Modeling of a broadband double-beam piezoelectric energy harvesting system

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Abstract. In this work, a mathematical model, that describes the mechanical and electrical behaviors of an energy harvesting system composed by a cantilever beam (primary beam) and a bimorph piezoelectric beam (energy harvester) attached to the primary one, is presented. From a mechanical excitation (acceleration) at the base of the primary beam, a voltage is generated in the piezoelectric. The primary beam is modeled as a multiple degrees of freedom system, while the harvester is modeled as an equivalent spring (dynamic stiffness), which has the mechanical and electrical characteristics of the piezoelectric beam. In order to obtain the equivalent stiffness, the harvester is modeled as a single degree of freedom system. The equations of motion of the composite system are obtained from Lagrange Equations. Using the generated voltage and the acceleration at the base of the primary beam signals, a Frequency Response Function (FRF) was calculated, which can be used to identify a physical system of this kind and to optimize its physical parameters in order to maximize power in a desired frequency range. The numerical results showed the ability of the proposed compound system to generate energy over a wide frequency range.

Keywords: energy harvesting, piezoelectric materials, mechanical vibrations, Lagrange Equations.

1 Introduction

The demand for autonomous electronic devices has grown in the last two decades, motivating researches aiming to obtain alternative ways of generating electricity [1], [2]. In this context, the concept of energy harvesting arises, representing the conversion of the energy available in nature into a useful form, such as electricity [3].

Energy harvesters generate a small amount of energy that can be used to power electronic devices or charge wireless sensors [4], [5]. Piezoelectric harvesters have gained prominence, mainly due to their high power density [6] and their usefulness in microtechnology manufacture [7].

Most piezoelectric energy harvesters are given by cantilever beam systems, due to their resonance at low frequencies and the high generation of electrical energy at resonance [8]. These systems can be unimorph or bimorph, characterized by the presence of piezoelectric material on one or both sides of the beam. The piezoelectric beam is attached to the structure subject to mechanical vibrations and the deformation induced in the piezoelectric is converted into a voltage output. However, using a single piezoelectric beam, power generation is limited to frequencies close to its resonance.

An option to ensure higher energy generation over a wide range of excitation frequencies, enabling easier tuning, is by using a cantilever beam (primary beam) and a piezoelectric beam, attached to the primary one. In this work, a mathematical model, that describes the mechanical and electrical behavior of this coupled system, using a bimorph piezoelectric, is presented.

2 Methodology

The system to be modeled is composed by a cantilever steel beam (primary beam) and a bimorph piezoelectric beam (energy harvester) attached to the steel one. The primary beam base is excited with a known acceleration and, from the mechanical vibrations transmitted to the piezoelectric, a voltage is generated and can be captured. From the base excitation input signal and the captured voltage signal, the voltage-acceleration Frequency Response Function (FRF) of the system is calculated.

2.1 Compound system model

In the compound system model, the steel beam is considered continuous, with its respective characteristic properties. The primary beam is excited with a displacement $g(t)$ at its base, and responds with a relative transverse displacement $w(x,t)$. Thus, the absolute displacement of the beam is given by $w(x,t) + g(t)$. The piezoelectric, represented by the spring, is set at a distance x_h from the base of the steel beam. However, unlike the classical model, represented by Fig 1(a), where the harvester would have displacement $u(x,t)$, the piezoelectric energy harvester is considered as a spring with an equivalent dynamic stiffness dependent on the excitation frequency (K_{eq}=$K_{eq}(\Omega)$), which has all its dynamic, mechanical and electrical characteristics, and represents the influence that the piezoelectric beam has on the primary system. This proposed theoretical system is represented by Fig. 1(b).

