An Integrated Formulation to Predict Pre and Post-Critical Behavior of Frames

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Abstract. To reduce the discretization influence and allow a minimal beam subdivision in geometric nonlinear analysis of a framed structure, using the finite element method (FEM), the present work evaluates an integrated formulation in the pre and post-critical phases. This updated Lagrangian formulation considers the Euler-Bernoulli beam theory with high-order terms of the strain tensor and a tangent stiffness matrix calculated with analytical interpolation functions. These functions are obtained from the solution of the equilibrium differential equation of a deformed infinitesimal element, which includes the influence of axial forces. In pre and post-critical stages, the nonlinear response of the proposed integrated formulation is evaluated with robust nonlinear solution schemes, and the results are compared with conventional formulations. Examples clearly show the efficiency of the integrated formulation to predict the pre-critical phase using a low discretization and consistent results with conventional formulations in the post-critical behavior.

Keywords: Complete tangent stiffness matrix; Analytical interpolation functions; Post-critical behavior

1 Introduction

In a linear elastic analysis of frame models with beam elements with constant cross-sections, Hermitian interpolation functions can accurately obtain the model displacements and rotations, independently of the adopted discretization. This fact occurs because the Hermitian interpolation functions correspond to the analytic solution of the problem (Martha [1], Rodrigues *et al.* [2,3,4], Burgos and Martha [5]).

In the context of nonlinear geometric analysis, these interpolation functions do not correspond to the solution of the differential equation of the problem. Souza [6] indicates that using these interpolation functions in the displacement-based finite element theory requires the discretization of the structure into several elements, reducing computational efficiency. Alternatively, higher-order elements can be used (So and Chan [7], Zheng and Dong [8], Rodrigues et al. [9]), as well as stabilization functions (Chen and Lui [10], Aristizábal-Ochoa [11-14]). Other authors develop the formulation using the deformed infinitesimal element equilibrium (Goto and Chen [15], Chan and Gu, [16], Balling and Lyon [17]). Furthermore, Rodrigues et al. [3] show the influence of considering high order terms on the Green-Lagrange strain tensor when performing a geometric nonlinear analysis.

The beam-column element considered in this work is formulated in Rodrigues *et al.* [2] and Rodrigues [18]. The tangent stiffness matrix is calculated considering the high-order terms of the strain tensor. Interpolation functions are obtained directly from the differential equation homogeneous solution of the problem, i.e., the equilibrium of a deformed infinitesimal element, which considers the influence of axial load in the formulation. Therefore, the interpolation functions correspond to the analytic solution of the problem.

The proposed element can efficiently describe the equilibrium path in a geometric nonlinear analysis up to the critical load of the problem considering only one element per bar. However nonlinear formulations provide paths that can be complex curves with critical points and multiple responses to a given load level, requiring a refined discretization, in addition to a robust incremental-iterative method to solve the problem (Rangel [19]).

This research integrates the formulation developed in Rodrigues *et al.* [2] with incremental-iterative methods implemented in the NUMA-TF program, an open-source MATLAB program to perform numerical analyses of reticulated structural models (Rangel [19]). Examples are calculated with a reduced structure discretization using the formulation presented by Rodrigues *et al.* [2]. The pre and post-critical behaviors are evaluated and compared with refined discretized models using conventional formulations.

2 Rodrigues *et al.* [2] Beam-Column Element

2.1 Differential Relation of the Beam-Column Element

Figure 1 shows a deformed infinitesimal element subjected to a distributed transversal load q and axial load P. The element equilibrium equations are written in Eq. (1), where dv is the infinitesimal element transversal displacement, $V(x)$ is the vertical component of the force acting on the cross-section, P the horizontal component, and $M(x)$ the bending moment.

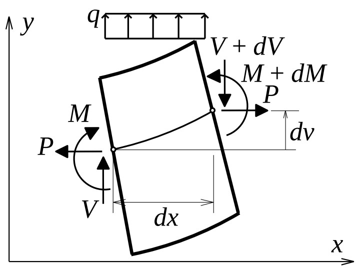

Figure 1. Equilibrium of the deformed infinitesimal element

$$\sum F_y \rightarrow \frac{dV(x)}{dx} = q(x) \quad ; \quad \sum M_o \rightarrow dM - (V + dV)dx - P.dv + q(x)\frac{dx^2}{2} = 0 \tag{1}$$

