A multiscale recursive numerical method for semilinear parabolic problems

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Abstract. We present a multiscale recursive numerical method in the context of time-dependent initial-boundary value problems for semilinear parabolic equations with discontinuous and high-contrast coefficients. We consider a backward Euler scheme for the temporal discretization along with an extension of the Recursive Mixed Multiscale Method based on domain decomposition technique, recently introduced in the literature by Ferraz [1], for the spatial discretization of the semilinear parabolic operator. Thus, at each time step, the spatial and temporal discretizations lead to large linearized systems of equations that involve solving local multiscale boundary value problems followed by the solution of a family of decomposed interface problems that showed excellent scalability. We will also briefly discuss some ideas of the proposed recursive multiscale approach for non-linear parabolic problems, by considering efficient approximation strategies along with the reuse of the multiscale basis functions and parallelization. Numerical examples with both homogeneous medium and heterogeneous high-contrast coefficients for semilinear problems are considered to show the behavior of the multiscale approach and our findings.

Keywords: Multiscale recursive approach, Semilinear parabolic equations, Multiscale approximations.

1 Introduction

In this paper, we design and implement a new parallel, multiscale hybrid-mixed method with domain decomposition for solving time-dependent initial-boundary value problems for semilinear parabolic equations with highly heterogeneous coefficients. Our approach is based on the multiscale Recursive Multiscale Mixed Method (RMuMM) for elliptic problems recently introduced in Ferraz [1] and retains all the good conservative properties and scalability parallelization issues discussed in Abreu & Ferraz et. al [2], for which all basis functions that can be computed independently. The RMuMM is a mixed Recursive multiscale method in constrat with the very interesting iterative MuMM by Francisco et. al. [3]. In the RMuMM, we decompose the global interface problem into a family of small interface problems that showed to adapt quite well into the multi-core platform presenting excellent scalability for elliptic problems (see Abreu et. al [2] and Ferraz [1] for more details).

In general, multiscale methods obtain coarsened models that incorporate the fine-grid details of the underlying continuum problem. The design and applicability of multiscale methods to a variety of static and dynamics problems, as well as the scalability and numerical analysis of such methods, is still an area of active research from fluid dynamics to multiphysics problems, see for example, Ferraz [1], Guiraldello et. al. [4], Duran et. al. [5], Araya et. al. [6] and the references cited therein.

The objective of the present work is to develop a specific multiscale method to deal with time-dependent initial-boundary value problems for semilinear parabolic equations subject to discontinuous and high-contrast coefficient, which is a challenging problem (see, e.g., Malqvist et. al [7] and references cited therein for a motivation on the development of multiscale techniques for parabolic problems of particular interest in the context of this work).

In this paper, we discretize the three-dimensional time-dependent semilinear parabolic model by using Raviart-

Thomas mixed-hybrid finite-element in the spatial variables, which has as an essential advantage of local conservation. We consider a backward Euler scheme for the temporal discretization along with an extension of the Recursive Mixed Multiscale Method based on domain decomposition technique. As a result of this procedure, at each time step, the spatial and temporal discretizations lead to large linearized systems of equations that involve solving local boundary value problems in a hierarchy of interfaces resulting from domain decomposition strategy and the followed by the solution of a family of decomposed interface problems. An efficient implementation of mixed multiscale methods can be achieved by writing the final solution in terms of a set of mixed multiscale basis functions (hereafter referred as MMBF's). The MMBF's are obtained from the resolution of linearized systems of equations that involve solving local multiscale boundary value problems as local interface problems, which are obtained by means of a conjugate gradient with an algebraic multigrid preconditioner, see, e.g., Liebmann [8]. The continuity equations at the in interfaces are defined in the coarse scale, such that the flux conservation is only satisfied on the $\bar{H}$ scale. For this reason post processing techniques needs to be used. Considering multiphase flow problems that multiscale methods frequently are applied, we used a simple post processing technique to recover flux conservation on fine scale (Francisco et. al. [3], Guiraldello et. al. [4] and Guiraldello et. al. [9]).

