# Nonlinear vibrations of a FG cylindrical shell on a circumferential discontinuous elastic foundation

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**Abstract.** The nonlinear vibrations of a simply supported cylindrical shell made by a functionally graded material with a circumferentially discontinuous elastic base is analyzed. The equilibrium equations are obtained from Donnell's nonlinear shallow shell theory. The modal solution to the transversal displacement field, used to discretize the equilibrium equations, is obtained by perturbation techniques. The discretized equations are analyzed, considering a harmonic excitation in the form of the combination of the lowest vibration modes of cylindrical shell. The chosen geometry of the cylindrical shell presents natural frequencies nearly commensurate to an internal resonance 1:1:1:1, due to the discontinuity of the elastic base in the circumferential direction. The nonlinear dynamic behavior is analyzed from the resonance curves that they are obtained by the continuation method and the basins of attraction. Several resonance peak regions are observed, due to the interaction between the modes of the transversal displacement field, showing the competition of multiple stable, quasi -periodic and chaotic solutions. Time responses, phase portraits and Poincaré sections are also used to understand the nonlinear dynamic behavior of the cylindrical shell.

**Keywords:** discontinuous elastic base, cylindrical shell, nonlinear dynamics, reduced order model.

## 1 Introduction

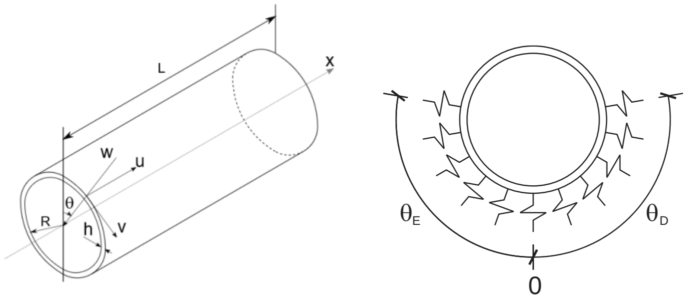
Structural mechanics of cylindrical shells is an important topic in engineering field with several studies considering the interaction between an elastic foundation and the cylindrical shell [1, 2]. However, a few works focus on cylindrical shells with circumferentially discontinuous elastic base. Important works in this field has been done by Amabili and Dalpiaz [3] and Tj et al. [4, 5] which investigate the linear vibrations. On the other hand, Nejad and Bideleh [6], Rodrigues [7] and Silva et al. [8] present an analysis of the nonlinear vibrations considering the discontinuity of the elastic base either on circumferential or longitudinal direction of cylindrical shell. It is found in the literature that, depending on the type of discontinuity of elastic base, the forced nonlinear vibrations in the system are higher than the nonlinear vibrations of a cylindrical shell with a continuous elastic base [7, 8]. So, based on previous work [9], this work deduces a reduced order models through a perturbation method to evaluate the global stability of a cylindrical shell resting on a circumferentially discontinuous elastic base. The obtained results show a complex behavior of forced response of shell with several bifurcations points and a strong competition between several dynamical attractors.

## 2 Problem formulation

The derived mathematical model considers a simply supported cylindrical shell on a discontinuous elastic base delimited by $\theta_E$ and $\theta_D$. The geometry of structure is a perfect cylindrical shell with constant radius $R$, thickness $h$, where $h << R$, and length $L$. Figure 1(a) shows the cylindrical shell's geometry and its coordinates axes $x$, $\theta$ and $z$ which are related to axial ($u$), circumferential ($v$) and transversal ($w$) displacement fields, respectively. Figure 1(b) illustrates the origin of cylindrical coordinate axis and the open angles $\theta_E$ and $\theta_D$ of elastic base. The shell is composed of a functionally graded material based on two materials *A* and *C*, which varies in the direction of thickness *h* and obeys a sandwich distribution where the physical parameters *E*, $\rho$ and $\nu$ are given by:

$$P = \left(P_A - P_C\right)V_A\left(z\right) + P_C, \text{ with } V_A\left(z\right) = \left(1 - \frac{4z^2}{h^2}\right)^{2N+1}, \tag{1}$$

where $P_A$ and $P_C$ are the properties of materials *A* and *C*, respectively, $V_A(z)$ is the sandwich function and *N* is the exponent of the graded function.