From the element equilibrium and the approximate relation between bending moment and curvature, $M(x) = EId\theta/dx$, wherein $\theta(x)$ corresponds to the cross-section rotation, the differential equation of the problem is obtained. Considering the Euler-Bernoulli beam theory, the cross-sectional rotation corresponds to the derivative of the transverse displacement, leading to Eq. (2).

$$EI\frac{d^2\theta(x)}{dx^2} - V(x) - P\frac{dv(x)}{dx} = 0 \rightarrow EI\frac{d^3\theta}{dx^3} - P\frac{d^2v(x)}{dx^2} = q(x) \rightarrow \frac{d^4v(x)}{dx^4} - \frac{P}{EI}\frac{d^2v(x)}{dx^2} = \frac{q(x)}{EI} \tag{2}$$

The homogeneous part of the differential equation solution is calculated considering an unloaded element, $q(x) = 0$. Thus, the solution for the transverse displacement and cross-section rotation is given according to Eq. (3).

$$v_h(x) = c_1e^{\mu x} + c_2e^{-\mu x} + c_3x + c_4\,, \qquad \theta_h(x) = c_1\mu e^{\mu x} - c_2\mu e^{-\mu x} + c_3, \quad com\ \mu = \sqrt{P/_{EI}} \tag{3}$$

Eq. (3) can be rewritten with hyperbolic functions for tensile forces and with trigonometric functions for compressive forces (Rodrigues *et al*. [18]).

2.2 Interpolation Functions

Figure 2 illustrates the deformed configuration of an isolated element obtained by interpolating the nodal displacements (transverse displacements and rotations) through interpolation functions, as in Eqs. (4) and (5).

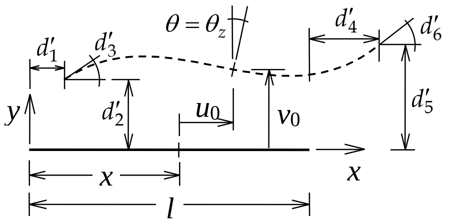

Figure 2. Deformed configuration of an isolated element

$$v_0(x) = N_2^v(x)d'_2 + N_3^v(x)d'_3 + N_5^v(x)d'_5 + N_6^v(x)d'_6 \rightarrow v_0(x) = \{N_v(x)\}\{v\} \tag{4}$$

$$\theta(x) = N_2^\theta(x)d'_2 + N_3^\theta(x)d'_3 + N_5^\theta(x)d'_5 + N_6^\theta(x)d'_6 \rightarrow\ \theta_z(x) = \{N_{\theta z}(x)\}\{v\} \tag{5}$$

Considering the transverse displacement and rotation obtained in Eq. (3), from the equilibrium of a deformed infinitesimal element, the Eq. (6) can be written. Boundary conditions are evaluated as per Eq. (7).

$$\begin{matrix} v_h(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} + c_3 x + c_4 \\ \theta_h(x) = c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x} + c_3 \end{matrix} \rightarrow \begin{Bmatrix} v_0(x) \\ \theta(x) \end{Bmatrix} = \begin{bmatrix} e^{\mu x} & e^{-\mu x} & x & 1 \\ \mu e^{\mu x} & -\mu e^{-\mu x} & 1 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = [X]\{C\} \tag{6}$$

$$\{d'\} = \begin{Bmatrix} d'_2 \\ d'_3 \\ d'_5 \\ d'_6 \end{Bmatrix} = \begin{Bmatrix} v_0(0) \\ \theta(0) \\ v_0(L) \\ \theta(L) \end{Bmatrix} \rightarrow \{d'\} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ \mu & -\mu & 1 & 0 \\ e^{L\mu} & e^{-L\mu} & L & 1 \\ \mu e^{L\mu} & -\mu e^{-L\mu} & 1 & 0 \end{bmatrix} . \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = [H]\{C\} \tag{7}$$

Therefore, using the Eqs. (6) and (7), the interpolation functions of the problem can be calculated with Eq. (8) and are presented in Rodrigues [18]. In the referred work, the interpolation functions are also written considering hyperbolic and trigonometric functions.

$$\begin{Bmatrix} v_0(x) \\ \theta(x) \end{Bmatrix} = [X][H]^{-1}\{d'\} \Rightarrow [N] = [X][H]^{-1} \tag{8}$$

2.3 Tangent Stiffness Matrix Considering Higher-Order Terms in the Strain Tensor

When the Euler-Bernoulli beam theory is considered, the displacement field of a beam element is given according to Eq. (9) and can be seen in Fig.3.

