Simplified analytical and finite element models of the vibration frequencies and buckling loads of a metallic tapered hollow tower.

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Abstract. Structural analysis of slender structures, goes through the study of the static, dynamic and instability. This requires the evaluation of natural frequencies and buckling loads. The Rayleigh method was an analytical formulation used to access these data. To verify the accuracy of the formulation we applied the analytical method to a cantilever metallic hollow tower, used as a small wind turbine tower, to estimate the first eigenfrequency and the buckling load, assuming different shape functions, the results are carefully compared with the results obtained by a finite-element model using the SAP2000 software. The frequency values are also compared with vibrations measurements in the real structure. Results showed that Rayleigh quotient are reliable to evaluate the frequency, with the first frequency in good concordance with FEM simulations and experimental results. Nevertheless, the analytical prediction of buckling loads shows some incongruent results with differences in the range of (3-50)% and highlights the importance of select the appropriate model to represent the structure.

Keywords: eigenfrequency, buckling, Rayleigh's method, FEM analysis.

1 Introduction

Tapered hollow metallic towers are slender structures usually used as support for wind turbines and telecommunication equipment because of their aesthetic and low need of implantation area compared with other solutions, such as lattice guyed towers. The tower should support the dead loads of equipment and additional loads caused by wind, as slender structures vibration and buckling loads (axially and lateral) are significant and can be the main cause of instability. Evaluation of these parameters is essential in the initial phase of the design process, and it is carried out through analytical models. The pre-design geometry is then evaluated in a finite element model (FEM), which consists of a simplification of the structure, but it must adequately represent the real system.

Previous studies for tapered towers buckling usually consider an I-beam section or a solid section [1]–[7], however, the thin hollow tower behavior is quite different essentially when analyzing the buckling load. Several studies proposed numerical formulations for frequency estimation [8]–[12]. Some simplified models for frequency estimation are more accurate than others, the Rayleigh's method and SDOF reduction method [8] proved to be reliable formulations. The Rayleigh method is quite important here since it allows the easy evaluation of both the frequency and critical load, and the results depend entirely on the shape function that is assumed to approximate the exact mode shape.

The main objective of this study is to evaluate the accuracy of the analytical models on the prediction of vibration frequencies and buckling loads when applied to structures with variable inertia. The analytical models applied to a small wind tower were compared with FEM models, which comprise models using beam and shell elements. The simplified FEM model consists of beam elements, to evaluate the uncertainty of model parameters or inaccuracies in modeling, the frequencies results were confronted with experimental results of in-situ vibration test, using Operational Modal Analysis (OMA).

2 Analytical formulation

The classical Rayleigh's Method is based on the principle of conservation of energy, basically a single degree of freedom system, is used to describe a continuous system, describing their properties as a function of generalized coordinates [13], [14].

Considering a cantilever beam with a distributed mass with the displacement of the tower in simple harmonic motion, given by:

$$y(z,t) = z_0 \cdot \sin \omega_n t \cdot \psi(z) \tag{1}$$

Where z_0 is the amplitude of the generalized coordinate system z(t), $\psi(z)$ is the assumed shape function and ω_n is the natural frequency to be determined.

The maximum potential energy (U) of the system over a vibration cycle is equal to the strain energy associated with the maximum displacement $y_0(z)$

$$U = \int_0^L \frac{EI(z)y''(z)^2}{2} dz = \int_0^L \frac{{z_0}^2 EI(z)[\psi''(z)]^2}{2} dz \tag{2}$$

Where, E is the elasticity modulus, I(z) is the second moment of inertia variation over the length L, and $y''(z)$ denotes the second derivative of the displacement y(z). The maximum kinetic energy of the nonuniformly distributed mass $m(z)$ is:

$$C = \int_0^L \frac{m(z)y'(z)^2}{2} dz = \int_0^L \frac{{z_0}^2 {\omega_n}^2 m(z)[\psi(z)]^2}{2} dz \tag{3}$$

