Remarks on nonlinear dynamics of a suspension bridge model

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Abstract. The following paper aims to analyze the Lazer-McKenna suspension bridge model, a system with a nonlinear response. The proposed model has as a difference the presence of a rubber band in a mass-spring-damper system, where the rubber band exerts force only against the extension. The dynamic response is analyzed in time and frequency domains with time tools like Phase Portrait, Poincare Maps and frequency tools like Fast Fourier transform, and Continuous Wavelets transform, where the nonlinearity is investigated. Both the time response, obtained by numerically solving the second order differential equation with Runge-Kutta methods, and the signal processing were obtained using the python programming language.

Keywords: Time-frequency Analysis, Nonlinear Dynamics, Chaos, Lazer-McKenna suspension bridge model.

1 Introduction

A famous problematic case in dynamics of structures was the Tacoma Narrows Bridge ,where the linear model failed to demonstrate certain patterns presented by it, later (1940) it collapsed and gave way. From this, there was a need to look for non-linear modeling, which exhibits characteristics that cannot be predicted using a simpler model. Over the years, this case has become a classic problem for introduction into the study of mechanical vibrations.

The purpose of this paper is to analyze and simulate a nonlinear model of a suspension bridge, since a linear model is insufficient to explain the behavior of large oscillations [1] . Lazer and McKenna proposed some differential equation models to predict the simplified and idealized oscillatory behavior of a suspension bridge [2]. The first model to be proposed [3], and later better detailed in [4] has as its main feature a system where large oscillations were observed for small applied periodic forces, thus enabling the initial understanding of how the collapse of the Tacoma Bridge may have occurred. The proposed model considers large amplitudes and introduces nonlinearities through a nonlinear spring, thus making the model more realistic.

The system used is the one initially formulated by Lazer McKenna. In the mathematical background, a harmonic oscillator is equated, but an elastic is added, similar to what was proposed by [5]. In numerical simulations, an investigation through time-frequency tools in the Python programming language was performed, by means of Fast Fourier transform, Short Time Fourier Transform, Continuous Wavelet Transform, Map and Phase Space, to identify the non-linear responses and possibly chaos.

2 Mathematical background

Many models were proposed by Lazer and McKenna [2] to explain the nonlinearities found in Tacoma narrow bridge. One of these models aimed to demonstrate the phenomenon of purely vertical oscillations of a suspension bridge, which could have large oscillations for small periodic forces by means of a second-order differential equation.

In the model, the roadbed was used as an initial assumption as a one-dimensional vibrating beam [2], being supported by the cable-stays, those with a non-linear connection with the roadbed, which act only against the extension, exerting no force against compression, being modeled as a nonlinear spring [1, 3, 4], also taking the horizontal cable of the bridge as a fixed object.

Below in Fig.1 an idealization of the model (adapted from [1]) is shown, in addition to the partial differential equation eq.(1) and the boundary conditions eq.(2):

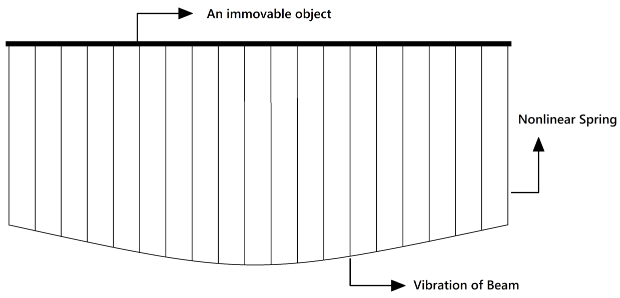

Figure 1. Bridge Simplification

$$mu_{tt} + EIu_{xxxx} + \delta' u_t = -ku^+ + W(x) + \varepsilon f(x,t) \tag{1}$$

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0 \tag{2}$$

In eq.(2) L is the length of the bridge, with hinged ends, whose positive vertical deflections are measured in the downward direction by $u(x,t)$, with a weakly viscous term and being exposed to three forces: the cable stays act as a nonlinear spring, represented by the constant k, the weight for unit length, $W(x)$, and an external force, $\varepsilon f(x,t)$, due to the origin may be, for example, a soldiers march or a gust force.

To verify the solution response with an initial value integrator, Lazer and McKenna proposed the simplification, to take the weight of the bridge as a non-constant, changing the form $W(x)$ to $W(x) = W_0 sin(\frac{\pi x}{L})$ [1, 4]. They also assumed that the force was given by $f(x,t) = g(t)sin(\frac{\pi x}{L})$.

Finally, looking for no-node solutions of the form $u(x,t) = y(t)sin(\frac{\pi x}{L})$, which oscillations were observed for low wind speeds at Tacoma narrow bridge [6]. By substituting the considerations into eq.(1), the term $sin(\frac{\pi x}{L})$ could be removed from the equation by dividing by getting:

$$\ddot{y} + \frac{\delta'}{m}\dot{y} + \frac{EI}{m}(\pi/L)^4 y + \frac{k}{m}y^+ = \frac{W_0}{m} + \frac{\varepsilon}{m}g(t) \tag{3}$$

