ANALYSIS OF A FINITE STAGGERED PERIODIC BEAM DYNAMICS UNDER DIFFERENT BOUNDARY CONDITIONS

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Abstract. Passive solutions for the vibrations control via metamaterials have gained relevance due to the possibility of developing a considerable vibration attenuation on lightweight and compact structures, with competitive performance in certain frequency bands. In this context, the modelling of these periodic systems has central importance in the design of such structures, allowing for sensitivity analyses, design refinement, and even, optimisation. Two families of modelling procedures can be highlighted: those that require the full system to be modelled, including the full structural domain, boundary conditions and external loads (if needed), and those that are based on a single unit-cell model under periodic boundary conditions, more compact than the latter but with some important limitations, e.g. the representation of global boundary conditions and external loads. Typical solutions for the modelling of dynamic systems make use of numerical methods, such as the Finite Element Method (FEM); however, this may become an unlikely alternative in systems that have a very large number of degrees of freedom due to the high computational costs involved. Regarding the modelling of periodic systems, the Wave Finite Element (WFE) approach has been presented as an interesting alternative due to the possibility of reducing the computation effort, as it is based on an FEM model of a single unit cell, with the force/displacement relationships between 2 adjacent unit cells determined through the transfer matrix. The WFE requires the resolution of an eigenvalue problem, which may present ill-conditioning depending on the properties of the system. In this study, the dynamics of staggered periodic beams are evaluated using the WFE approach to calculate the forced response in the frequency domain of such systems subjected to a bending load. The influence of different boundary conditions imposed on an increasing number of unit cells (the so-called metastructure) is evaluated against vibration attenuation performance. In addition, the results obtained via FE and WFE are compared and contrasted both, in terms of accuracy, and in terms of computational effort.

Keywords: metamaterials, attenuation band, computational efficiency

1 Introduction

Periodic structures consist of the arrangement of identical elements (unit cells) that repeats along a certain direction. These structures are observed in several systems, for example, photonic crystals, acoustic barriers, resonant structures, and the wave propagation in these mediums is of great interest due to the appearance of characteristic physical phenomena, such as internal resonances and band gaps, which indicate the possibility of promising engineering solutions in diverse applications [5, 10]. By exploring phenomena of electromagnetic waves in mechanical waves, space is opened for innovative and promising solutions, although not yet mature from the market point of view, *eg*, negative refraction [8], acoustic lenses, super- and hyper-lenses [3, 4, 12], acoustic camouflage, non-reciprocal acoustic devices [2], etc. The present work seeks to contribute to this study through the modeling of one-dimensional dynamic systems subject to different boundary conditions, proposing numerical-computational solutions that aim at computational efficiency accuracy and.

The importance of these studies is justified by the fact that periodic structures are common in engineering environments and therefore, optimised solutions for the study of physical phenomena involved in wave propagation by these systems should be explored [4]. One such alternative and potentially optimised solution is the Wave Finite Elements (WFE) approach [1], which combines the general theory for wave propagation in periodic systems, presented by Mead [6], with the FEM approach to determine the mass, stiffness and damping matrices for a unit cell.

Usual solutions via FEM [9] to determine the forced response in mechanical systems consist in discretization of the whole system through the distribution of nodes along the structure and, for high frequencies, a large number of nodes is required, which considerably increases the computational efforts. In solutions via the WFE approach, the nodes are distributed in a single unit cell of the system, so that the state vector that enables displacements/rotations and forces/moments in the following cell can be determined through the transfer matrix (T), which saves time and computational resources while providing accurate responses at high frequencies.

Finally, when dealing with periodic systems, Floquet-Bloch theory allows for a quite efficient modelling when an infinite domain is considered. The boundary conditions present in any real finite structure will pose movement restriction (hence wave reflections) that will interfere with performance. Therefore, this article aims to investigate the application of the WFE approach in finite periodic structures. The specific objectives are listed below:

- Determine the forced response on a finite staggered periodic beam composed by Euler-Bernoulli elements using the WFE approach for 2 distinct boundary conditions, benchmarking the results against FEM;
- Analyse the accuracy and efficiency of each method for an increasing number of unit cells;
- Evaluate the impact of boundary conditions on the system performance, while evaluating the accuracy in obtaining the frequency response provided by the WFE and FEM.

2 WFE approach

In this section, the general formulation of the WFE approach in modelling periodic structures will be presented, the main equations that describe the dynamic equilibrium in a unit cell will be described and, through the determination of the transfer matrix, the dynamic behaviour of the complete structures will be determined.

