Modelling of the P-δ effect using interpolating functions

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Abstract. P-Delta is a second-order effect that arises from the consideration of loads acting on the deflected configuration of the structure. This effect is especially relevant in slender structures, which present lateral displacements large enough to significantly increase the bending moment caused by an axial load P acting upon a displacement Delta (hence P-Delta). There are typically two sources of P-Delta, known as P-Δ (P-"big-delta") and P-δ (P-"small-delta"). The P-big-delta result is easier to obtain in any geometrically nonlinear analysis, as it is a global effect associated with displacements of the member ends. On the other hand, the P-small-delta effect is associated with local displacements relative to the original shape of the element. The usual way to capture this behavior is to subdivide the elements, thus transforming the problem into a P-Δ effect within each segment. Since discretization can sometimes be unwanted, especially when dealing with students who still do not grasp this concept, a solution to overcome it is interesting from a didactic point of view. This work proposes the use of different sets of shape functions to interpolate the bending moment along the element's length, to account for the P-small-delta effect. Shape functions obtained directly from the solution of the differential equation of an axially loaded deformed infinitesimal element and traditional Hermitian polynomials are used. Comparisons were made with analytical and numerical solutions. Initial results for Euler Bernoulli beam theory indicate the ability of the formulation to capture the P-δ effect successfully.

Keywords: Shape functions, P small delta, P big delta

1 Introduction

A geometric nonlinear analysis using the finite element method (FEM) requires discretization of the structural members. Among other reasons, discretization is needed since cubic interpolation functions are not the homogeneous solution of the problem differential equation.

Some works use the differential equation of the problem to develop an "exact" element, for example, using the equilibrium of infinitesimal element as in Goto and Chen [1], Chan and Gu [2] and in Rodrigues et al. [3]. Most of these researchers consider Timoshenko beam theory, i.e., the shear deformation is considered. This influence is important to analyze structures with small slenderness or composite materials. This work will use Euler-Bernoulli beam theory for the sake of simplicity.

Geometric nonlinearity is especially relevant in slender structures, which present lateral displacements which are large enough to significantly change the way loads act upon them. The increase in the bending moment caused by an axial load acting upon a displacement is known as the P-Delta effect. There are two sources of P-Delta, known as P-Δ (P-"big-delta") and P-δ (P-"small-delta"). The P-big-delta result is a global effect associated with displacements of the member ends, while the P-small-delta effect is associated with local displacements relative to the original shape of the element. The usual way to capture this local behavior is to subdivide the elements, thus transforming the problem into a P-Δ effect within each segment. Since discretization can sometimes be unwanted, especially when dealing with students who still do not grasp this concept, a solution to overcome it is interesting from a didactic point of view.

This work proposes the use of interpolation functions coming from the exact solution of the differential equation of an axially loaded beam to represent the bending moment along the element's length, to account for the P-small-delta effect. These functions are obtained following the work of Burgos and Martha [4]. All examples were implemented in an improved version of Ftool (Two-dimensional Frame Analysis Tool) (Martha [5]), a widely used software for structural analysis. This improved version contains a wide range of options and parameters for nonlinear analysis, including the most used incremental-iterative methods and is presented in Rangel and Martha [6]. Some examples for single columns were analyzed and comparisons were made against analytical solutions. Initial results indicate the ability of the formulation to capture the P-δ effect successfully.

2 Axially loaded beam differential equation

2.1 Equilibrium of the deformed infinitesimal element

Figure 1 shows a deformed infinitesimal element subjected to a distributed load q and an axial load P. Equilibrium equations are obtained according to eq. (1), in which dv is transversal displacement, $V(x)$ is vertical force, P horizontal load and $M(x)$ is the bending moment acting upon the cross-section.