(a) (b)

Figure 1. Shell characteristics (a) geometry and (b) circumferential discontinuity of elastic base.

Donnell's nonlinear shallow shell theory is considered to obtain the nonlinear kinematic equations, as shown in eq. (2), which are described in terms of internal membrane forces and bending moments as:

$$\begin{aligned}
&-N_{x,x} - \frac{1}{R}N_{x\theta,\theta} = 0, \; -\frac{1}{R}N_{\theta,\theta} - N_{x\theta,\mathrm{x}} = 0, \\
&\rho_1\ddot{w} - M_{x,\mathrm{xx}} - \frac{2}{R}M_{x\theta,\mathrm{x}\theta} - \frac{1}{R^2}M_{\theta\theta,\theta\theta} + \frac{N_\theta}{R} + \left(N_x\beta_{x0} + N_{x\theta}\beta_{\theta 0}\right)_{,x} \\
&+\frac{1}{R}\left(N_{x\theta}\beta_{x0} + N_\theta\beta_{\theta 0}\right)_{,\theta} - P_B - P + 2\eta_1\rho_1\omega_0\dot{w} = 0.
\end{aligned} \tag{2}$$

where $\eta_1$ is the viscous damping, $\omega_0$ is the natural frequency of cylindrical shell and $\rho_1$ is the average density of the material distributed in the thickness of the shell.

In eq. (2) *P* is the lateral harmonic force and $P_B$ is reaction of elastic base, assumed as a Winkler model, which are given by:

$$P\left(t\right) = P_L\left(\sum_i P_i^C\cos\left(i\theta\right) + P_i^S\sin\left(i\theta\right)\right)\sin\left(\frac{m\pi x}{L}\right)\cos\left(\omega_1 t\right), \tag{3}$$

$$P_B = -K_W w\left[H\left(\theta - \theta_E\right) - H\left(\theta - \theta_D\right)\right]. \tag{4}$$

where $P_L$ is force amplitude, *m* is the number of half-waves in the longitudinal direction, $\omega_1$ is the frequency of excitation of the lateral pressure and *t* is the time. In eq. (4) $K_W$ is the stiffness modulus of the elastic base which obeys $K_W=K_{nW}A_{11}/R^2$, being $K_{nW}$ the nondimensional parameter of stiffness of elastic base, $A_{11}$ the membrane stiffness of elastic constitutive matrix and *H*( ) the Heaviside function to accomplish the circumferential discontinuity.

The internal forces - $N_x$, $N_\theta$ and $N_{x\theta}$ - and moments - $M_x$, $M_\theta$, $M_{x\theta}$ -described in eq. (2) are given in terms of deformations field and curvatures of the mid-surface of cylindrical shell, as given by:

$$\begin{Bmatrix} N_x \\ N_\theta \\ N_{x\theta} \\ M_x \\ M_\theta \\ M_{x\theta} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{11} & 0 & B_{12} & B_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & C_{11} & C_{12} & 0 \\ B_{12} & B_{11} & 0 & C_{12} & C_{11} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{\theta 0} \\ \varepsilon_{x\theta 0} \\ \kappa_{x0} \\ \kappa_{\theta 0} \\ \kappa_{x\theta 0} \end{Bmatrix}, \quad \begin{aligned} &\left(A_{ij}, B_{ij}, C_{ij}\right) = \int_{-h/2}^{+h/2} Q_{ij}\left(1, z, z^2\right) dz \\ &Q_{11} = \frac{E}{1-\nu^2},\ Q_{12} = \frac{E\nu}{1-\nu^2},\ Q_{66} = \frac{E}{2(1-\nu)} \end{aligned} . \tag{5}$$

where $A_{ij}$, $B_{ij}$, $C_{ij}$ ($i$,$j$ = 1, 2, 6) are the terms of the elastic constitutive matrix that consider the effect of functionally graded material and the physical parameters, $E$, $\rho$ and $\nu$ are described in eq. (1).