(a) (b)

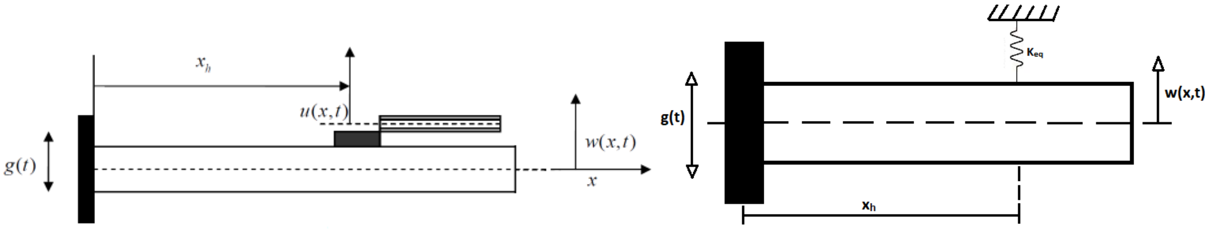

Figure 1. Compound system model

For a given excitation frequency Ω, the kinetic (T) and potential (U) energy of the system are given by eq. (1) and eq. (2), respectively:

$$T = \frac{1}{2}\rho_b A_b \int_0^{L_b} [\dot{w}(x,t) + \dot{g}(t)]^2 \, dx \ , \tag{1}$$

$$U = \frac{1}{2} E_b I_b \int_0^{L_b} w''(x,t)^2 dx + \frac{1}{2} K_{eq}(\Omega)[w(x_h,t) + g(t)]^2 \, . \tag{2}$$

ρ_b, A_b, E_b, L_b are the specific mass, cross-sectional area, Young's modulus and primary beam length, respectively, $w''(x,t)$ is the second order derivative of w with respect to x and I_b is the moment of inertia of the cross section. The proposed solution to the problem is given by eq. (3):

$$w(x,t) = \sum_{r=1}^{n} a_r(t)\phi_r(x) \, , \tag{3}$$

where $a_r(t)$ and $\phi_r(x)$ are the generalized coordinates and assumed vibration modes of the primary beam, respectively, and n cantilever beam assumed modes are considered.

Applying the Lagrange Equations, given by eq. (4), and coupling the damping matrix C^a, one arrives at eq. (5):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{a}_j}\right) - \frac{\partial T}{\partial a_j} + \frac{\partial U}{\partial a_j} = 0 \, , \tag{4}$$

$$M^a \ddot{\boldsymbol{a}} + C^a \dot{\boldsymbol{a}} + K^a \boldsymbol{a} = \boldsymbol{f}_1 \, , \tag{5}$$

where the $n \ x \ n$ matrices M^a and K^a, and the $n \ x \ 1$ vector $\boldsymbol{f}_1$, are given by:

$$M_{rl}^a = \rho_b A_b \int_0^{L_b} \phi_r \phi_l \, dx \, , \tag{6}$$

$$K_{rl}^a = E_b I_b \int_0^{L_b} \phi_r'' \phi_l'' \, dx + K_{eq}\phi_r(x_h)\phi_l(x_h) \, , \tag{7}$$

$$f_{1,r} = -m_b \ddot{g} \int_0^{L_b} \phi_r \, dx - K_{eq}\phi_r(x_h) g \, . \tag{8}$$

In order to obtain the solution of the homogeneous differential equation, one can define $\lambda = \Omega^2$ and ${\Omega_i}^2 = k_i/m_i$ (where Ω_i are the system natural frequencies), obtaining the eigenvalue problem given by eq. (9):

$$\lambda_i M^a \boldsymbol{\theta}_i = K^a \boldsymbol{\theta}_i \,, \tag{9}$$

where $\Theta = [\boldsymbol{\theta}_i]$ is the system eigenvectors matrix.

From the orthogonality properties of the eigenvectors given by eqs. (10) and (11), it is possible to normalize the eigenvectors $\boldsymbol{\theta}_i$, as given by eq. (12), such that ${\Theta^0}^{\boldsymbol{T}} M^a \Theta^0 = I$ and ${\Theta^0}^T K^a \Theta^0 = \Lambda$, where I is the identity matrix and Λ is called spectral matrix:

$$\Theta^T M^a \Theta = \begin{bmatrix} \ddots & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & \ddots \end{bmatrix}, \tag{10}$$

$$\Theta^T K^a \Theta = \begin{bmatrix} \ddots & 0 & 0 \\ 0 & k_i & 0 \\ 0 & 0 & \ddots \end{bmatrix}, \tag{11}$$

$$\boldsymbol{\theta}_i^0 = \frac{\boldsymbol{\theta}_i}{\sqrt{m_i}} . \tag{12}$$