$$u(x,y) = u_0(x) - \theta(x).y = u_0(x) - \frac{v_0(x)}{dx}.y \qquad v(x,y) = v_0(x) \tag{9}$$

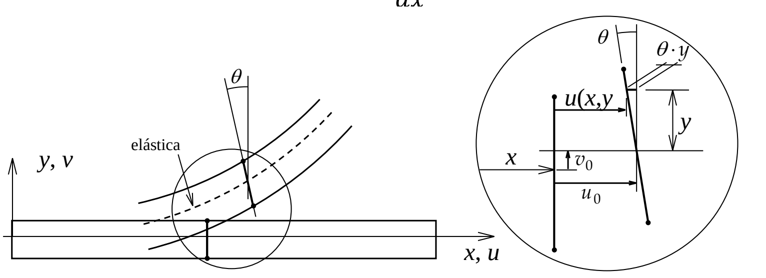

Figure 3. Beam displacement field

From the displacement field, e considering the linear ($\varepsilon_{xx}, \gamma_{xy}$) and nonlinear ($\eta_{xx}, \eta_{xy}$) parts of the Green-Lagrange strains tensor, the elastic and geometric stiffness matrices of an element can be calculated by the virtual work principle, according to the Eqs. (10) e (11) (Rodrigues *et al.* [2]).

$$\delta U_L = \int_V \varepsilon_{xx}.E\delta\varepsilon_{xx}dV + \int_V \gamma_{xy}.G\delta\gamma_{xy}dV \rightarrow \quad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - y\frac{\partial^2 v}{\partial x^2} \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{10}$$

$$\delta U_{NL} = \int_V \tau_{xx}\delta\eta_{xx}dV + \int_V \tau_{xy}\delta\eta_{xy}dV \rightarrow \tag{11}$$

$$\rightarrow \eta_{xx} = \frac{1}{2}\left(\frac{\partial u^2}{\partial x} + \frac{\partial v^2}{\partial x}\right) = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + y^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right) - y\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial x}$$

$$\rightarrow \eta_{xy} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = y\frac{\partial^2 v}{\partial x^2}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}$$

Considering the higher-order terms of the strain tensor, generally disregarded in usual formulations, and using the interpolation functions, Eqs. (10) and (11) are rewritten by Eqs. (12) and (13).

$$\delta U_L = \int_A\left(\int_0^L\left(\frac{\partial u}{\partial x} - y\frac{\partial^2 v}{\partial x^2}\right)E\left(\delta\frac{\partial u}{\partial x} - y\delta\frac{\partial^2 v}{\partial x^2}\right)dx\right)dA = \left(\int_0^L\frac{\partial u}{\partial x}\delta\frac{\partial u}{\partial x}dx\right)EA + \left(\int_0^L\frac{\partial^2 v}{\partial x^2}\delta\frac{\partial^2 v}{\partial x^2}dx\right)EI_z \tag{12}$$

$$\delta U_L = \{\delta u\}^T\int_0^L EA\{N_u'\}\{N_u'\}^T dx\{u\} + \{\delta v\}^T\int_0^L EI_z\{N_v''\}\{N_v''\}^T dx\{v\}$$

$$\delta U_{NL} = \int_A\left(\int_0^L t_{xx}\delta\left(\frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + y^2\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right) - y\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial x}\right)dx\right)dA + \int_A\left(\int_0^L t_{xy}\delta\left(y.\frac{\partial^2 v}{\partial x^2}.\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}.\frac{\partial u}{\partial x}\right)dx\right)dA \tag{13}$$

$$\delta U_{NL} = \frac{1}{2}\int_0^L\left[P\delta\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) + P\frac{I_z}{A}\delta\left(\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right)\right]dx + \int_0^L\left[M_z\delta\left(\frac{\partial^2 v}{\partial x^2}\frac{\partial u}{\partial x}\right) - Q_y\delta\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right)\right]dx$$

$$\delta U_{NL} = \{\delta u\}^T\int_0^L P\{N_u'\}\{N_u'\}^T dx\{u\} + \{\delta v\}^T\int_0^L P\{N_v'\}\{N_v'\}^T dx\{v\} + \{\delta v\}^T\int_0^L P\frac{I_z}{A}\{N_v''\}\{N_v''\}^T dx\{v\}$$

$$+\{\delta v\}^T\int_0^L M_z\{N_v''\}\{N_u'\}^T dx\{u\} + \{\delta u\}^T\int_0^L M_z\{N_u'\}\{N_v''\}^T dx\{v\} - \{\delta v\}^T\int_0^L Q_y\{N_v'\}\{N_u'\}^T dx\{u\}$$

$$-\{\delta u\}^T\int_0^L Q_y\{N_u'\}\{N_v'\}^T dx\{v\}$$

The interpolation functions used are based on the problem differential equation, Eq. (8), topic 2.2. The tangent stiffness matrix corresponds to the sum of Eqs. (12) and (13).