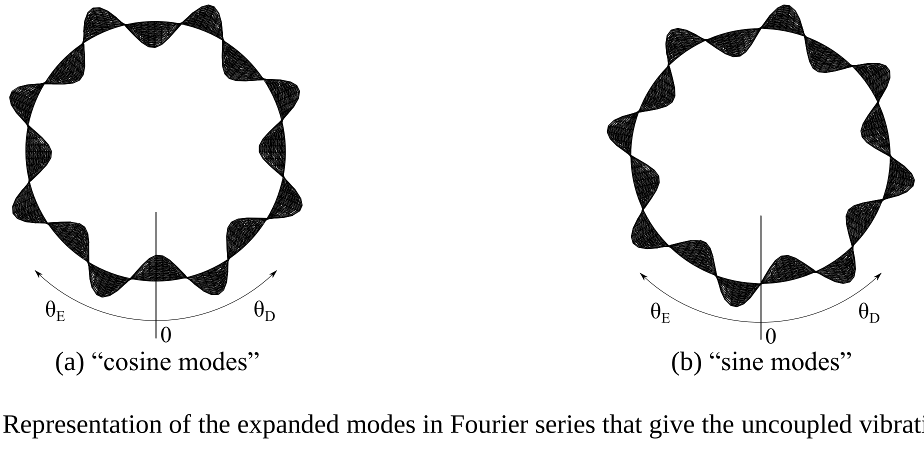
To obtain the reduced-order model and following numerical results, a consistent numerical procedure, derived by Pereira et al [9], is applied where the transversal field displacement is obtained by perturbation techniques. The axial and circumferential field displacements is described as a function of transversal displacement field [10]. However, in this work an analysis considering a full model will be derived considering all basic modes of vibration mode. As shown in Amabili and Dalpiaz [3], a great number of terms is necessary to describe the transversal displacement in Fourier series to ensure the convergence of linear vibration mode of a cylindrical shell with circumferential discontinuity in its elastic base, but as observed by Pereira et al [9] to achieve nonlinear dynamics of cylindrical shell this consideration demands a lot of computational effort.

Then, to ensure only the most important modes on vibration mode, reduced order model is derived to describe the nonlinear dynamic behavior of a FG cylindrical shell. From a FEM software, a strategy to obtain the vibration mode of a cylindrical shell resting on a discontinuous elastic base is conducted. Thus, the main terms of a linear vibration mode are obtained from application Fourier transform on the results of the Abaqus® FEM software. So, for the quantification of the participation of each modal expansion, the Parseval theorem is applied as show in eq. (6), where the transversal velocity field is analogous to the transversal displacement field. Then, it is established the relation of kinetic energy of shell and the terms of Fourier transform, where $\omega_0$ is the natural frequency of the shell and F(i$\omega$) is the amplitude of the frequency obtained by the Fourier transform.

$$\int_L \int_h \int_{2\pi R} \rho \dot{f}\left(\theta\right)^2 d\theta\, dz\, dx = \frac{\omega_0^2}{2\pi} \int_L \int_h \int_\omega \rho F\left(\mathrm{i}\omega\right)^2 d\omega\, dz\, dx \tag{6}$$

To obtain the main terms through the FEM model, we use 2500 shell elements (S4R) and consider a simply supported perfect cylindrical shell with radius $R$=0.6 m, length $L$=0.6 m, thickness $h$=0.003 m and $\theta_D$=-$\theta_E$=22.5º. The shell is composed of the following materials: steel, called material A, and a ceramic material, called material C. The properties of these materials are: $E_A$= 205.1 × $10^9$ N/m$^2$, $\rho_A$= 8900 kg/m$^3$, $\nu_A$= 0.31, $E_C$= 322.3 × $10^9$ N/m$^2$, $\rho_C$= 2370 kg/m$^3$ and $\nu_C$= 0.24.