2.1 Cell dynamics

Consider a periodic system composed of identical unit cells arranged symmetrically in a given direction, as shown in Figure 1. These structures are described as waveguides and can be analysed using the WFE approach.

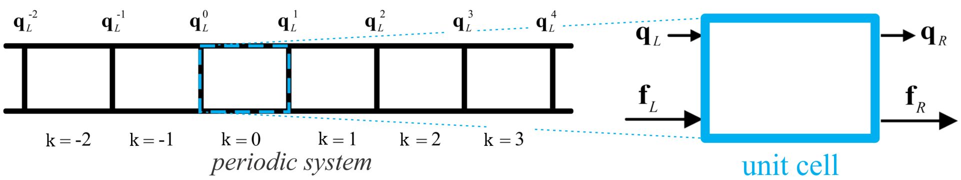


Figure 1. Representation of a generic periodic system and highlighted unit cell with the displacement/rotations vectors $\mathbf{q}$ and forces/moments $\mathbf{f}$ at the left $[\,]_L$ and right $[\,]_R$ sides.

The cell dynamics is calculated by $\mathbf{D}_e\mathbf{q} = \mathbf{f}$, where the cell dynamic stiffness matrix is $\mathbf{D}_e = \mathbf{K}_e + j\omega\mathbf{C}_e - \omega^2\mathbf{M}_e$, for a frequency ω, being K_e, C_e, M_e the stiffness, damping and mass matrix of the cell; the vector $\mathbf{q}$ represents the displacements/rotations and $\mathbf{f}$ the forces/moments. Considering the the left (L), right-hand (R) and the inner (I) DoF_s of the periodic element, the Equation 1 can be established:

$$\begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LI} & \mathbf{D}_{LR} \\ \mathbf{D}_{IL} & \mathbf{D}_{II} & \mathbf{D}_{IR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RI} & \mathbf{D}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_I \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{f}_L \\ \mathbf{f}_I \\ \mathbf{f}_R \end{bmatrix} \tag{1}$$

given the hypothesis that there is no application of forces/moments in the internal DoF_s of the substructure ($\mathbf{f}_I = 0$), it is possible to describe the Equation (1) only in terms of the right and left DoF_s:

$$\begin{bmatrix} \mathbf{D}^*_{LL} & \mathbf{D}^*_{LR} \\ \mathbf{D}^*_{RL} & \mathbf{D}^*_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{f}_L \\ \mathbf{f}_R \end{bmatrix} \tag{2}$$

so that the condensed dynamic stiffness matrix (D^*) is expressed by $\mathbf{D}^*_{BB} = \mathbf{D}_{BI}\mathbf{D}_{II}\mathbf{D}^{-1}_{IB}$ where B is the DoF corresponding to the L or R end of the element. Given the continuity condition in the structure, it is possible to determine the relation between the vectors $\mathbf{q}$ and $\mathbf{f}$ referring to the left and right ends of the substructure k and the following one $k+1$:

$$\mathbf{q}_L^{k+1} = \mathbf{q}_R$$
$$\mathbf{f}_L^{k+1} = -\mathbf{f}_R$$

thus, it is possible to describe the relation between the vectors $\mathbf{q}$ and $\mathbf{f}$ at two subsequent ends through the transfer matrix ($\mathbf{T}$):

$$\mathbf{T}\begin{bmatrix}\mathbf{q}_L^k \\ \mathbf{f}_L^k\end{bmatrix} = \begin{bmatrix}\mathbf{q}_R^k \\ -\mathbf{f}_R^k\end{bmatrix} = \begin{bmatrix}\mathbf{q}_L^{k+1} \\ \mathbf{f}_L^{k+1}\end{bmatrix} \tag{3}$$

which displays the dimensions of a matrix ($2nx2n$), where n is the number DoF_s at the L or R end:

$$\mathbf{T} = \begin{bmatrix} -\mathbf{D}_{LR}^{*-1}\mathbf{D}_{LL}^* & \mathbf{D}_{LR}^{*-1} \\ -\mathbf{D}_{RL}^* + \mathbf{D}_{RR}^*\mathbf{D}_{LR}^{*-1}\mathbf{D}_{LL}^* & -\mathbf{D}_{RR}^*\mathbf{D}_{LR}^{*-1} \end{bmatrix} \tag{4}$$

from the theory of periodic structures [6], the dynamic behavior is determined through the eigenproblem below:

$$\mathbf{T}\begin{bmatrix}\mathbf{q}_L \\ \mathbf{f}_L\end{bmatrix} = \mu \begin{bmatrix}\mathbf{q}_L \\ \mathbf{f}_L\end{bmatrix} \tag{5}$$