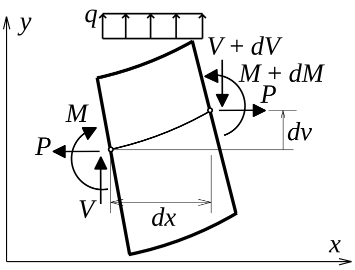

Figure 1. Equilibrium of the infinitesimal element

$$\sum F_y \rightarrow \frac{dV}{dx} = q(x) \text{ and } \sum M \rightarrow dM - (V + dV)dx - P \cdot dv + q(x) \cdot \frac{dx^2}{2} = 0 \tag{1}$$

Using the relation between bending moment and curvature, $M(x) = EId\theta/dx$, in which $\theta(x)$ is the cross-section rotation, the differential equation for axially loaded beams is obtained. For Euler-Bernoulli beam theory the cross-section rotation is the derivative of the lateral displacements, hence:

$$EI\frac{d^2\theta}{dx^2} - V(x) - P\frac{dv}{dx} = 0 \rightarrow EI\frac{d^3\theta}{dx^3} - P\frac{d^2v}{dx^2} = q(x) \rightarrow \frac{d^4v}{dx^4} - \frac{P}{EI}\frac{d^2v}{dx^2} = \frac{q(x)}{EI} \tag{2}$$

The homogenous solution of eq. (2) is given according to eq. (3) for any axial force value, eq. (4) for tension (positive axial force), or eq. (5) for compression (negative axial force). When using eqs. (4) and (5), if P is a positive value, parameter μ^2 is given by P/EI and if P is negative, $\mu^2 = -P/EI$. When using eq. (3) there is no need to modify the parameter μ since the exponential function accepts complex arguments.

$$\begin{aligned} v_h(x) &= Ae^{\mu x} + Be^{-\mu x} + Cx + D \\ \theta_h(x) &= A\mu e^{\mu x} - B\mu e^{-\mu x} + C \end{aligned} \tag{3}$$

$$\begin{aligned} v_h(x) &= A\sinh(\mu x) + B\cosh(\mu x) + Cx + D \\ \theta_h(x) &= A\mu\cosh(\mu x) + B\mu\sinh(\mu x) + C \end{aligned} \tag{4}$$

$$\begin{aligned} v_h(x) &= A\sin(\mu x) + B\cos(\mu x) + Cx + D \\ \theta_h(x) &= A\mu\cos(\mu x) - B\mu\sin(\mu x) + C \end{aligned} \tag{5}$$

2.2 P-Δ and P-δ effects

Figure 2 shows a loading situation in which P-Δ (big delta) and P-δ (small delta) are clearly depicted. If the axial force is sufficiently large, bending moment rising from its action upon relatively small displacements becomes relevant, which is known as the P-Δ effect. This effect can be seen in the figure as the additional moment at the base of the column. Since the displacement is not linearly distributed along the length of the column, a linear distribution of the bending moment (P- Δ contribution in the figure) is not accurate. For any intermediate section the additional bending moment is given by the axial load P acting upon the section's lateral displacement. This effect is known as P-δ (P-small delta). Both effects appear naturally when using the exact solution given by eq. (5) with appropriate boundary conditions.

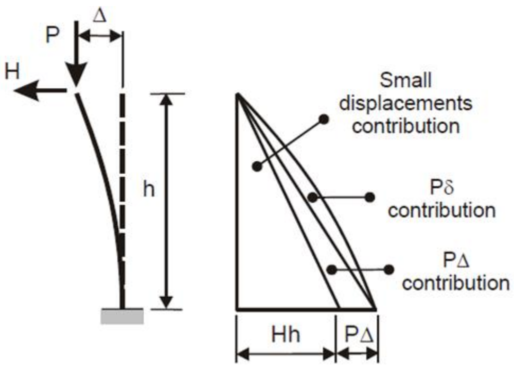

Figure 2. P-Δ and P-δ contributions in a column [7]

After applying boundary conditions, the expression for the bending moment at any point y of the column is:

$$M(y) = Hh\frac{\left[\tan(\mu h)\cos(\mu y) - \sin(\mu y)\right]}{\mu h}. \tag{6}$$

The bending moment at the base of the column is obtained by substituting y=0 in eq. (6). The first order moment given by Hh is multiplied by a factor that tends to infinity as the axial load approaches its critical value:

$$M(0) = Hh\frac{\tan(\mu h)}{\mu h}. \tag{7}$$

The P-δ part of eq. (6) can be obtained by subtracting the small displacements and the P-Δ contributions:

$$M_{P-\delta}(y) = \frac{H}{\mu}\left\{\sin(\mu y) + \left(1 - \frac{y}{h} - \cos(\mu y)\right)\tan(\mu h)\right\}. \tag{8}$$

In this work $M_{P\text{-}\delta}$ will be obtained using shape functions that interpolate the lateral displacement.