In the current stage, it is worth mentioning that our recursive multiscale mixed-hybrid approach is developed keeping in mind the behavior of the solutions with respect to accuracy and scalability as well as the robustness to reuse of basis functions for time-dependent problems and aspects of efficiency in the parallelization in multi-core platforms. A simple parallelizable predictor-corrector strategy for linearization the source term at each time step is presented. An extension of this overall multiscale recursive approach seems applicable to non-linear parabolic equations and it is now in progress.

The paper is organized as follows. In Section 2, we introduce the time-dependent semilinear parabolic problem with discontinuous coefficients and some basic notation to the spatial-temporal discretizations of the differential operators. In Section 3, we present and discuss the parallel multiscale numerical simulations to show the viability of the formulation. In Section 4, we highlight our concluding remarks keeping in mind the reuse of basis functions for time-dependent non-linear parabolic problems.

2 Model and Discretization

Let Ω be a regular domain in $\mathbb{R}^d, d = \{2, 3\}$ with Lipschitz boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$ and $T > 0$. Consider the following semilinear parabolic equation,

$$\begin{aligned} &C\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = f(p), && \mathbf{u} = -A\nabla p, && \text{in } \Omega \times (0,T], \\ &\mathbf{u}\cdot\mathbf{n} = g_N, \text{ in } \Gamma_N \times (0,T], && p = g_D, \text{ in } \Gamma_D \times (0,T], && p(\mathbf{x},0) = g_0, \text{ in } \Omega, \end{aligned} \tag{1}$$

where $p = p(\mathbf{x}, t)$, $C = C(\mathbf{x})$, $A = A(\mathbf{x})$, e $\mathbf{n}$ is the outward unitary vector over $\partial\Omega$.

Temporal Dicretization. We consider a backward Euler scheme for the temporal discretization

$$\frac{C}{\Delta t}(p^{n+1} - p^n) + \nabla \cdot \mathbf{u}^{n+1} = f^{n+1}, \qquad \mathbf{u}^{n+1} = -A\nabla p^{n+1}, \qquad \text{in } \Omega \times (t^n, t^{n+1}], \tag{2}$$

where $f^{n+1} = f(p^{n+1})$ and the superscript n refers to the function evaluated at time step t^n. Since the PDE coefficients C and A are only a spatial function, they do not change at each time step.

Spatial Discretization. The Recursive Multiscale Mixed Method (RMuMM) introduced in Ferraz [1] is a mixed multiscale method based on a non-overlapping domain decomposition over three length scales where subdomains are coupled through weak continuity of Robin conditions, $\lambda = \beta\mathbf{u}\cdot\mathbf{n} + p$, across the interfaces of subdomains (Abreu et. al.[2] Francisco et. al [3] and Guiraldello et. al. [4]), where β is the positive function known as the Robin parameter. The three different length scales are: h, the mesh size of an underlying fine grid, where the solution is sought; H, the mesh size for the subdomain partition, where multiscale basis functions are defined and $\bar{H}$, an intermediate length scale on the subdomain interfaces where flux conservation is directly imposed through the Robin condition, such that $h \leq \bar{H} \leq H$ (Francisco et. al. [3] and Abreu et. al. [2]).

We start by decomposing the domain Ω into m non-overlapping subdomains Ω^i , $i = 1, \ldots, m$, with reference size H, each with a well-defined Lipschitz boundary $\partial\Omega^i$, $\Omega = \bigcup_{i=1}^{m} \Omega^i$, $\quad \Omega^k \cap \Omega^i = \emptyset, \quad i \neq k, \quad \Gamma^i = \Gamma \cap \partial\Omega^i, \quad \Gamma^{ik} = \Gamma^{ki} = \partial\Omega^i \cap \partial\Omega^k$. The skeleton of the decomposition is $\Gamma = \cup_i \partial\Omega^i \setminus \partial\Omega$, and $\Gamma^{ik} = \Gamma^{ki}$ is the interface between Ω^i and Ω^k. Additionally, let us define two types of normal vectors. One denoted by $\check{\mathbf{n}}^i$ is simply the normal vector pointing outward of subdomain Ω^i. The second, denoted as $\check{\mathbf{n}}$ with no superscript, identify the direction of fluxes over each interface of Γ.