Equation the maximum potential energy to the maximum kinetic energy, the squared frequency is found to be:

$$\omega_n^2 = \frac{\int_0^L EI(z)[\psi''(z)]^2 dz}{\int_0^L m(z)[\psi(z)]^2 dz} \tag{4}$$

when the structure has a tip mass M_t on the top the formula become:

$$\omega_n^2 = \frac{\int_0^L EI(z)[\psi''(z)]^2 dz}{\int_0^L m(z)[\psi(z)]^2 dz + M_t} \tag{5}$$

For the axial buckling problem, a similar analogy can be used but now the work done by the force P applied to the system is stored as stretching strain energy, as the system allow a bending deformation $\psi(z)$, the change in the potential energy is:

$$\Delta U = \int_0^L \frac{EI(z)[\psi''(z)]^2}{2} dz - \int_0^L \frac{P[\psi'(z)]^2}{2} dz \tag{6}$$

The critical load is found when $\Delta U = 0$

$$P = \frac{\int_0^L EI(z)[\psi''(z)]^2 dz}{\int_0^L [\psi'(z)]^2 dz} \tag{7}$$

Timoshenko [15] used another formulation for the strain energy and the critical load is calculated by:

$$P' = \frac{\int_0^L [\psi'(z)]^2 dz}{\int_0^L \frac{[\delta - \psi(z)]^2}{EI(z)} dz} \tag{8}$$

Where δ is the deflection on the free end of the tower, this formula is known as the Timoshenko quotient more accurate since depends only on the approximation of the deflected shape, which is easier to obtain than the $\psi''^{(z)}$ according to Lang [16].

In the scope of the lateral instability, the buckling load can be calculated by a direct method according to Barros [17]

$$P_L = \frac{E\psi(L)}{\int_0^L \frac{\psi(z)z}{I(z)} dz} \tag{9}$$

3 Tower and investigation description

3.1 Description of the tower

A steel tower presented in Fig. 1 with height (L) of 17.8 m is located at School of Technology and Management of the Polytechnic Institute of Bragança (Portugal), consists of a steel S275 structure with a hexdecagonal section with a diameter d_b = 0.5890 m at the base and d_t = 0.1954 m at the top and constant wall thickness e = 4 mm.

Figure 1 - Small wind turbine tower, Bragança - Portugal

The tower is fixed to the gravity-based foundation by 16 anchor bolts connected to a flange at the base, the flange has an external diameter of 0.7960 m and thickness of 20 mm, the main properties are shown in Tab 1.

Table 1 - Tower properties

Density (ρ)	7850.000	[kg/m³]
Young's modulus	210.000	[GPa]
Poisson's ratio	0.300	--
Yield strength	275.000	[MPa]
L	17.800	[m]
$\boldsymbol{d_t}$	0.195	[m]
$\boldsymbol{d_b}$	0.589	[m]
$\boldsymbol{e}$	0.004	[m]
$\boldsymbol{A_b}$	$\frac{\pi}{4}({d_b}^2 - (d_b - 2e)^2)$	[m^2]
$\boldsymbol{I_b}$	$\frac{\pi}{64}({d_b}^4 - (d_b - 2e)^4)$	[m^4]

$\boldsymbol{I_t}$	$\frac{\pi}{64}(d_t{}^4 - (d_t - 2e)^4)$	[m^4]
$\boldsymbol{\eta}$	$1 - \left(\frac{I_t}{I_b}\right)^{1/3}$	
$\boldsymbol{m(z)}$	$\rho A_b(1 - \eta\frac{z}{17.8})$	[kg/m]
$\boldsymbol{I(z)}$	$I_b\left(1 - \eta\frac{z}{17.8}\right)^3$	[m^4/m]
$\boldsymbol{M_t}$	75.000	[kg]
$\boldsymbol{M_{tower}}$	664.277	[kg]

3.2 Operational Modal Analysis

To determine the structure's dynamic characteristics, the experimental modal analysis is carried out using a data acquisition system composed of Sirius modular data acquisition system, data processing computer SBOX and DewesoftX software, with 8 shear accelerometers placed in the x and y direction at height z=1 and 2.2 m. The first frequency is 1.55 Hz and the second is 7.4 Hz.