3 P-δ modelling

3.1 Interpolation functions

In the context of the direct stiffness method, the analytical behavior of a frame element can be approximated by a discrete behavior. The discrete solution is represented by nodal displacements, while the continuous solution is obtained by interpolating the nodal displacements using shape functions. Figure 3 shows the deformed configuration of an element obtained from interpolation of nodal values. Axial displacement $u(x)$ uses nodal values d_1 and d_4 while transversal displacement $v(x)$ uses d_2, d_3, d_5 and d_6.

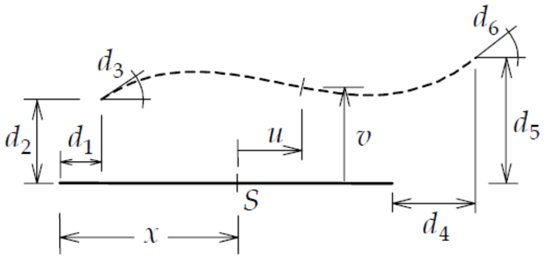

Figure 3. Deformed configuration of a frame element

The deformed shape of an element in terms of lateral displacement and rotations can be written based on the nodal values using interpolation functions, eq. (9):

$$\begin{aligned} v(x) &= N_2^v(x)d_2 + N_3^v(x)d_3 + N_5^v(x)d_5 + N_6^v(x)d_6 \\ \theta(x) &= N_2^\theta(x)d_2 + N_3^\theta(x)d_3 + N_5^\theta(x)d_5 + N_6^\theta(x)d_6 \end{aligned} \rightarrow \begin{Bmatrix} v(x) \\ \theta(x) \end{Bmatrix} = [N]\{d\} \tag{9}$$

The interpolation functions are calculated directly using the homogenous solution of the problem differential equation, i.e., from the equilibrium of a deformed infinitesimal element subjected to a compressive force, as in eq. (5). Displacements and rotations can be written in matrix form:

$$\begin{Bmatrix} v(x) \\ \theta(x) \end{Bmatrix} = [X]\{\alpha\}$$

$$[X] = \begin{bmatrix} \sin(\mu x) & \cos(\mu x) & x & 1 \\ \mu\cos(\mu x) & -\mu\sin(\mu x) & 1 & 0 \end{bmatrix}, \quad \{\alpha\} = \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} \tag{10}$$

Boundary conditions are then imposed by evaluating the homogeneous solution at the bar's nodes:

$$\{d\} = \begin{Bmatrix} d_2 \\ d_3 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} v(0) \\ \theta(0) \\ v(L) \\ \theta(L) \end{Bmatrix} = \begin{Bmatrix} B + D \\ A\mu + C \\ A\sin(\mu L) + B\cos(\mu L) + CL + D \\ A\mu\cos(\mu L) - B\mu\sin(\mu L) + C \end{Bmatrix}$$

$$\{d\} = [H]\{\alpha\}, \quad [H] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \mu & 0 & 1 & 0 \\ \sin(\mu L) & \cos(\mu L) & L & 1 \\ \mu\cos(\mu L) & -\mu\sin(\mu L) & 1 & 0 \end{bmatrix} \tag{11}$$

Finally, using eqs. (10) and (11), the interpolation functions can be obtained from:

$$\begin{Bmatrix} v(x) \\ \theta(x) \end{Bmatrix} = [X][H]^{-1}\{d\} \Rightarrow [N] = [X][H]^{-1} \tag{12}$$

3.2 Tangent stiffness matrix

Due to space limitation, the expressions of interpolation functions will not be presented here. They can be found in [8]. By applying the usual procedure to calculate the stiffness matrix in the context of a Finite Element Analysis, based on the Virtual Work Principle ([9]), the following expressions are obtained:

$$[k] = \int_0^L [B]^{\mathrm{T}}[E][B]dx, \quad [B] = \begin{bmatrix} \dfrac{d(N_2^\theta)}{dx} & \dfrac{d(N_3^\theta)}{dx} & \dfrac{d(N_5^\theta)}{dx} & \dfrac{d(N_6^\theta)}{dx} \\ \dfrac{d(N_2^v)}{dx} & \dfrac{d(N_3^v)}{dx} & \dfrac{d(N_5^v)}{dx} & \dfrac{d(N_6^v)}{dx} \end{bmatrix}, \quad [E] = EI\begin{bmatrix} 1 & 0 \\ 0 & -\mu^2 \end{bmatrix} \tag{13}$$