The domain decomposition formulation of the RMuMM is performed directly in the discrete form of the system in eq.(1). Consider a regular mesh discretizing Ω^i, with reference size $h << H$, where we define the lowest order Raviart-Thomas spaces for velocity and pressure [10], say $\mathbf{U}_h^i \subset H(\mathrm{div}, \Omega^i)$ and $V_h^i \subset L^2(\Omega^i)$, respectively. Consider also the vector space $\mathbf{U}_{h,g_n}^i \subset \mathbf{U}_h^i$ of the functions in $\mathbf{U}_h^i$ satisfying the Neumann boundary conditions in eq.(1). The variational formulation of the RMuMM introduces the unknown λ defined only on Γ. The interface space for λ is defined as the piecewise constant functions $F_{\bar{H}}$ on the $\bar{H}$ scale, see Figure 1.

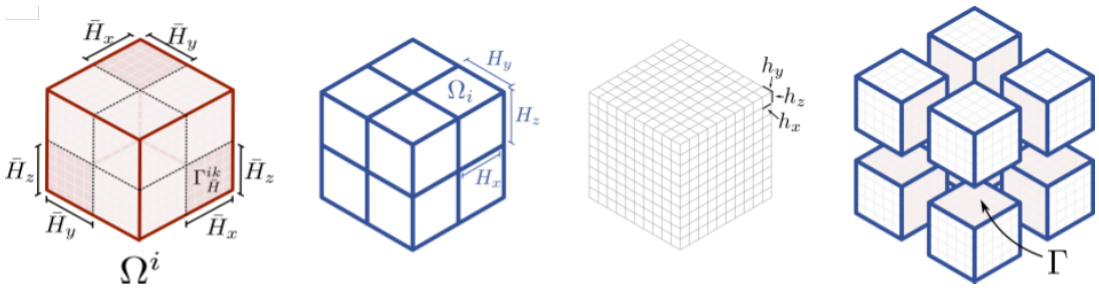

Figure 1. Representations of a three-dimensional domain decomposition of Ω. On the leftmost image, we show the $\bar{H}$ scale. The second and the third pictures show the coarse scale H and the fine scale h, respectively. Rightmost picture depicts Γ, the skeleton of the decomposition and is composed by subdomain interfaces.

The discrete variational formulation of the RMuMM for the semilinear parabolic problem (2) is to find, for each Ω^i, the triple $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \lambda_{\bar{H}}^{n+1}) \in \mathbf{U}_{h,g_n}^i \times V_h^i \times F_{\bar{H}}$, for $i = 1, \ldots, M$, such that

$$
\begin{aligned}
(A^{-1}\mathbf{u}_h^{n+1}, \mathbf{v})_{\Omega_i} - (p_h^{n+1}, \nabla\cdot\mathbf{v})_{\Omega_i} + (\beta_{\bar{H}}\,\mathbf{u}_h^{n+1}\cdot\check{\mathbf{n}}^i, \mathbf{v}\cdot\mathbf{n}^i)_{\partial\Omega_i\cap\Gamma} \quad &+ \\
(\lambda_{\bar{H}}^{n+1}, \mathbf{v}\cdot\check{\mathbf{n}}^i)_{\partial\Omega_i\cap\Gamma} \quad &= \quad -(g_D, \mathbf{v}\cdot\check{\mathbf{n}}^i)_{\partial\Omega_i\cap\partial\Omega_D}\,, \qquad (3)\\
\left(\frac{C}{\Delta t}p_h^{n+1}, q\right)_{\Omega_i} + (\nabla\cdot\mathbf{u}_h^{n+1}, q)_{\Omega_i} \quad &= \quad \left(f^{n+1}, q\right)_{\Omega_i} + \left(\frac{C}{\Delta t}p_h^{n}, q\right)_{\Omega_i}\,, \qquad (4)\\
\sum_{i=1}^{M}\left(\mathbf{u}_h^{n+1}\cdot\check{\mathbf{n}}^i, M_{\bar{H}}\right)_{\Gamma_i} \quad &= \quad 0\,, \qquad (5)\\
\sum_{i=1}^{M}\left(\beta_{\bar{H}}\,\mathbf{u}_h^{n+1}\cdot\check{\mathbf{n}}^i + \lambda_{\bar{H}}^{n+1}, M_{\bar{H}}\check{\mathbf{n}}^i\cdot\check{\mathbf{n}}\right)_{\Gamma_i} \quad &= \quad 0\,, \qquad (6)
\end{aligned}
$$