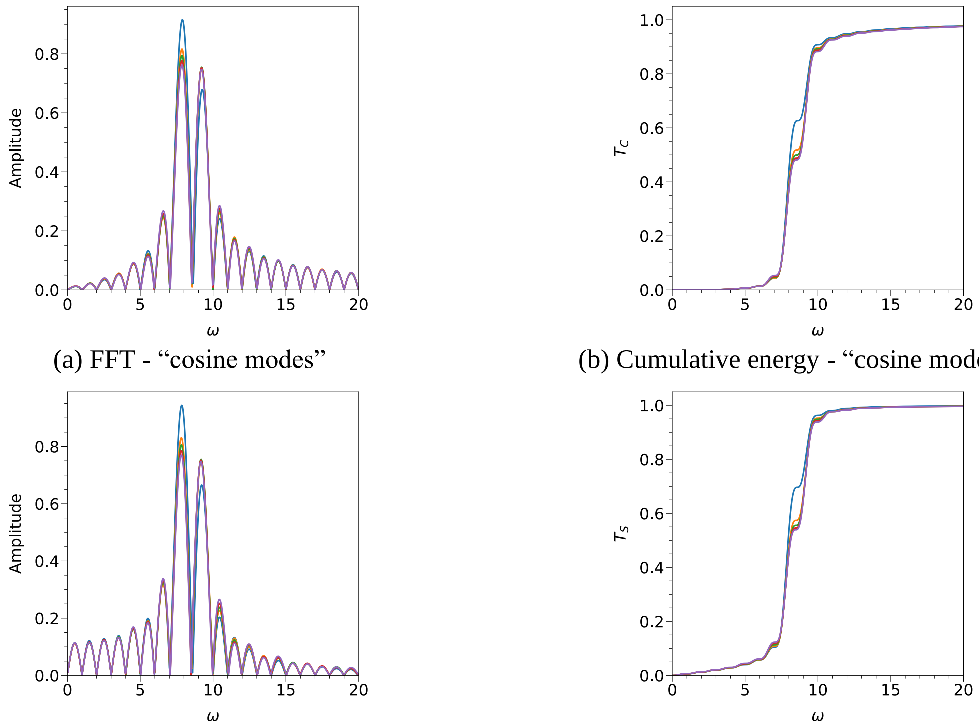
Expanding the transversal vibration mode, obtained from FEM software, in Fourier series, it can obtain two uncoupled vibration modes as presented in Fig. 2 where Fig. 2(a) shows the "cosine modes" and Fig. 2(b) illustrates the "sine modes". In this work, we named the vibration mode as "cosine modes" and "sine modes" according to the used functions to expand the vibration mode in Fourier series.




(a) "cosine modes" (b) "sine modes"

Figure 2. Representation of the expanded modes in Fourier series that give the uncoupled vibration mode

Figure 3 presents the frequency spectrums obtained from FFT with its cumulative energy in frequency domain for both “cosine modes” and “sine modes”, considering different values for stiffness of elastic base. It is observed in Fig. 3 that the main terms of Fourier series are the same for both “cosine modes” and “sine modes” (ω =8, ω =9) where the cumulative kinetic energy represents 80% and 85% total kinetic energy, respectively. So, it can conclude the region 7 < ω < 10 contains the main modes to describe the vibration mode of cylindrical shell, independently of the evaluated stiffness of elastic base.