2.2 Dynamic stiffness matrix of the whole structure

Given a generic structure composed by N unit cells, that is, a finite periodic structure, the dynamic stiffness matrix for the system ($\mathbf{D}_T$) is defined by the relation between forces and displacements referring to the first and last interface:

$$\begin{bmatrix}\mathbf{f}_1 \\ \mathbf{f}_{N+1}\end{bmatrix} = \mathbf{D}_T \begin{bmatrix}\mathbf{q}_1 \\ \mathbf{q}_{N+1}\end{bmatrix} \tag{6}$$

being $\mathbf{f}_1$ and $\mathbf{f}_{N+1}$ the vectors of forces/moments referring, respectively, to the left end of the first cell and the right end of the Nth, while $\mathbf{q}_1$ and $\mathbf{q}_{N+1}$ refer to the displacements/rotations at the same ends ; $\mathbf{D}_T$ is dependent on the number of unit cells (N) and is calculated by the equation 7:

$$\mathbf{D}_T = \begin{bmatrix}\mathbf{D}_{LL}^* & 0 \\ 0 & \mathbf{D}_{RR}^*\end{bmatrix} + \begin{bmatrix}\mathbf{D}_{LR}^* & 0 \\ 0 & \mathbf{D}_{RL}^*\end{bmatrix}\begin{bmatrix}\mathbf{P}_l^{N-1} & \mathbf{P}_r \\ \mathbf{P}_l & \mathbf{P}_r^{N-1}\end{bmatrix}\begin{bmatrix}\mathbf{P}_l^N & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{P}_r^N\end{bmatrix}^{-1} \tag{7}$$

the matrices $\mathbf{P}_l$ and $\mathbf{P}_r$ are calculated by the diagonal matrix of eigenvalues $\mathbf{\Lambda}$ and the eigenvectors referents to the incident ($\mathbf{\Phi}_q^{inc}$) and reflected ($\mathbf{\Phi}_q^{ref}$) reflected wave modes:

$$\mathbf{P}_l = \mathbf{\Phi}_q^{ref}\mathbf{\Lambda}\mathbf{\Phi}_q^{ref^{-1}}$$
$$\mathbf{P}_r = \mathbf{\Phi}_q^{inc}\mathbf{\Lambda}\mathbf{\Phi}_q^{inc^{-1}}$$

The complete solution for the eigenproblem of the Equation 5 is presented in detail in [1], using the solution proposed in [13] for generalized eigenvalue problems, being done indirectly due to the ill conditioning in the calculation of the eigenvectors of the transfer matrix [11]; the physical meaning of the matrices $\mathbf{P}_l$ and $\mathbf{P}_r$, which are dependent on the solution of the self-problem, is also explored.

3 Numerical experiments

In a periodic structure, represented by the Figure 2 composed by N unit cells defined by Euler-Bernoulli staggered beams, the forced response in the frequency domain was calculated using the FEM and the WFE approach; therefore, the results were analyzed in view of the accuracy and computational efficiency for both methods.