The unique values for stiffness coefficients (only for the beam part) are given below:

$$\begin{aligned}
&k_{22} = EI\,\frac{\mu^3 \sin(\mu L)}{D}, \quad k_{23} = EI\,\frac{\mu^2\left[1-\cos\left(\mu L\right)\right]}{D} \\
&k_{33} = EI\,\frac{\mu\left[\sin\left(\mu L\right)-\mu L\cos\left(\mu L\right)\right]}{D}, \quad k_{36} = EI\,\frac{\mu\left[\mu L-\sin\left(\mu L\right)\right]}{D} \\
&D = 2-2\cos\left(\mu L\right)-\mu L\sin\left(\mu L\right), \quad \mu=\sqrt{-\frac{P}{EI}}
\end{aligned} \tag{14}$$

The expressions in eq. (14) are meant to be used in a two-cycle framework [10]. In this work, Taylor expansions of the exact expressions are used, up to the cubic term of load *P*, in the context of an updated lagrangian scheme. These expressions can be found in [11].

3.3 Bending moment representation

In a geometrically nonlinear analysis, a body subjected to external loads takes on different configurations as it moves through space and changes its shape. There are different approaches to represent the deformed configuration of an element at step "i' (current) of the analysis: Lagrangian (total or updated) and corotational (Bathe [12]). Figure 4 shows nodal forces acting upon the deformed configuration of the element, considering only pure deformations, i.e., separating the rigid body displacements. Nodal forces are also rearranged as to be aligned with this configuration. In this approach, only rotation related interpolation functions are needed to represent the displacement along the element.

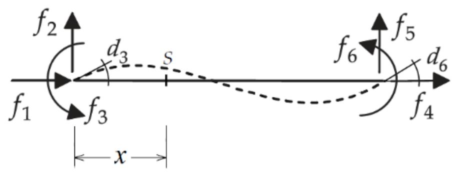

Figure 4. Nodal forces acting on pure deformational configuration

The P-δ contribution of axial force f_1 at any section S can be thought of as the bending moment caused by that force considering the displacement at point *x*. This displacement can be written in terms of nodal values and interpolation functions:

$$M_{P-\delta}\left(x\right)=f_1 v\left(x\right)=f_1\left[N_3^v d_3+N_6^v d_6\right] \tag{15}$$

4 Examples

The expression in eq. (15) was implemented in Ftool (Two-dimensional Frame Analysis Tool) (Martha [5]), considering cubic and trigonometric interpolation functions. The user can choose from a menu whether one or the other are used. The consideration of the P-δ contribution is also optional for the user.

4.1 Cantilever column

The first example is the classic column used for demonstrating the P-Δ effect, as exposed in section 2.2. Loading, material, and geometric parameters are shown in Figure 5. The solution for the bending moment along the column using minimal discretization (1 finite element) is also shown. The nonlinear geometric analysis was performed using 4th order geometric matrix (Taylor series), Updated Lagrangian Description and a load control solution algorithm with adjusted increment type (Rangel and Martha [6]).

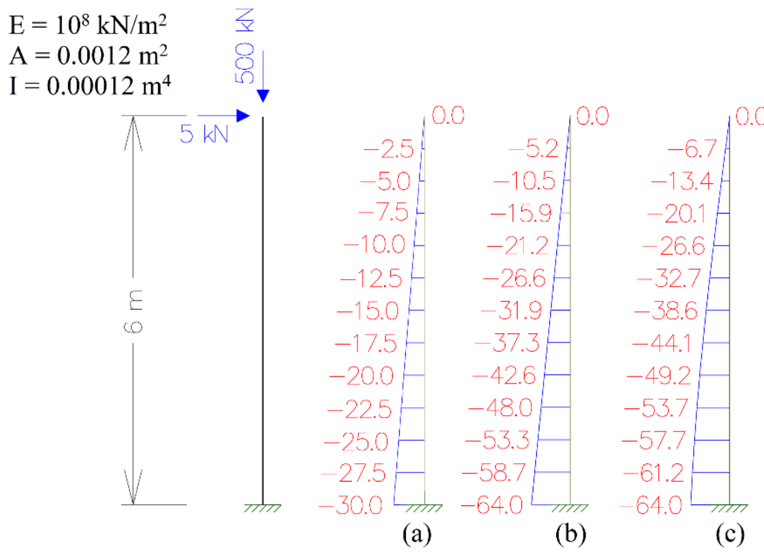

Figure 5. Example 1: (a) 1st order analysis; (b) 2nd order without P-δ; (c) 2nd order with P- δ