for all $(\mathbf{v}, q) \in \mathbf{V}_{h0}^i \times Q_h^i$, for all $M_{\bar{H}} \in F_{\bar{H}}$, and $\beta_{\bar{H}} > 0$. Here we drop the superscript i from the variables for simplicity of notation. For herein the variables are always associated with a subdomain. Through the Robin condition $\lambda_{\bar{H}}^{n+1}$, the parameter β is also attached to the $\bar{H}$-scale and shall be denoted herein by $\beta_{\bar{H}}$.

The solution $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ of eqs.(3)-(6) is written as a linear combination of the local solutions, called multiscale basis functions and a particular solution given by,

$$
\mathbf{u}_h^{n+1} = \sum_{j=1}^{n^i} X_j\boldsymbol{\Phi}_{k_j}^{n+1} + \bar{\mathbf{u}}_h^{n+1}, \qquad p_h^{n+1} = \sum_{j=1}^{n^i} X_j\Psi_{k_j}^{n+1} + \bar{p}_h^{n+1}. \qquad (7)
$$

An efficient implementation of mixed multiscale methods can be achieved by writing the final solution in terms of a set of mixed multiscale basis functions (Abreu et. al. [2], Ganis and Yotov [11], Francisco et. al. [3] and Guiraldello et. al. [9]). The MMBF's are a set of local problems similar to eqs.(3)-(4), denoted here as $\{\boldsymbol{\Phi}_{k_j}^{n+1}, \Psi_{k_j}^{n+1}\}_{1\le k_j\le n^i}$, constructed by properly setting $\lambda_{\bar{H}}$. It does not present any source or external boundary data. Consider $\{\phi^j\}_{1\le j\le n}$ a finite element basis for the coarse interface space $F_{\bar{H}}$, where $n = \dim(F_{\bar{H}})$. Then, $\lambda_{\bar{H}}$ can be written as $\lambda_{\bar{H}} = \sum_{j=1}^{n} X_j\phi^j$, where the coefficients X_j are to be determined later. Notice that $F_{\bar{H}}$ is a constant by parts space. Define n as the *global* number of the interface degrees of freedom and n^i as the number of interface degrees of freedom associated with Ω^i whose support is on the boundary Γ^i. For every $j \in \{1, \ldots, n^i\}$, the MMBFs in Ω^i, are given by the following set of local problems: Find $(\boldsymbol{\Phi}_{k_j}^{n+1}, \Psi_{k_j}^{n+1}) \in \mathbf{U}_{h,0}^i \times V_h^i$, such that

$$
(K^{-1}\boldsymbol{\Phi}_{k_j}^{n+1}, \mathbf{v})_{\Omega^i} - (\Psi_{k_j}^{n+1}, \nabla\cdot\mathbf{v})_{\Omega^i} + (\beta_{\bar{H}}\,\boldsymbol{\Phi}_{k_j}^{n+1}\cdot\check{\mathbf{n}}^i, \mathbf{v}\cdot\check{\mathbf{n}}^i)_{\Gamma^i} = (\phi^j\check{\mathbf{n}}^i\cdot\check{\mathbf{n}}, \mathbf{v}\cdot\check{\mathbf{n}}^i)_{\Gamma^i}, \qquad (8)
$$