Computational functions with the stiffness matrix are available in open repositories for positive or negative axial force (Rodrigues *et al.* [20, 21]). Furthermore, for computational efficiency, the obtained matrices can be written using Taylor series expansion. In this work, up to 4 terms were considered (4tr). These functions have been implemented in the NUMA-TF program (Rangel [19]).

2.4 Internal Forces in a Updated Lagrangian Formulation Considering Large Displacements

To solve problems with large displacements, it is necessary to distinguish between rigid body motion and natural deformation, Fig.4. The forces at the end of the load step $\{^2F\}$ are calculated according to Eq. (14), considering the forces at the start of the load step $\{^1F\}$, and its increment {dF} until the respective iteration, obtained by the tangent stiffness matrix $(K_e + K_a)$ and the natural displacement increments $(d\Delta_n)$.

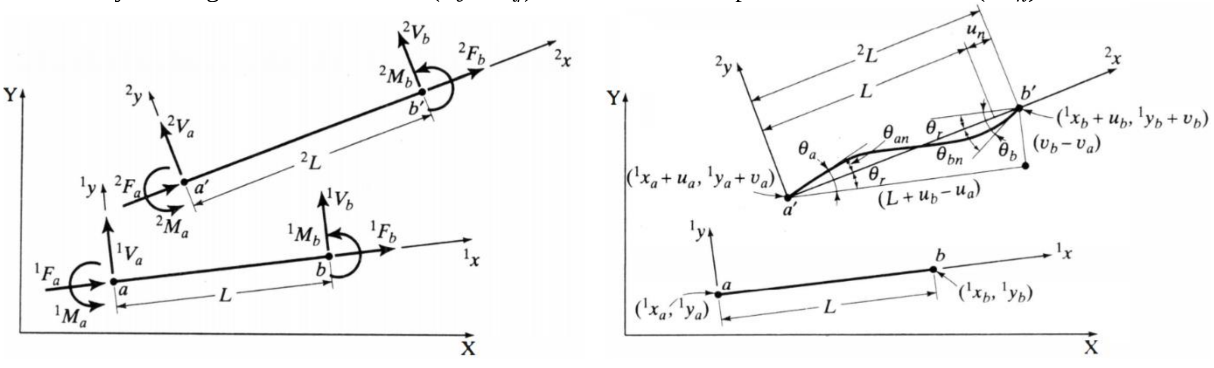

Figure 4. Element forces and displacements (McGuire *et al.* [22])

$$\{\,^2F\} = \{\,^1F\} + \{dF\},\ \ \{dF\} = \left[K_e + K_g\right]\{d\Delta_n\} = \left[K_e + K_g\right][0\ \ 0\ \ \theta_{an}\ \ u_n\ \ 0\ \ \theta_{bn}]^T \tag{14}$$

3 Numerical Applications

The results obtained by the proposed integrated formulation (el-Large-4tr) were compared with numerical solutions of a usual formulation with discretized structure into 10 elements (10el-Small-2tr) and also with a reduced discretization (el-Small-2tr e el-Large-2tr). Examples consider structures with a length of L=1 m, Young's modulus of E=10^7 kN/m^2, and slenderness ratio of L/h = 10 (Euler-Bernoulli beam theory).

3.1 Isolated Column and Roorda Frame

Considering two elements in each bar, the proposed formulation was evaluated in an isolated column and a Roorda frame, Fig.5. The reached equilibrium paths are shown in Fig.6 and Fig.7.