Performing the coordinate transformation described by eq. (13), pre-multiplying eq. (9) by theta and moving to the frequency domain, one arrives at eq. (14):

$$a(t) = \Theta^0 p(t) \,, \tag{13}$$

$$[-\Omega^2 I + i\Omega\Gamma + \Lambda]P(\Omega) = \Theta^T F_1(\Omega) \,, \tag{14}$$

where:

$$\Gamma = \begin{bmatrix} \ddots & 0 & 0 \\ 0 & 2\xi_i\Omega_i & 0 \\ 0 & 0 & \ddots \end{bmatrix}, \tag{15}$$

$$\Lambda = \begin{bmatrix} \ddots & 0 & 0 \\ 0 & {\Omega_i}^2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}, \tag{16}$$

and ξ_i are the system's modal damping ratios.

Solving eq. (14) and returning with the coordinate transformation given by eq. (13), it is possible to obtain the solution for $A(\Omega) = \Theta^0 P(\Omega)$:

$$A(\Omega) = \Theta \begin{bmatrix} \ddots & 0 & 0 \\ 0 & \frac{1}{-\Omega^2 + i\Omega(2\xi_i\Omega_i) + {\Omega_i}^2} & 0 \\ 0 & 0 & \ddots \end{bmatrix} \Theta^T F_1(\Omega) \,, \tag{17}$$

where:

$$F_1(\Omega) = 2[\Omega^2 I_\phi^a - k_{eq\phi}^a] G(\Omega) \,, \tag{18}$$

$$I_\phi^a = m_b \int_0^{L_b} \phi_r \, dx \,, \tag{19}$$

$$k_{eq\phi}^a = K_{eq} \phi_r(x_h) \,. \tag{20}$$

The voltage-acceleration FRF of the system is given by eq. (21):

$$H_V(\Omega) = \frac{V_p(\Omega)}{Y(\Omega)} = \frac{V_p(\Omega)}{-\Omega^2 G(\Omega)} , \tag{21}$$

where $V_p(\Omega)$ is the voltage amplitude generated by the piezoelectric and $Y(\Omega) = -\Omega^2 G(\Omega)$ is the acceleration amplitude excitation at the base of the primary system.

In order to obtain the voltage amplitude generated by the energy harvester, it is necessary to find the displacement at its base $(x_b(t))$, which is equal to the absolute displacement of the primary system at the coupling point. Thus, using eq. (3), one arrives at eqs. (22) and (23) in time and frequency domain, respectively:

$$x_b(t) = w(x_h, t) + g(t) = \sum_{r=1}^{n} a_r(t)\phi_r(x_h) + g(t) \,. \tag{22}$$

$$X_b(\Omega) = \sum_{r=1}^{n} A_r(\Omega)\phi_r(x_h) + G(\Omega) \,. \tag{23}$$

Therefore, in order to obtain the displacement at the base of the harvester, it is necessary to find the value of $K_{eq}(\Omega)$ and use the equations presented in this section. On the other hand, in order to obtain the value of $K_{eq}(\Omega)$, a mathematical model for the piezoelectric energy harvester is necessary, which will also be used to obtain the generated voltage and, consequently, the voltage-acceleration FRF.

2.2 Energy harvester model

This paper aims to model a bimorph energy harvester, consisting of 2 layers of piezoelectric element, with thickness h_p each, separated by 1 layer of metallic substrate, with thickness h_s, as shown in Fig. 2. The harvester has length L_h and width b_h.

