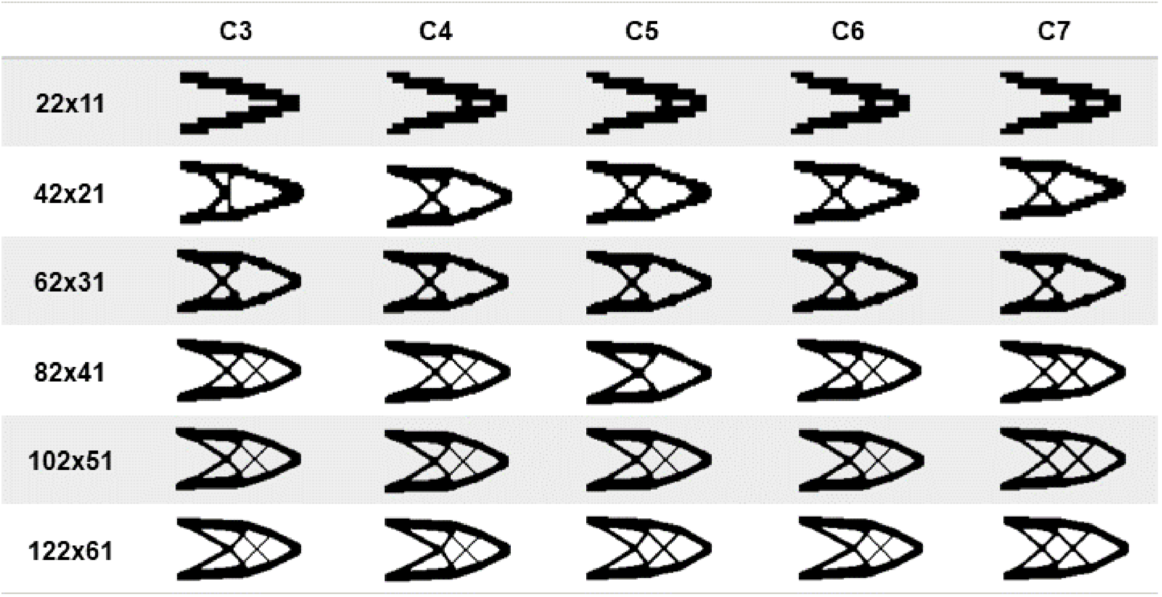
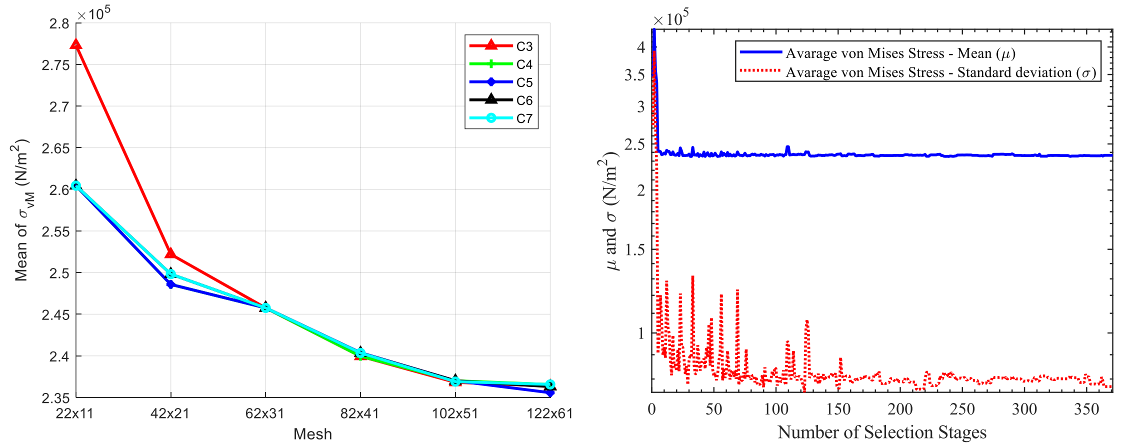


Figure 5. Optimal topologies for the cantilever beam by PDS.

Figure 6 (a) shows the evolution of the mean of the average von Mises stress of the selected population, in relation to the meshes and defined criteria. The reduction of the $\boldsymbol{\sigma}_{vm}^{population}$ mean shows that the mesh refinement is essential for better results, which was expected for problems of this type. Figure 6 (b) shows the results for case C7 and 122x61 mesh, in which the mean and standard deviation of the average von Mises stress of the population at each stage of selection are plotted. In this case, convergence was achieved with 372 selection stages.



(a) (b)

Figure 6. (a) the evolution of the mean of the average von Mises stress of the selected population, (b) the mean and standard deviation of the average von Mises stress for C7 criterion and mesh 122x61.

The formulation of the finite-volume theory employed in this example has its bases on the zeroth-order Cartesian formulation for bidimensional structures of the finite-volume theory presented by Cavalcante and Pindera [16, 17]. This optimal topology is like several results found in the literature for this type of problem – [3], [12], and [15] – indicating the potential of the PDS. The mesh dependence was reduced by applying the filtering technique, considering the results obtained by Véras and Cavalcante [1].

5 Conclusions

A new approach for Topology Optimization (TO) for two-dimensional continuum elastic structures based on the Progressive Directional Selection (PDS) method is presented in this investigation. The numerical example considered herein has shown that optimal topologies can be achieved by the proposed method. However, it is important to continue investigating the PDS method's application in other topological optimization problems and thoroughly comparing it with results from other optimal topologies found in the literature.

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