3.3 Numerical model

The analysis of the structure using the finite element method is performed using the SAP2000, considering the structure as a beam element and shell elements. The tower is modeled with the geometry presented above, considering the bottom boundary condition the type BC1f (CEN, 2007) in the bottom for the beam element, in the location of the 16 anchor bolts for the shell structure, the top boundary is considered free with and without the action of a tip mass of 75 kg, which is the weight of the wind turbine. The simulation is performed considering the linear and nonlinear behavior of the material. The buckling factor was found applying a unitary concentrated load on top of the structure in the axial direction (-z) and lateral direction (x or y).

4 Results and discussion

As already shown in the literature the effectives of the Rayleigh method is intrinsically dependent on the equation used as approximate mode shape. Figure 2 presents the normalized mode shape obtained by FEM shell model and the three equations used as approximation for the deformed shape.

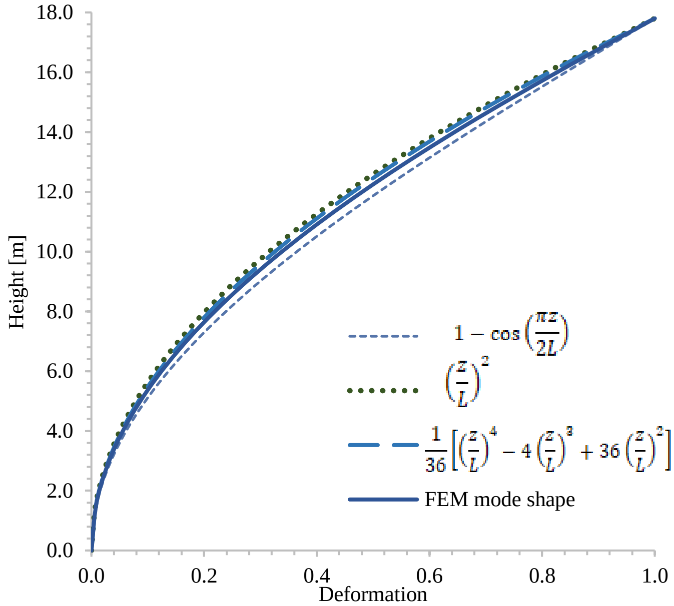


Figure 2 – Normalized mode shapes

Table 2 presents the results of the frequency evaluation, with the last row presenting the relative error between the values of the frequencies with tip mass and the experimental one. The results are congruent with the experimental value, as expected all the results are higher than the exact value. The frequency results of the FEM models are similar for both elements. When analyzing the buckling load presented in Tab. 3, the direct method could provide an upper bound for the critical load, however the relative error between the analytical model and the FEM shell models are higher for the Rayleigh quotient than for the Timoshenko quotient, as presented in the literature for the axial load. The lateral load analyzed by the direct method also shown significantly higher differences.