$$
\left(\frac{C}{\Delta t}\Psi_{k_j}^{n+1}, q\right)_{\Omega_i} + (\nabla\cdot\boldsymbol{\Phi}_{k_j}^{n+1}, q)_{\Omega^i} = 0, \qquad (9)
$$

hold for all $(\mathbf{v}, q) \in \mathbf{U}^i_{h,0} \times V^i_h$. In the variational formulations above the function ϕ^j depend on the interface space considered. Now the particular solution $(\bar{\mathbf{u}}^{n+1}_h, \bar{p}^{n+1}_h) \in \mathbf{U}^i_{h,g_N} \times V^i_h$ is obtained by a similar problem as the MMBF's, but no Robin boundary conditions ($\lambda_{\bar{H}} = 0$) and the sources terms and boundary conditions of the original problem are considered: Find $(\bar{\mathbf{u}}^i_h, \bar{p}^i_h) \in \mathbf{U}^i_{h,0} \times V^i_h$, such that

$$(K^{-1}\bar{\mathbf{u}}^{n+1}_h, \mathbf{v})_{\Omega_i} - (\bar{p}^{n+1}_h, \nabla\cdot\mathbf{v})_{\Omega_i} + (\beta_{\bar{H}}\,\bar{\mathbf{u}}^{n+1}_h\cdot\check{\mathbf{n}}^i, \mathbf{v}\cdot\mathbf{n}^i)_{\partial\Omega_i\cap\Gamma} = -(g_D, \mathbf{v}\cdot\check{\mathbf{n}}^i)_{\partial\Omega_i\cap\partial\Omega_D}\,, \tag{10}$$

$$\left(\frac{C}{\Delta t}\bar{p}^{n+1}_h, q\right)_{\Omega_i} + (\nabla\cdot\bar{\mathbf{u}}^{n+1}_h, q)_{\Omega_i} = \left(f^{n+1}, q\right)_{\Omega_i} + \left(\frac{C}{\Delta t}\bar{p}^{n}_h, q\right)_{\Omega_i}\,, \tag{11}$$

hold for all $(\mathbf{v}, q) \in \mathbf{U}^i_{h,0} \times V^i_h$. The local problems (8)-(11) can be solved by any discretization that delivers both pressure and normal fluxes at the skeleton Γ of the decomposition. In this work we use hybrid mixed finite elements (Thomas and Raviart [10]).

To construct the X's coefficients of the linear system we make use of the MMBF's (8)-(11) solutions. The procedure consists of substituting the solution of eq.(7) in the coarse scale continuity conditions (5)-(6). The next step is to substitute the unknowns by the finite element basis linear combinations and test M_H appearing in eqs.(5)-(6) for all basis functions spanning $F_{\bar{H}}$. Finally, we obtain a linear global system for all the interfaces unknowns.

The RMuMM (Ferraz [1] and Abreu et. al. [2]) reformulates the global interface linear system as a family of smaller problems that fit well into multi-core parallel machines and can be solved recursively, showing excellent scalability. This is obtained by using a two-subdomain RMuMM decomposition of Ω successively on each subdomain, decomposing it into two smaller adjacent subdomains (as a new two-subdomain decomposition) until a last stage. In this last stage, the linear systems are solved by LU decomposition for its small size. The details of the interface solver is not in the scope of this paper, since the temporal derivative addition does not alter the interface system solution structure, only the MMBF's. We refer the reader to Abreu et al. [2] for a complete description of the recursive interface formulation. Also, more details about the variational formulation, as well as the well-posedness of the discrete system, can be seen in Abreu et. al. [2] and Guiraldello et. al. [9]. The recursive formulation was implemented in C, C++ and openMPI. The MMBF's are obtained by means of a conjugate gradient with an algebraic multigrid preconditioner C++ solver by Liebmann [8], with a tolerance of 10^{-8}. In Ferraz [1] and Abreu et. al. [2], the authors showed great scalability for the RMuMM implementation for the elliptic version of problem (1). For the semilinear parabolic equation, we expect the scalability to remain excellent since the MMBFs computation does not change, only the particular local solution is affected by the changing source term. Our implementation consider this by computing the homogeneous MMBFs only once in the beginning of the implementation.