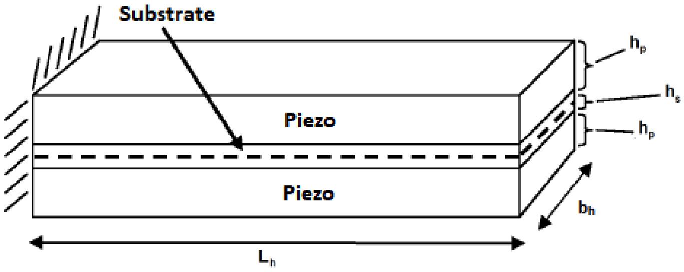

Figure 2. Piezoelectric energy harvester [9]

In this model, the piezoelectric energy harvester is considered as a single degree of freedom system, composed by an equivalent mass (m_h), rigidity (k_h) and damping (c_h^*), which has both the mechanical part (c_h) and the electrical part coupled [8]. This system is excited at the massless base with a force $f(t)$, and responds with a displacement $x_h(t)$ at the mass m_h and with $x_b(t)$ at the base, which has no mass ($m_b = 0$), as shown in Fig. 3 (a). A methodology for finding m_h and k_h was developed by Rao [10]. Furthermore, $c_h = 2\xi_h m_h \Omega_n$, where ξ_h is the modal damping ratio and Ω_n is the natural frequency of the single degree of freedom system.

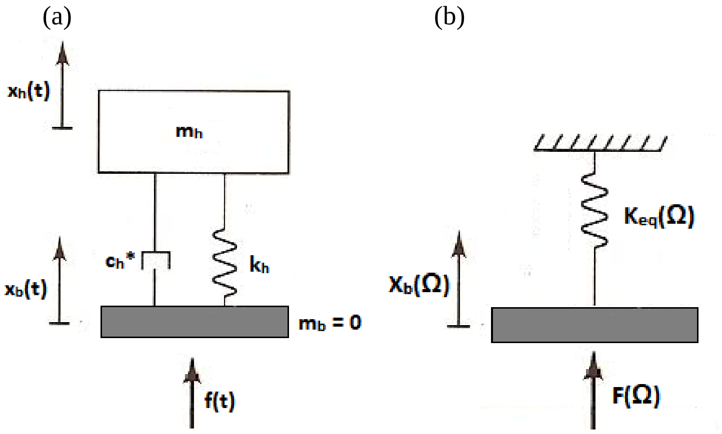

Figure 3. (a) System I - single degree of freedom harvester model and (b) System II - equivalent dynamic stiffness model

In order to find the equivalent dynamic stiffness ($K_{eq}(\Omega)$) imposed by this model, the system shown in Fig. 3(a) (System I) must be transformed into a system with an equivalent spring, in the frequency domain, as shown in Fig. 3(b) (System II) [11].

Applying Newton's 2nd law to System I, it is possible to obtain eq. (24):

$$f(t) - k_h(x_b - x_h) - c_h^*(\dot{x}_b - \dot{x}_h) = 0\,, \tag{24}$$

where

$$c_h^* = c_h + \frac{\tilde{\theta}^2}{i\Omega C_p + \frac{1}{R_l}}\,. \tag{25}$$

Besides, according to Kundu and Nemade [8], the governing equations of the harvester are given by:

$$m_h \ddot{x}_h + c_h(\dot{x}_h - \dot{x}_b) + k_h(x_h - x_b) + \tilde{\theta} v_p = 0\,, \tag{26}$$

$$C_p\dot{v}_p - \tilde{\theta}(\dot{x}_h - \dot{x}_b) + \frac{v_p}{R_l} = 0\,, \tag{27}$$

where R_l is the equivalent resistance of an external circuit supplied by the harvester, $v_p = v_p(t)$ is the generated voltage, C_p is the total capacitance of the harvester, calculated from the connection of two equal parallel capacitors $\left(C_p = 2C_{\tilde{p}} = 2\frac{\varepsilon_{33}^{\sigma} b_h L_h}{h_p}\right)$, where $\varepsilon_{33}^{\sigma}$ is the dielectric permittivity constant at constant voltage. Moreover, $\tilde{\theta}$ is the electromechanical coupling vector, given by $\tilde{\theta} = e_{31}b_h(h_p + h_s)\alpha'(x)$, where $\alpha'(x)$ is the spatial derivative of the harvester's characteristic vibration form ($\alpha(x)$ is the first mode of a cantilever beam) and e_{31} is the effective piezoelectric voltage constant, given by $e_{31} = d_{31}c_{11}^{E}$, with d_{31} being the piezoelectric constant and c_{11}^{E} being the Young's modulus of the piezoelectric material at constant electric field.