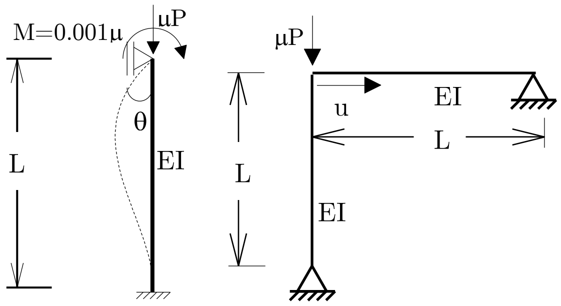

Figure 5. Isolated column and Roorda frame

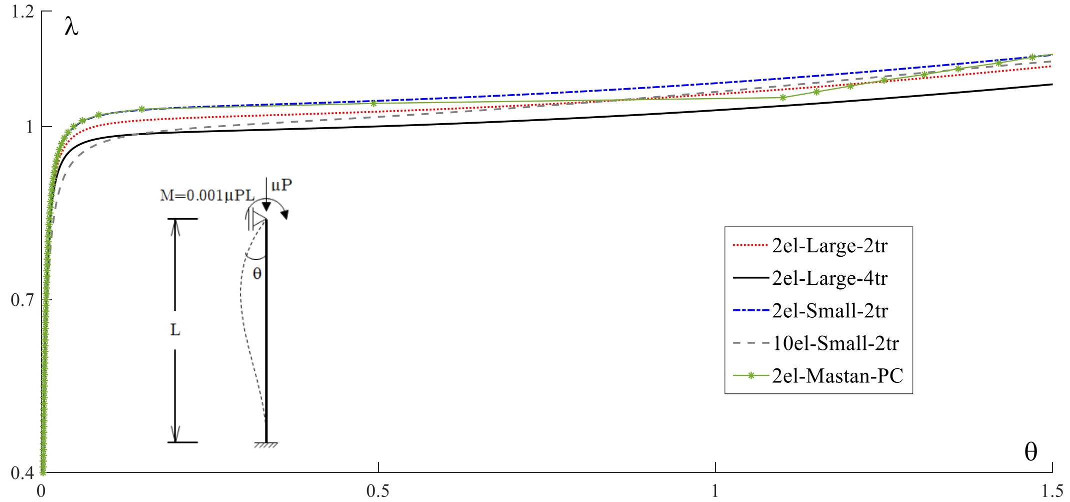

Figure 6. Equilibrium path for isolated column

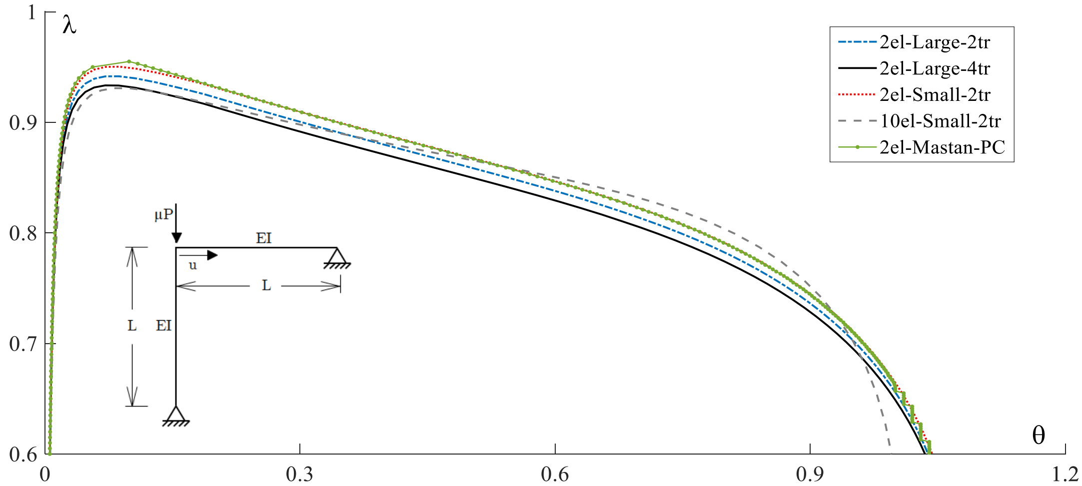

Figure 7. Equilibrium path for Roorda frame

3.2 Two Floors Frame

The example evaluates the proposed formulation, considering just one element in each bar, in a two floors frame, Fig.8. The equilibrium path obtained is represented in Fig.9.