(a) FFT - “cosine modes” (b) Cumulative energy - “cosine modes”

(c) FFT - “sine modes” (d) Cumulative energy - “sine modes”

Figure 3. Frequency spectrum and cumulative energy in frequency domain. (a), (c) “Cosine mode” and (b), (d) “Sine mode”. (— $K_{nW}$ = 0.003, — $K_{nW}$ = 0.015, — $K_{nW}$ = 0.03, — $K_{nW}$ = 0.06 and — $K_{nW}$ = 0.12)

Then, the considered seed solution for perturbation method [7-10] is given by:

$$w_0 = \sum_{i=8,9} \bar{W}_{i,1}^{C}(\tau)\cos(i\theta)\sin(q\xi) + \sum_{i=8,9} \bar{W}_{i,1}^{S}(\tau)\sin(i\theta)\sin(q\xi), \tag{7}$$

where $q = m\pi$, $\xi = x/L$, with $0 \le \xi \le 1$, $m = 1$ and coefficients $W^C$, $W^S$ are amplitudes related with “cosine modes” and “sine modes”, respectively. Then, using the seed solution of eq. (7), the modal solution for transversal displacement field was derived by perturbation method, where all degrees of freedom arises from the modal couplings of the quadratic and cubic terms and, which in its turn, are present in the nonlinear equilibrium equation, are shown in eq. (8), being $\tau = t\omega_0$.

$$\begin{aligned} w = &\sum_{\beta=1,3} \sum_{i=C3} \left\{ \left[ W_{i,\beta}^{C}(\tau)\cos(i\theta) + W_{i,\beta}^{S}(\tau)\sin(i\theta) \right] \sin(\beta q\xi) \right\} + \\ &\sum_{i=C2} \left\{ \left[ W_{i,0}^{C}(\tau)\cos(i\theta) + W_{i,0}^{S}(\tau)\sin(i\theta) \right] \left[ \frac{3}{4} - \cos(2q\xi) + \frac{1}{4}\cos(4q\xi) \right] \right\}; \end{aligned} \tag{8}$$

$$\begin{aligned} u = &\sum_{\beta=1,3,5,7} \sum_{i=C3} \left\{ \left[ U_{i,\beta}^{C}(\tau)\cos(i\theta) + U_{i,\beta}^{S}(\tau)\sin(i\theta) \right] \cos(\beta q\xi) \right\} + \\ &\sum_{\beta=2,6} \sum_{i=C4} \left\{ \left[ U_{i,\beta}^{C}(\tau)\cos(i\theta) + U_{i,\beta}^{S}(\tau)\sin(i\theta) \right] \times \left[ \sin(\beta q\xi) - \frac{\beta}{\beta+2}\sin((\beta+2)q\xi) \right] \right\}; \end{aligned} \tag{9}$$

$$v = \sum_{\beta=1,3,5,7} \sum_{i=C3} \left\{ \left[ V_{i,\beta}^{C}(\tau)\cos(i\theta) + V_{i,\beta}^{S}(\tau)\sin(i\theta) \right] \sin(\beta q \xi) \right\} + \sum_{\beta=0,4} \sum_{i=C4} \left\{ \left[ V_{i,\beta}^{C}(\tau)\cos(i\theta) + V_{i,\beta}^{S}(\tau)\sin(i\theta) \right] \times \left[ \cos(\beta q \xi) - \cos((\beta+2) q \xi) \right] \right\}. \tag{10}$$

where $C2$ = {0, 1, 16, 17, 18}, $C3$ = {7, 8, 9, 10, 24, 25, 26, 27}, $C4$ = {0, 1,2, 15, 16, 17, 18, 19, 32, 33, 34, 35, 36} are a set of circumferential modal modes for respectively, second, third and fourth order of perturbation method in circumferential direction of cylindrical shell.

To obtain a consistent system of displacements fields in the *u* and *v* directions, with their appropriate modal couplings. The procedure detailed in Gonçalves et al. [10] was applied to obtain the displacement fields *u* and *v*, eqs. (9) and (10), where equations of displacements field satisfies boundary conditions of a simply supported cylindrical shell that are given by:

$$u = 0 \text{ at } x = \frac{L}{2}; \quad v, w = 0 \text{ at } x = 0, L. \tag{11}$$

From nonlinear equations of eq. (2), a standard Galerkin procedure is applied to discretize the partial differential equations. Its observed that amplitudes $U^C$, $U^S$, $V^C$ and $V^S$ assemble a linear system that depends of the modal transversal amplitude $W^C$ and $W^S$. Thus, a nonlinear system of second-order equations in relation to τ written only in terms of $W^C$ and $W^S$ can be obtained [10].