Post Processing. The continuity equations in Γ are defined in the coarse scale, such that the flux conservation is only satisfied on the $\bar{H}$ scale. For this reason post processing techniques needs to be used. Considering multiphase flow problems that multiscale methods frequently are applied, we used a simple post processing technique to recover flux conservation on fine scale (Francisco et. al. [3], Guiraldello et. al. [4] and Guiraldello et. al. [9]).

In each time step we solve a local problem in the subdomains (for the cost of *one* MMBF) with average Neumann boundary condition at subdomain interfaces following Guiraldello et. al. [4].

2.1 Predictor–Corrector Strategy for the Non-linear Source Term

Since the source term in eq.(1) is non-linear, we apply a predictor-corrector strategy to linearize the source term at each time step. We begin with an explicit Euler scheme to find an approximation p^*,

$$\frac{C}{\Delta t_p}(p^* - p^n) + \nabla\cdot\mathbf{u}^n = f^n, \qquad \mathbf{u}^n = -A\nabla p^n, \qquad \text{in } \Omega\times[t^n, t^{n+1}]. \tag{12}$$

Then we approximate $f^{n+1} \approx f(p^*)$, where the time step $\Delta t_p = \alpha\Delta t$, $0 < \alpha \leq 1$. For the spatial discretization, we make use of the Raviart-Thomas discretized subdomain equations of the implicit version to construct an explicit approximation for p^*,

$$p^* = -\frac{\Delta t_p}{h\,C}\left(\sum_k \mathbf{u}\cdot\mathbf{n}\Big|_{\Gamma^{ik}_h} - f^n - \frac{C}{\Delta t_p}p^n\right), \tag{13}$$

on each element; here Γ^{ik}_h is the interface between element i and its adjacent k. Thus, at each time step, the spatial and temporal discretizations lead to a linearized systems of equations for the solution of the local MMBFs, and interface problem in the coarse-grid. This approximation is robust since the post-processing step guarantee that the

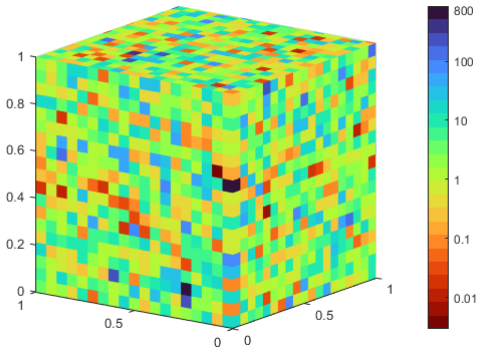

Figure 2. Heterogeneous log-normal A coefficients of eq.(1).

normal fluxes are continuous over the subdomains interfaces. It is also compatible with our parallelism strategy. Notice that this predictor-corrector can be easily applied to a fully non-linear problem, where the coefficients C and A of eq.(1) are functions of p. In this case the formulation is exactly the same, approximating $C^{n+1} \approx C(p^*)$ and $A^{n+1} \approx A(p^*)$. However, differently from the semilinear case, the RMuMM scalability can be affected. Since the non-linear PDE coefficient changes, so does the MMBFs requiring its re-computation at each time step. This leads to choosing strategies that are computational efficient by a reusing the MMBFs already computed such as fixing the MMBFs for a number of time steps and then recomputing. This strategies will be explored in further works.

3 Multiscale Numerical Simulations

In this section we present two preliminary tree-dimensional numerical examples to qualitative test our method. In both cases the domain is set to the unit cube $\Omega = [0,1] \times [0,1] \times [0,1]$ and $T = 0.1s$. The domain Ω is divided into 16 subdomains, each with a regular mesh of $20 \times 20 \times 20$ elements. The temporal interval is $\Delta t = 0.01s$.