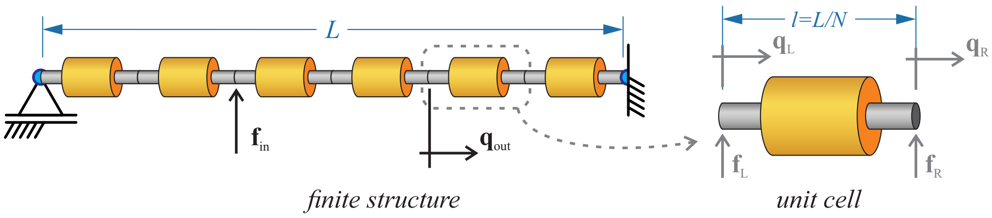
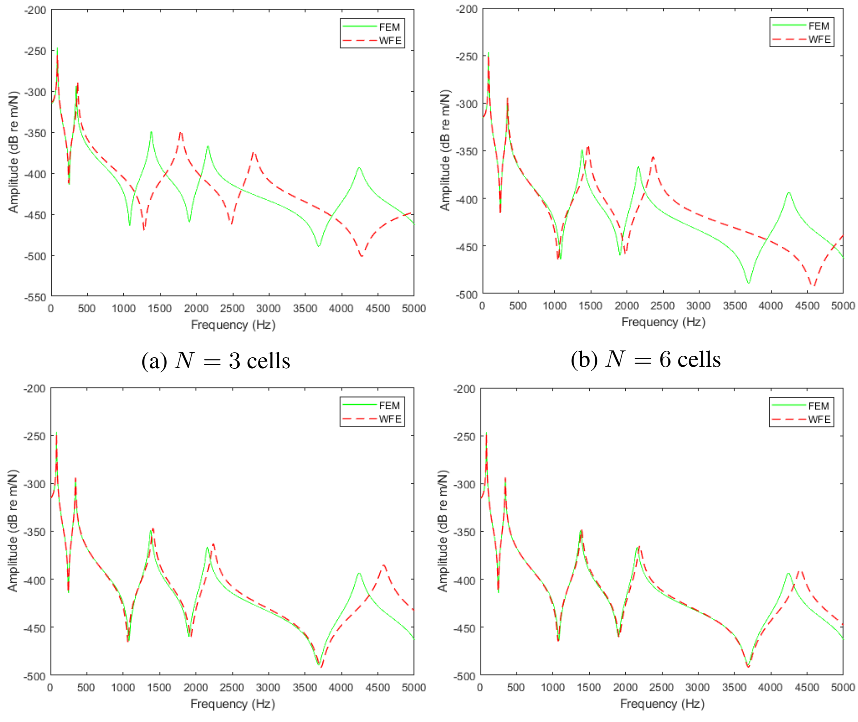


Figure 2. Finite structure with clamped-supported boundary condition and external load, and unit cell with internal DoFs.

3.1 Forced response of a N-cell structure

For the numerical simulation, a beam of length $L = 1m$ was considered, consisting of unit cells, discretized into 5 nodes, thus containing 4 finite elements of Euler-Bernoulli beams per cell, as represented in the Figure 2, where $d_1 = 0.05m$ and $d_2 = 0.1$. The material properties are: modulus of elasticity, $E = 210GPa$; density, $\rho = 7850kg/m^3$ and hysteretical damping, $\eta = 0.02$; so that $\mathbf{D} = (1 + j\eta)\mathbf{K} - \omega^2\mathbf{M}$. The forced responses were analyzed in the frequency range 5Hz - 5kHz, with discretization intervals of 5Hz.

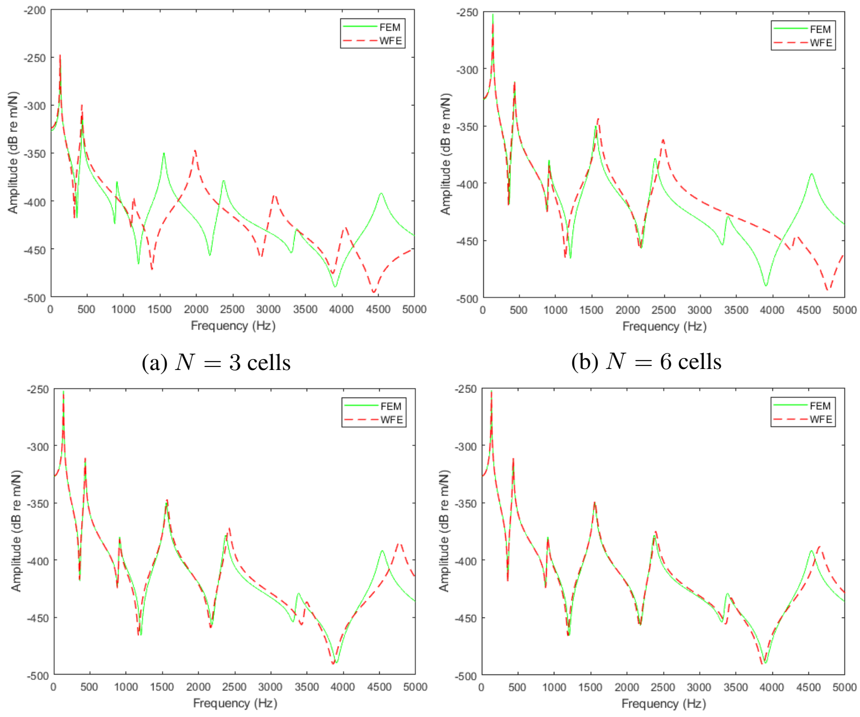
In the FEM solutions, a fixed number of unit cells was considered, $N = 30$; for the solutions using the WFE approach, this parameter was varied, assuming values $N = 3, 6, 9 and 12$, keeping the length $L = 1m$ fixed, so that it was possible to compare the dynamic behaviour of the structure as a function of the number of cells. Two boundary conditions were considered for the system: a symmetric simply supported case and a asymmetric clamped-supported case (Fig. 2. For the frequency response analyses, a transverse force is applied at $L/3$ while the response is evaluated at $2L/3$; since the number of unit cells is always a multiple of 3, it means force and response will always occur at a unit cell interface.