## 3 Numerical results

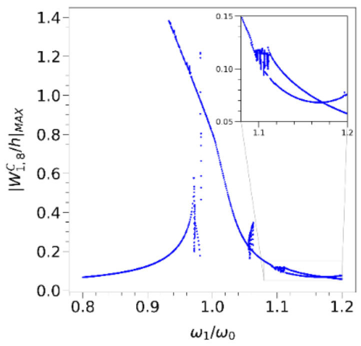
To obtain the numerical results, the physical and geometric parameters are the same that were defined in the previous section. The nondimensional stiffness value of the elastic base is $K_{nW}$ = 0.003 and viscous damping is $\eta_1$ = 0.001. To describe lateral harmonic pressure, the seed solution for perturbation method is used as follows:

$$P(t) = P_L \left( \sum_i P_8^C \cos(8\theta) + P_9^C \cos(9\theta) + P_8^S \sin(8\theta) + P_9^S \sin(9\theta) \right) \sin(q\xi) \cos\left( \frac{\omega_1}{\omega_0} \tau \right), \tag{12}$$

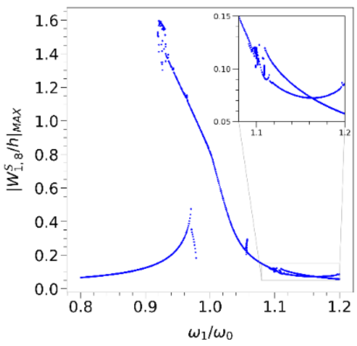
where $\mathrm{P}_8^{\mathrm{C}}$, $\mathrm{P}_8^{\mathrm{S}}$, $\mathrm{P}_9^{\mathrm{C}}$ and $\mathrm{P}_9^{\mathrm{S}}$ are a 0-1 factor that directly excites the basic modes of the seed solution, $\omega_1$ is the excitation frequency and $P_L$ = 5000 N/m² is the amplitude of the lateral pressure. In following numerical results of forced response, $P(t)$ excites the main mode of lowest natural frequency of cylindrical shell which occurs for vibration mode ($m$, $n$) = (1, 8) in "cosine mode". Thus, it is considered $\mathrm{P}_8^{\mathrm{C}}$ = 1 and $\mathrm{P}_8^{\mathrm{S}}$ = $\mathrm{P}_9^{\mathrm{C}}$ = $\mathrm{P}_9^{\mathrm{S}}$ = 0.

Figure 4 shows a comparison of the nonlinear resonance curve obtained from the full model for displacements fields, developed in this work, and from an uncoupled model for displacements fields given by Pereira et al [9], which used as seed solution for perturbation method the "cosine modes" or the "sine modes" to describe the linear vibration mode of cylindrical shell. The resonance curves were obtained through a brute force method where it is observed that all models present softening behavior. However, Figs. 4(c) and (f) show a new path of solution arise in full model with a chaotic behavior in the resonance region. This behavior does not occur in the uncoupled models of Pereira et. al [9], Figs. 4(a), (b), (d) and (e). It is important to note that the secondary peak on resonance curve is maintained in all analyzed models, in the same way, the competition between periodic and quasi-periodic solutions are observed in both models.

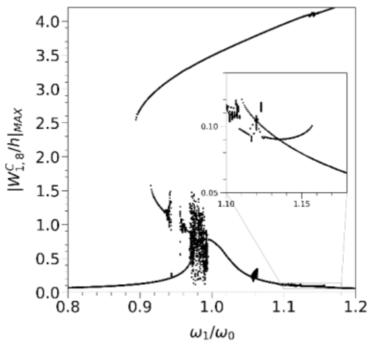
Figure 5 shows the basins of attraction – BoA - for full model considering two excitation frequencies: $\omega_1/\omega_0$ = 0.92 in Fig. 5(a) and $\omega_1/\omega_0$ = 1.10 in Fig. 5(c). Also, the phase-portrait for all identified dynamical attractors for both BoA are shown in Fig. 5(b) and (d). It can observe in Figs. 5(a) and (c) that BoA presents a large fractal domain in both projection's planes, indicating a high sensibility of the structural system to initial conditions. For $\omega_1/\omega_0$ = 0.92, near to main resonance peak, Fig. 5(a), the uncoupled solution dominates the BoA while, near to the second resonance peak, Fig. 5(c), the coupled solutions taken this place.