In the frequency domain, eqs. (24), (25) and (26) become, respectively:

$$F(\Omega) = k_h(X_b - X_h) + c_h^* i\Omega(X_b - X_h)\,, \tag{28}$$

$$-\Omega^2 m_h X_h = k_h(X_b - X_h) + c_h i\Omega(X_b - X_h) - \tilde{\theta}V_p\,, \tag{29}$$

$$\left(i\Omega C_p + \frac{1}{R_l}\right)V_p + \tilde{\theta} i\Omega(X_b - X_h) = 0\,. \tag{30}$$

Solving this system of equations for V_p and $F(\Omega)$, one arrives at eqs. (31) and (32):

$$V_p = \frac{-\tilde{\theta} i\Omega(X_b - X_h)}{i\Omega C_p + \frac{1}{R_l}}\,, \tag{31}$$

$$F(\Omega) = (k_h + i\Omega c_h^*)\left[X_b\left(1 - \frac{k_h + i\Omega c_h^*}{-\Omega^2 m_h + k_h + i\Omega c_h^*}\right)\right]. \tag{32}$$

Applying Newton's 2nd law to System II, it is possible to obtain eq. (33):

$$F(\Omega) = K_{eq}(\Omega)X_b(\Omega)\,. \tag{33}$$

Thus, from eqs. (32) and (33), it is possible to obtain the expression for the equivalent dynamic stiffness:

$$K_{eq}(\Omega) = \frac{F(\Omega)}{X_b(\Omega)} = \frac{(k_h + i\Omega c_h^*)(-\Omega^2 m_h)}{-\Omega^2 m_h + k_h + i\Omega c_h^*}\,. \tag{34}$$

From eq. (21), one can obtain the expression for the voltage-acceleration FRF:

$$H_V(\Omega) = \frac{V_p(\Omega)}{-\Omega^2 G(\Omega)} = \frac{\tilde{\theta} i(X_b - X_h)}{\Omega G(\Omega)\left(i\Omega C_p + \frac{1}{R_l}\right)}\,, \tag{35}$$

where

$$X_h = \frac{k_h + i\Omega c_h^*}{-\Omega^2 m_h + k_h + i\Omega c_h^*} X_b\,. \tag{36}$$

3 Numerical Results

In order to obtain numerical results for the presented model, a system compound by a cantilever steel beam (primary beam) and a bimorph piezoelectric beam with brass substrate (Q220-H4BR-2513YB, Piezo.com) was selected. The harvester was set at the primary beam at $x_h = 82.0\ mm$. The physical and geometrical constants of both beams are shown in Tab. 1.

Table 1. Physical system constants

Constant	Steel beam	Piezoelectric	Brass substrate
Length L [mm]	114.0	63.5	63.5
Width b [mm]	32.5	32.0	32.0
Thickness h [mm]	0.5	0.19	0.13
Young Modulus E [GPa]	190	50	100
Density ρ [kg/m³]	7850	7800	8300
Capacitance C_p [nF]	-	540	-
Piezoelectric constant d_{31} [m/V]	-	-320 x 10^{-12}	-
Modal damping ratio ξ [-]	0.0080	0.0020	0.0020

Figure 4 shows the voltage-acceleration FRF modulus for different values of R_l, from 10 to 100 Hz, showing the first two resonant frequencies and the ability of the system to generate a sustained voltage of 13 V/(m/s²) (for $R_l = 1000\ k\Omega$), between these two frequencies. On the other hand, Fig. 5 shows the equivalent stiffness K_{eq} modulus, which brings out the single degree of freedom nature of the piezoelectric model.