(a) $N = 3$ cells
(b) $N = 6$ cells
(c) $N = 9$ cells
(d) $N = 12$ cells

Figure 3. Receptance(m/N) at point $x = 2L/3$ for a simply supported beam at both ends with N cells (- - -) versus the receptance for a beam with 30 cells (——).

The results illustrated in the Figures 3 and 4 show that, compared to the structure of 30 unit cells whose forced response was determined by FEM, systems with fewer cells show similar behaviour at low frequencies, which indicates that, for this spectrum, reduced modal base models are more effective [7]; in the case of high frequencies, the responses diverge more quickly when dealing with systems with a smaller number of unit cells, whose response was obtained by the WFE approach. It can also be seen that the same pattern is observed for both the symmetric and the asymmetric systems, indicating that the WFE approach can be used to determine the



(a) $N = 3$ cells (b) $N = 6$ cells

(c) $N = 9$ cells (d) $N = 12$ cells

Figure 4. Receptance (m / N) at point $x = 2L/3$ for a beam embedded in the left and simply supported in the right end with N - - -) cells versus the receptance for a beam with 30 cells —–).

frequency response in periodic structures regardless of the presence of symmetry in the boundary conditions.

3.2 Computational efficiency of FEM and WFE approach in periodic structures

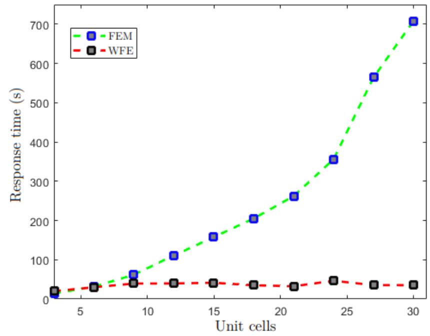
The difficulty in determining the forced response at high frequencies for periodic systems with a high number of cells via FEM is the fact that it is necessary to discretize the system into several elements, which requires a considerable numerical effort, consuming time and computational memory.Thus, the WFE approach is evaluated in this topic for its computational efficiency, for this, the time in determining the forced response via FEM and WFE was measured as a function of the number of unit cells in the periodic structure for the same conditions and frequency spectrum of the previous topic.

It is possible to conclude from the results presented in the Figure 5 that the time spent in determining the forced response for the two boundary conditions via FEM grows considerably in systems with a high number of cells, on the other hand, through the WFE approach, the time spent remains practically constant with the increase in the number of cells, which is due to the fact that the nodes are distributed in only one of the unit cells and not in the whole structure as in traditional applications via FEM.

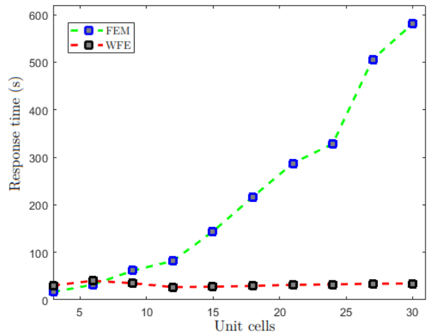
4 Conclusion

This paper presented a comparison of frequency responses obtained via two methods, namely, the WFE and FEM. A benchmark structure composed of 30 identical unit cells has been modelled via FEM under two distinct external boundary conditions: simply supported / clamped and simply supported on both ends. The effect of increasing the number of unit cells, while keeping total mass and length constant was evaluated, at the same time.

From the simulated results, it can be concluded that the WFE approach can present a much higher computational efficiency than the direct application of FEM in periodic structures consisting of a high number of unit cells, however in this study only simple one-dimensional structures were considered, it is necessary to assess the feasibility of application in more complex structures such as plates, trusses and membranes. In addition, it was found that for low frequencies, periodic systems consisting of 3 unit cells present a dynamic behaviour similar to structures with 30 cells, opening the possibility for applications of reduced modal base models as a strategy to reduce computational efforts.



(a) Simply supported at both ends.



(b) Embedded in the left and simply supported in the right.

Figure 5. Time (s) to compute the receptance (m / N) using the FEM and the WFE approach as a function of the number of unit cells in the two boundary conditions.

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