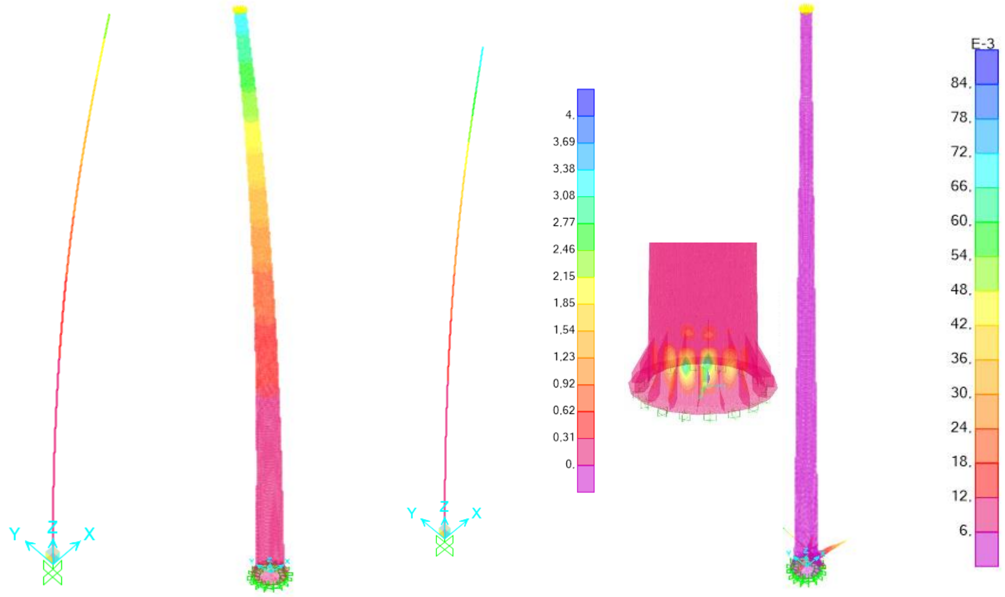
Table 2 – First eigenfrequency

Rayleigh Method				FEM		Experimental
Deformed shape $\psi(z)$	$1-\cos\left(\frac{\pi z}{2L}\right)$	$\left(\frac{z}{L}\right)^2$	$\frac{1}{33}\left[\left(\frac{z}{L}\right)^4-4\left(\frac{z}{L}\right)^3+36\left(\frac{z}{L}\right)^2\right]$	Beam	Shell	
Without M_t	2.17 Hz	2.18 Hz	2.18 Hz	2.17 Hz	2.15 Hz	-
With M_t	1.66 Hz	1.62 Hz	1.63 Hz	1.61 Hz	1.60 Hz	1.55 Hz
Error	7.35 %	4.22%	4.99%	3.87%	3.16%	-

Table 3 - Buckling loads

Axial load								
	Analytical						**FEM**		
Deformed shape $\psi(z)$	$1-\cos\left(\frac{\pi z}{2L}\right)$		$\left(\frac{z}{L}\right)^2$		$\frac{1}{33}\left[\left(\frac{z}{L}\right)^4-4\left(\frac{z}{L}\right)^3+36\left(\frac{z}{L}\right)^2\right]$		Beam		Shell
	P_{cr} (kN)	Error (%)	P_{cr} (kN)	Error (%)	P_{cr} (kN)	Error (%)	$P_{b,FEM}$ (kN)	Error (%)	$P_{s,FEM}$ (Kn)
Rayleigh quotient P	286.07	49.47	229.70	20.02	239.51	25.14	191.68	0.15	191.39
Timoshenko quotient P'	206.45	7.87	197.03	2.95	198.26	3.59			
Lateral load									
	Analytical						**FEM**		
Deformed shape $\psi(z)$	$1-\cos\left(\frac{\pi z}{2L}\right)$		$\left(\frac{z}{L}\right)^2$		$\frac{1}{33}\left[\left(\frac{z}{L}\right)^4-4\left(\frac{z}{L}\right)^3+36\left(\frac{z}{L}\right)^2\right]$		Beam		Shell
	P_{Lcr} (kN)	Error (%)	P_{Lcr} (kN)	Error (%)	P_{Lcr} (kN)	Error (%)	$P_{b,LFEM}$ (kN)	Error (%)	$P_{s,LFEM}$ (Kn)
Direct method Pl	60.01	12.57	62.79	17.78	62.29	16.84	-	-	53.51

This is due to the buckling behavior of the structure, as a slender thin tower the buckling shape for an axial load is similar to the shape of a Euler cantilever beam, but when the load is applied in the lateral direction the buckling shape presents a shell buckling shape, as seen in Fig 3. (d).



(a) Beam axial (b) Shell axial (c) Beam lateral (d) Shell lateral

Figure 3 – First buckling mode FEM model

5 Conclusions

The metallic hollow cantilever tapered tower analyzed has the first natural frequency of 1.55 Hz, axial buckling load 191.39 kN, and lateral buckling load of 53.31 kN. This frequency and buckling loads are accurately evaluated by the application of the shell element in the finite-element model, even though this element is time-consuming, the use of beam elements leads to inaccurate values for instability analyses and does not reflect the real behavior of the structure.