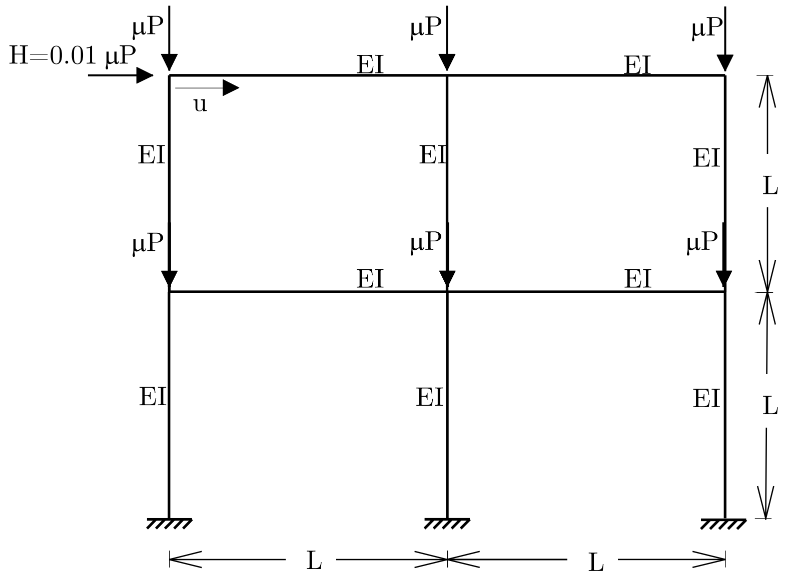

Figure 8. Two floors frame

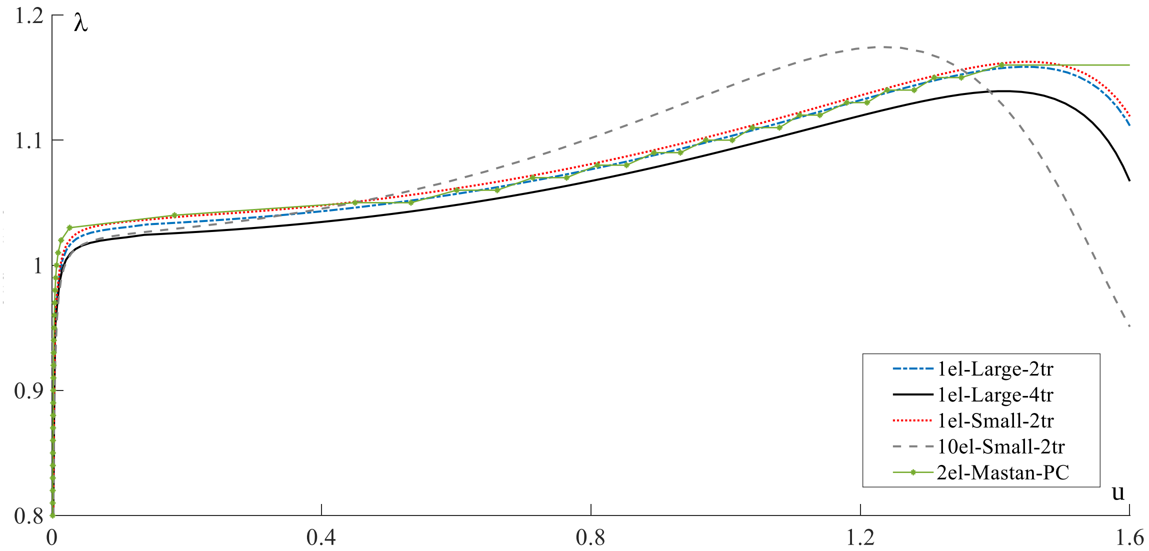

Figure 9. Equilibrium path for two floors frame

Analyzing the equilibrium paths, it can be seen that in the pre-critical phase, the proposed formulation (el-Large-4tr) with a reduced discretization of the structure, associated with incremental-iterative methods to solve nonlinear systems, provides accurate response related to the reference solution, i.e., discretized structure with the conventional formulation. In the post-critical phase, the formulation loses efficiency, requiring discretization to improve the results. However, maintain the same quality as a usual formulation, which shows the consistency of the element evaluated.

4 Conclusions

The developed examples show that the proposed formulation can efficiently describe the pre-critical phase of plane frames with a reduced discretization of the structure. The equilibrium path constructed with the proposed formulation reaches the best approximation with the response with the discretized structure when compared to the usual formulations. There is no effective improvement in the equilibrium path in the post-critical phase when using the proposed element. However, the result is consistent with the usual formulation.

The study is extended to 3D elements and considering shear deformation to understand the proposed element performance better. In addition, the shape functions are being used to obtain the P-δ effect on bars represented with a single element.

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