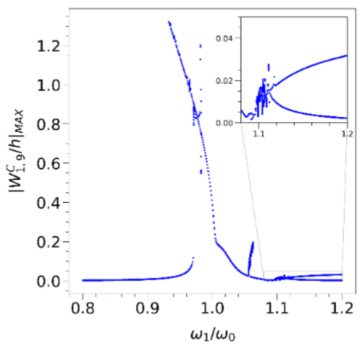

(a) $(m, n)$ = (1,8) “cosine modes” [9]



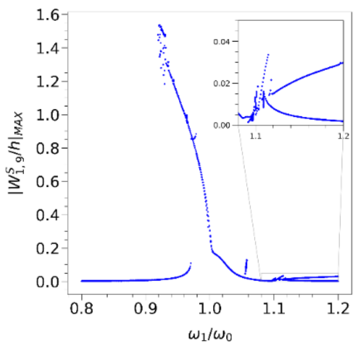

(b) $(m, n)$ = (1,8) “cosine modes” [9]



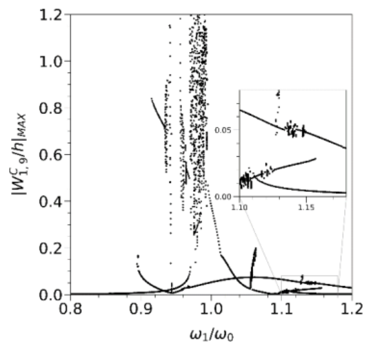

(c) $(m, n)$ = (1,8) “full model”




(d) $(m, n)$ = (1,9) “sine modes” [9]

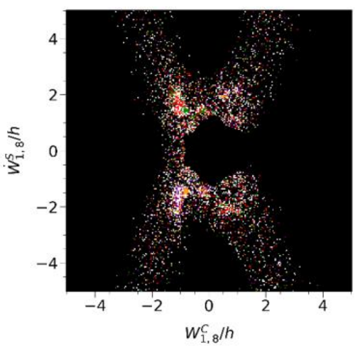



(e) $(m, n)$ = (1,9) “sine modes” [9]

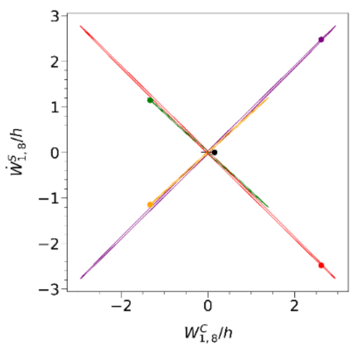



(f) $(m, n)$ = (1,9) “full model”

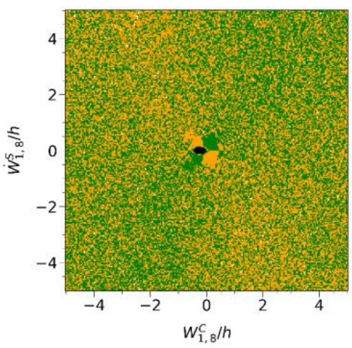
Figure 4. Resonance curves for a simply supported shell resting on elastic base with circumferential discontinuity for three seeds of perturbation method.