Rayleigh's method shows accurate results for frequency estimation but not for the axial buckling load, where the Timoshenko method proved to be more reliable. The dispersion of the buckling loads' results reflects the importance of accurate assessment of the structural behavior, nevertheless, further investigation on the instability of tapered tower with tip mass should be carried out.

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References

[1] S. M. Darbandi, R. D. Firouz-Abadi, and H. Haddadpour, "Buckling of Variable Section Columns under Axial Loading," *J. Eng. Mech.*, vol. 136, no. 4, pp. 472–476, 2010.
[2] M. A. Rosa and C. Francisconi, "The optimized rayleigh method and mathematica in vibrations and buckling problems," *J. Sound Vib.*, vol. 191, no. 5, pp. 795–808, 1996.

[3] F. Mohri, C. Bouzerira, and M. Potier-Ferry, "Lateral buckling of thin-walled beam-column elements under combined axial and bending loads," *Thin-Walled Struct.*, vol. 46, no. 3, pp. 290–302, 2008.
[4] K. S. Lee and H. J. Bang, "A study on the prediction of lateral buckling load for wind turbine tower structures," *Int. J. Precis. Eng. Manuf.*, vol. 13, no. 10, pp. 1829–1836, 2012.
[5] B. K. Lee and S. J. Oh, "Elastica and buckling load of simple tapered columns with constant volume," *Int. J. Solids Struct.*, vol. 37, no. 18, pp. 2507–2518, 2000.
[6] A. de M. Wahrhaftig, M. A. da Silva, and R. M. L. R. F. Brasil, "Analytical determination of the vibration frequencies and buckling loads of slender reinforced concrete towers," *Lat. Am. J. Solids Struct.*, vol. 16, no. 5, 2019.
[7] L. Marques, A. Taras, L. Simões Da Silva, R. Greiner, and C. Rebelo, "Development of a consistent buckling design procedure for tapered columns," *J. Constr. Steel Res.*, vol. 72, pp. 61–74, 2012.
[8] A. Dick, R. C. Barros, and M. T. B. César, "Simplified methods for frequency estimation of small wind turbine tower.," in *8th ECCOMAS Thematic Conference on Computational Methods in Structural Dynamics and Earthquake Engineering*, 2021.
[9] R. D. Blevins, *Formulas for natural frequency and mode shape.* 1979.
[10] C. W. Bert, "Application of a version of the Rayleigh technique to problems of bars, beams, columns, membranes, and plates," *J. Sound Vib.*, vol. 119, no. 2, pp. 317–326, 1987.
[11] R. Balagopal, N. Prasad Rao, and R. P. Rokade, "Simplified Model to Predict Deflection and Natural Frequency of Steel Pole Structures," *J. Inst. Eng. Ser. A*, vol. 99, no. 3, pp. 595–607, 2018.
[12] S. M. Avila, M. A. M. Shzu, M. V. G. Morais, and Z. J. G. Del Prado, "Numerical modeling of the dynamic behavior of a wind turbine tower," *J. Vib. Eng. Technol.*, vol. 4, no. 3, pp. 249–257, 2016.
[13] J. W. S. B. Rayleigh, *The Theory of Sound*, 2nd ed., no. v. 1. Macmillan, 1894.
[14] A. de M. Wahrhaftig and A. de M. Wahrhaftig, "Proposta para o cálculo da frequência natural de vibração sob não-linearidade geométrica," *Ação do Vent.*, no. 1877, pp. 63–76, 2017.
[15] S. P. Timoshenko and J. M. Gere, *Theory of elastic stability*, 2nd ed. McGraw-Hill, 1985.
[16] H. A. Lang, "Note on Rayleigh's method and the non-uniform strut," *Q. Appl. Math.*, vol. V, no. 4, pp. 1947–1948, 1948.
[17] R. C. Barros, J. M. Freitas, J. C. Matos, and A. C. Lopes, "Dimensionamento estrutural de postes metálicos de iluminação", IV Congresso de Construção Metálica e Mista, ISBN: 972983765-1, pp. 261-274, CMM, Lisboa, Portugal, Dezº 2003.