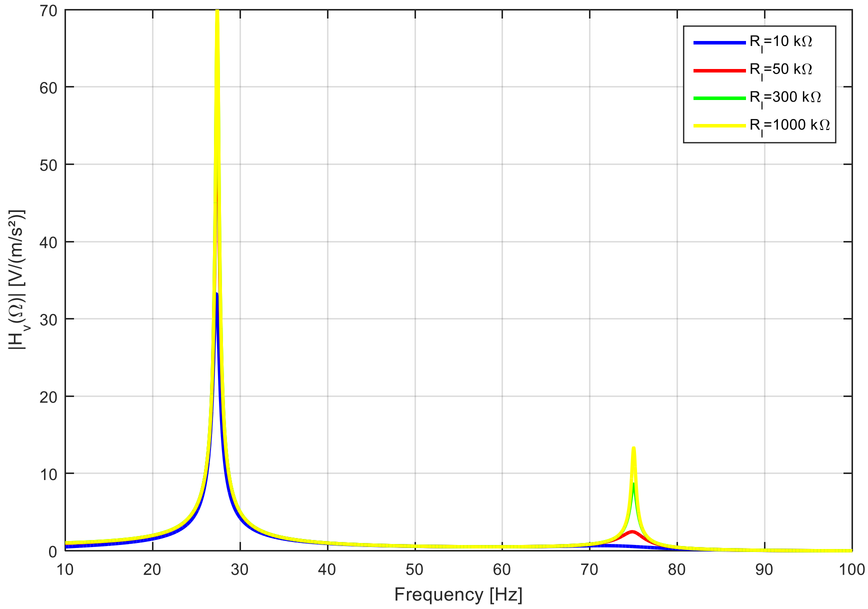

Figure 4. System I - single degree of freedom harvester model

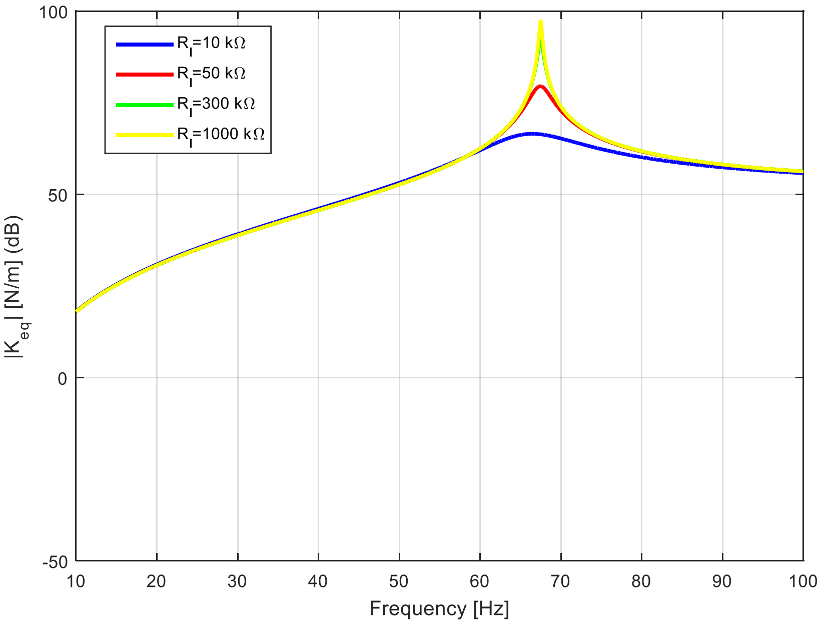

Figure 5. System I - single degree of freedom harvester model

4 Conclusions

The energy harvesting system presented is capable of generating energy in a wider range of excitation frequencies than a system composed by a single piezoelectric beam. The coupling of a steel beam to the harvester is capable of adding and altering the natural frequencies of the system, in order to tune the system with external excitation frequencies.

The modeling of the energy harvester as a spring with an equivalent dynamic stiffness dependent on the excitation frequency $\left(K_{eq} = K_{eq}(\Omega)\right)$, which has all its dynamic, mechanical and electrical characteristics, is unprecedented and can be used in conjunction with other models. In addition, the model presented can be used to identify a physical system of this type and/or to optimize its physical parameters, such as the length of the primary beam or external electrical resistance, in order to generate the maximum power of energy in a desired frequency range (operating range).

On the other hand, the single degree of freedom model for the harvester is an approximation, and not as robust as a continuous model, for example. Furthermore, the presented linear model is valid for small accelerations, in the order of up to 0.1 g, as shown by Stanton *et al* [12] and Leadenham and Erturk [13]. For higher accelerations, nonlinear models should be considered.

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