The expected value for the bending moment at the base of the column, according to eq. (7) is 67.94 kNm. The error of 5.6% is acceptable considering only 1 finite element was used in this model. The main improvement is the difference in the bending moment at the midpoint of the column. Since normally the distribution is linear, the solution presented in Figure 5b is 17% lower than the one in Figure 5c, in which P-δ is accounted for. The exact value obtained from the differential equation solution is 41.51 kNm.

4.2 Simply supported beam

The second example is a simply supported beam subjected to a compressive load and applied bending moments at its ends. This is an interesting example since even a nonlinear analysis would not be able to capture the bending moment at the midpoint of the bar using only 1 element discretization. Figure 6 shows the model and nonlinear solutions with (6a) and without (6b) P-δ. Material and geometric parameters are the same as the previous example. Parameters regarding the nonlinear solution algorithm are also the same as example 4.1.

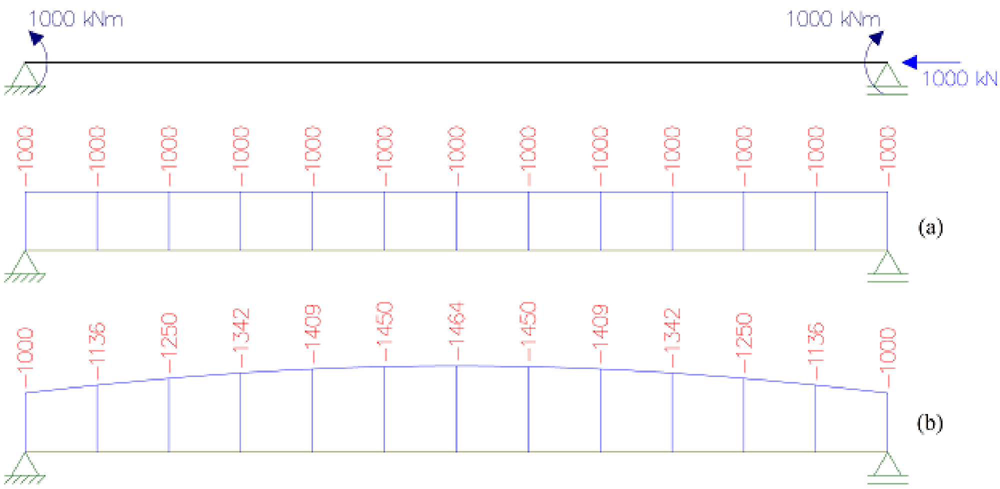

Figure 6. Example 2: (a) 2nd order without P-δ; (b) 2nd order with P- δ

The expected value for the bending moment at the midpoint of the beam is 1543.5 kNm. The error of 5.2% is acceptable considering only 1 finite element was used in this model.

5 Conclusions

This work proposed the use of interpolation functions obtained from the exact solution of the differential equation of an axially loaded beam to represent the bending moment along the element's length, to account for the P-small-delta effect.

Taking advantage of nonlinear analysis algorithms that were implemented in an improved version of Ftool (Two-dimensional Frame Analysis Tool), these interpolation functions were easily incorporated to a context menu in which the user has the option to turn on the local P-delta effect when plotting the bending moment diagram.

Analytical expressions for the P-small delta part of the bending moment were developed to compare with numerical results. Some simple examples for single columns were analyzed and comparisons were made against these analytical solutions.

Since Ftool is widely used in undergraduate courses, the main idea is to use minimal discretization. Of course, there are limitations when using only 1 finite element per bar in a nonlinear analysis framework. Nevertheless, results indicate the ability of the formulation to capture the P-δ effect successfully.

Future works involve extending the formulation to Timoshenko beam theory and to problems with distributed axial load.

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