We consider one problem with two coefficients A of eq.(1), one homogeneous and other heterogeneous coefficients generated by a normal probability distribution function, see Fig. 2. The later was generated by a log-normal model proposed by Glimm and Sharp [12] with a contrast of the maximum and minimum values of 10^5 on a mesh of $20 \times 20 \times 20$ elements projected on the whole subdomain.

In these domains we solve the problem (1) with right hand side $f(p) = p^3 - p$ and initial data $p(\mathbf{x}, 0) = x(1-x)y(1-y)z(1-z)$. The boundary conditions are give by $p = 0$ on the boundary. The multiscale simulations were made for three $\bar{H}$-scales of $\bar{H} = h$, $\bar{H} = H/2$ and $\bar{H} = H$ where we plotted the p variable and the $\mathbf{u}$ streamlines. The temporal evolution for homogeneous domain is seen on Fig. 3 top row. We only show the $\bar{H} = H$ since the smooth solution with the homogeneous coefficients is not affected by the interface space coarseness and reproduces perfectly the fine scale solution.

Figure 3 shows the temporal evolution of three $\bar{H}$-scales in the heterogeneous domain, from fine (second row) to coarse (bottom row).

We present a transient solution p slice with added gradient $\mathbf{u}$ streamlines. The homogeneous case forms a circular region around the center of the domain with symmetrical and uniform streamlines. Our method captured the expected behavior in time of parabolic damping like in a heat conduction problem, which reflects a typical transient to stationary dynamic as time evolves.

The heterogeneous case form perturbations of this solution following the preferential paths according to the coefficients variability, as expected. From the second row forward we see, respectively, the fine scale ($\bar{H} = h$) solution, an intermediate scale ($\bar{H} = H/2$) solution and, the coarse scale ($\bar{H} = H$) solution for three different times. The $\bar{H}$-scale choice can interfere in the solution, i.e., the coarser the scale more information on the subdomain interfaces will be lost, still the coarse scale simulations captured main features and information of the solution.

On the other hand, comparing the computational times presented in Fig. 3 shows a great advantage in using coarse scales. For this method, the MMBFs computational time is predominant so the fewer local problems we have to compute, faster the method will be (Abreu et. al. [2]). This way a balance between computational efficiency and accuracy is to be studied.

The preliminary results here showed that the multiscale strategy is well suited for the solution of the semilinear parabolic problem. Continuously work on qualitative experiments and applications in realistic porous media flow problems will be explored in further works, as well as the extension to fully non-linear parabolic problems.