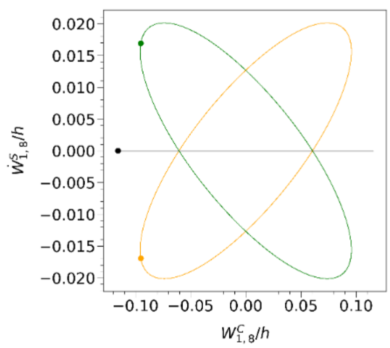

(a) $\omega_1/\omega_0$ = 0.92 – “full model”




(b) $\omega_1/\omega_0$ = 0.92 – “full model”




(c) $\omega_1/\omega_0$ = 1.10 – “full model”




(d) $\omega_1/\omega_0$ = 1.10 – “full model”

Figure 5. Basins of attraction and phase-portrait with Poincare map for a simply supported shell resting on elastic base with circumferential discontinuity using the full model for perturbation method.

# 4 Conclusions

In this paper, we studied the nonlinear vibrations of the shell with a circumferential discontinuity through the frequency-response curve and basins of attraction. The low dimensional model was derived, considering all coupling modes in perturbation method. An investigation of the influence of these basic modes was discussed. Resonance curves for nonlinear forced vibrations showed that the consideration of the basic modes changed the behavior of nonlinear oscillations of non-resonant path from stable to unstable solutions. The full model did not change the softening behavior of resonance curves as observed in the peak of resonance and quasi-periodic solutions.

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# References

[1] M. Amabili, M.P. Païdoussis, Review of studies on geometrically nonlinear vibrations and dynamics of circular cylindrical shells and panels, with and without fluid-structure interaction, Applied Mechanics Reviews. 56 (2003) 349–356. https://doi.org/10.1115/1.1565084.
[2] F. Alijani, M. Amabili, Non-linear vibrations of shells: A literature review from 2003 to 2013, International Journal of Non-Linear Mechanics. 58 (2014) 233–257. https://doi.org/10.1016/j.ijnonlinmec.2013.09.012.
[3] Amabili, M.; Dalpiaz, G. (1997) Free Vibrations of Cylindrical Shells with Non-Axisymmetric Mass Distribution on Elastic Bed. Meccanica, 32, 71–84.
[4] Tj, H. G.; Mikami, T.; Kanie, S.; Sato, M. Free vibrations of fluid-filled cylindrical shells on elastic foundations. Thin-Walled Structures, v. 43, n. 11, p. 1746–1762, 2005.
[5] Tj, H. G.; Mikami, T.; Kanie, S.; Sato, M. Free vibration characteristics of cylindrical shells partially buried in elastic foundations. Journal of Sound and Vibration, v. 290, n. 3–5, p. 785–793, 2006.
[6] Nejad, F. B.; Bideleh, S. M. M. Nonlinear free vibration analysis of prestressed circularcylindrical shells on the Winkler/Pasternak foundation. Thin-Walled Structures, v. 53, p.26–39, 2012.
[7] Rodrigues, P. C. Influência da descontinuidade da base elástica nas curvas de ressonância de uma casca cilíndrica, 2019. Goiânia: Universidade Federal de Goiás.
[8] Silva, F. M. A.; Rodrigues, P. C.; Gonçalves, P. B. Nonlinear Oscillation of a FG Cylindrical Shell on a Discontinuous Elastic Foundation. In New Trends in Nonlinear Dynamics - Proceedings of the First International Nonlinear Dynamics Conference (NODYCON 2019); Lacarbonara, W., Balachandran, B., Ma, J., Machado, J. A. T., Stepan, G., Series Eds.; Springer: Roma, 2020; Vol. 3, pp 79–88.
[9] Pereira, J. K. A., Soares, R. M.; SILVA, F. M. A. Influence of circumferential discontinuity of an elastic foundation on the nonlinear dynamics of cylindrical shells with functionally graded material. in *XLI Ibero-Latin-American Congress on Computational Methods in Engineering* vol. 1 1–7 (2020).
[10] Gonçalves, P. B.; Silva, F. M. A.; Del Prado, Z. J. G. N. Low-dimensional models for the nonlinear vibration analysis of cylindrical shells based on a perturbation procedure and proper orthogonal decomposition. Journal of Sound and Vibration, v. 315, n. 3, p. 641–663, 2008.