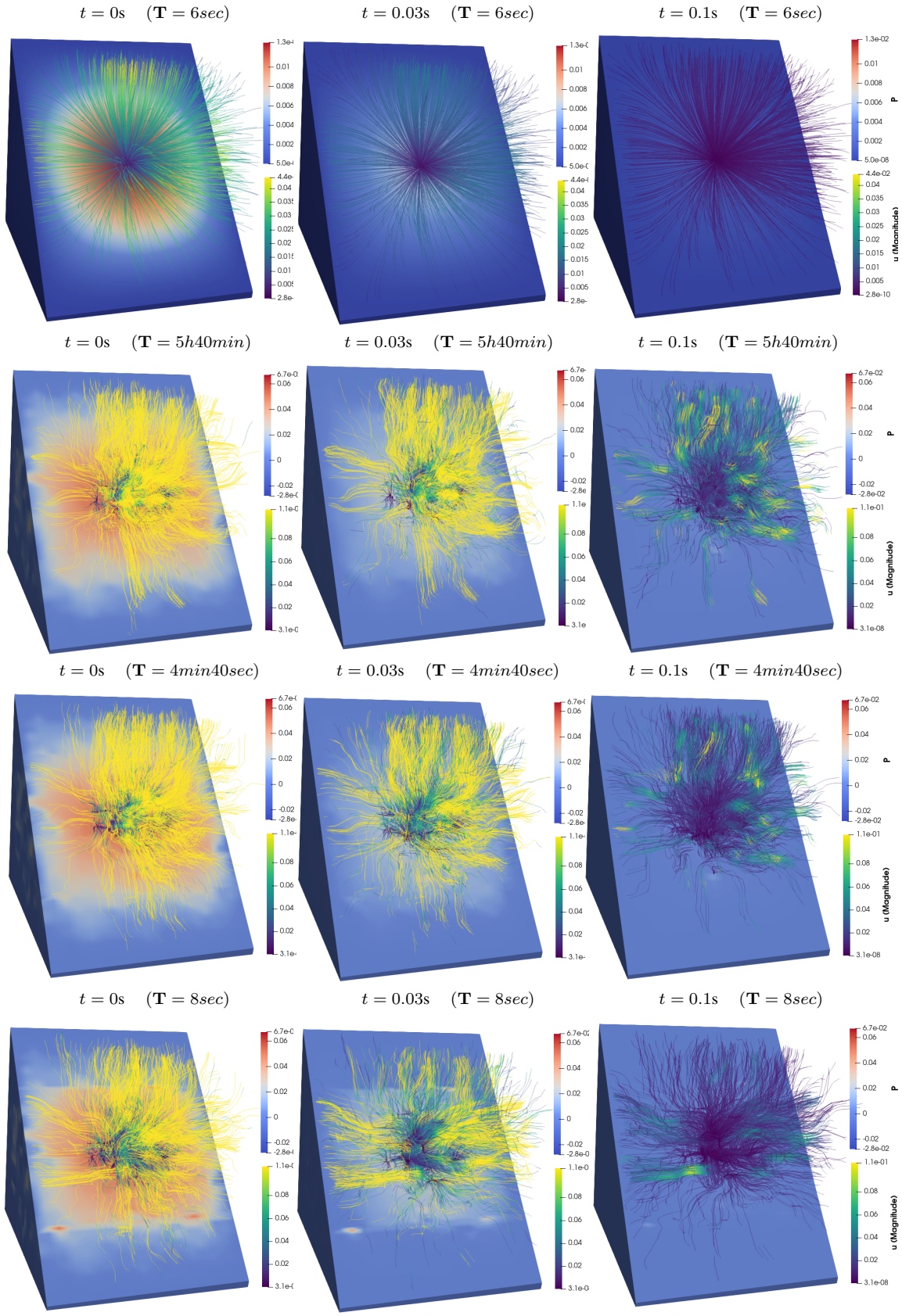

Figure 3. Solution p, $\mathbf{u}$ streamlines, and computational times $\mathbf{T}$ of eq.(1) for distinct $\bar{H}$-scales. From top to bottom: homogeneous case, heterogeneous $\bar{H} = h$ (fine) scale, heterogeneous $\bar{H} = H/10$ (intermediate) scale, and heterogeneous $\bar{H} = H$ (coarse) scale.

4 Conclusions

We introduced a parallel multiscale recursive numerical method for time-dependent initial-boundary value problems involving semilinear parabolic equations with highly heterogeneous coefficients. We also highlight some ideas of the proposed multiscale approach for non-linear parabolic problems. We presented multiscale simulations to show qualitative the viability of the parallel multiscale formulation, where the method captured the parabolic damping behavior that reflects a typical transient to stationary dynamic as time evolves. For a highly heterogeneous domain, our solution in a coarse scale was comparable to the fine scale solution, but it was calculated with much less computational time.

Further work based on the proposed multiscale method for parabolic non-linear problems with the reuse of MMBFs and scalability of the scheme will be explored. In particular, we will a perform a thoroughly study on the efficiency of parallel multiscale algorithm (e.g., strong and weak scalability issues) for time-dependent initial-boundary value problems for non-linear and semilinear parabolic equations with discontinuous and high-contrast coefficients in the same lines as presented in Abreu, Ferraz et. al. [2] motivated, for instance, for flow and transport in subsurface rocks, develpment of robust numerical techniques for upscaling multiphase flow in porous media (e.g., upscaling of pressure and saturation equations) and for a range of problems from fluid dynamics to multiphysics problems.

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