Finally, separating by the y-axis asymmetry (y^+ as $y > 0$, and y^- as $y < 0$) and taking $a = \frac{1}{m}(EI(\pi/L)^4 + k)$, $b = \frac{EI}{m}(\pi/L)^4$, $\delta = \frac{\delta'}{m}$, $g = \frac{W0}{m}$, $\lambda = \frac{\varepsilon}{m}$ e $g(t) = sin(\mu t)$, is obtained:

$$\ddot{y} + \delta\dot{y} + ay^+ - by^- = g + \lambda \sin \mu t \tag{4}$$

This type of asymmetric system is analogous to a harmonic oscillator with a linear spring and rubber band [5, 7], similarly to Fig.2:

Applying Newton's second law formulation to the model shown in Fig.2 and modeling the rubber band as a nonlinear spring, which exert a linear force only against the extension, not applying any force against compression, in addition to assuming a periodic external force in the form $F_{ext} = \varepsilon \sin \mu t$ and isolating the $\ddot{y}$ term, obtaining the following equation of motion:

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y + \frac{E}{m}y^+ = g + \frac{F_{ext}}{m} \tag{5}$$

Parameterizing the y term by the asymmetry and grouping the others terms together, similar to eq.(4), there is: $\delta = \frac{c}{m}$, $a = \frac{k+E}{m}$, $b = \frac{k}{m}$, $\lambda = \frac{\varepsilon}{m}$, the eq.(6) present the following form:

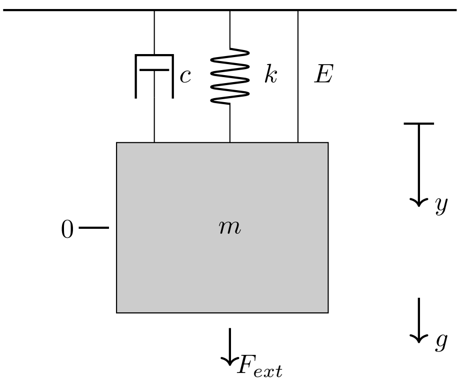

Figure 2. Mass-spring-rubber model

$$\ddot{y} + \delta\dot{y} + ay^{+} - by^{-} = g + \lambda sin(\mu t) \tag{6}$$

Being the nomenclature of eq.(6), a represents the parameters of the rubber band constant plus the spring constant, b represents only the spring, δ is the viscous damping, λ is the intensity of the external force, all divided by the mass, besides that g is the gravity and μ is the frequency of the external force.

3 Numeric Simulation

It is known that every linear system with damping and a periodic sinusoidal force results in a periodic solution in which all orbits converge when t increases [5]. However, this is not the case for nonlinear systems. For the numerical simulations, eq.(6) will be numerically integrated using the fourth-order Runge-Kutta integrator from the SciPy library. In the simulations, we will use two signals (time domain responses of the analyzed system). In signal processing, according to the methodology of Varanis [8] and Lynch [9], t was taken in the range between $2500 \geq t \geq 3000$ so that there were no remnants of the transient period.

In numerical simulations, as shown in [3, 4], for nonlinear behavior, one can increase the asymmetry of the system by varying the stiffness constant of the rubber band, in the a term, and as shown in [7], the amplitude of λ has an important contribution for richer dynamics. The following arbitrary values were taken in the simulation: a=12.75 $[N/mkg]$, b=1.75$[N/mkg]$, g=9.81 $[m/s^2]$, μ=0.85 $[rad/s]$ and δ=0.01 $[Ns/mkg]$ and with initial conditions $y(0) = a/g$ and $y'(0) = 0$. These parameters will be the same for both cases, however, the only difference is the λ parameter: 4 $[N/kg]$ for the linear case and 15.9 $[N/kg]$ for the nonlinear one.

From Fig.3a and Fig.4a one gets an initial view of the periodic behavior of the signal, in contrast to Fig.3b and Fig.4b with a rich dynamic behavior.

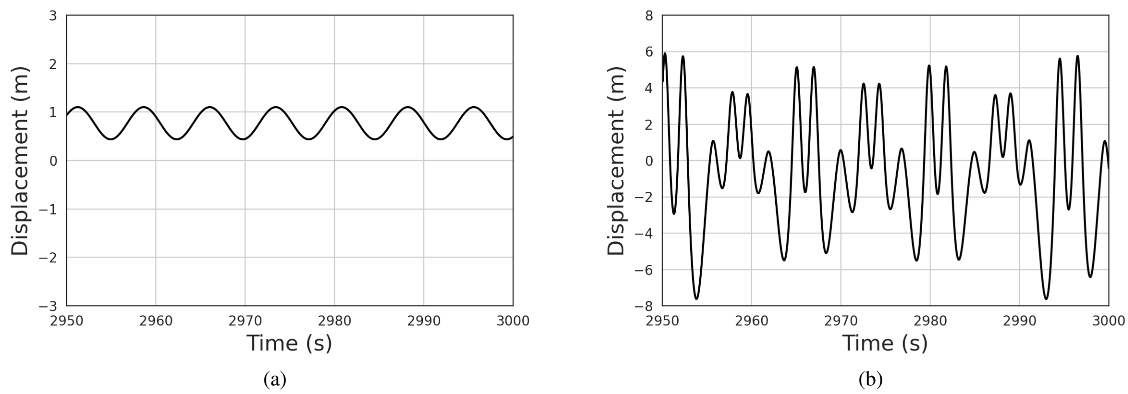

Figure 3. a)Time Domain Response of linear case, b) Time Domain Response of nonlinear case.

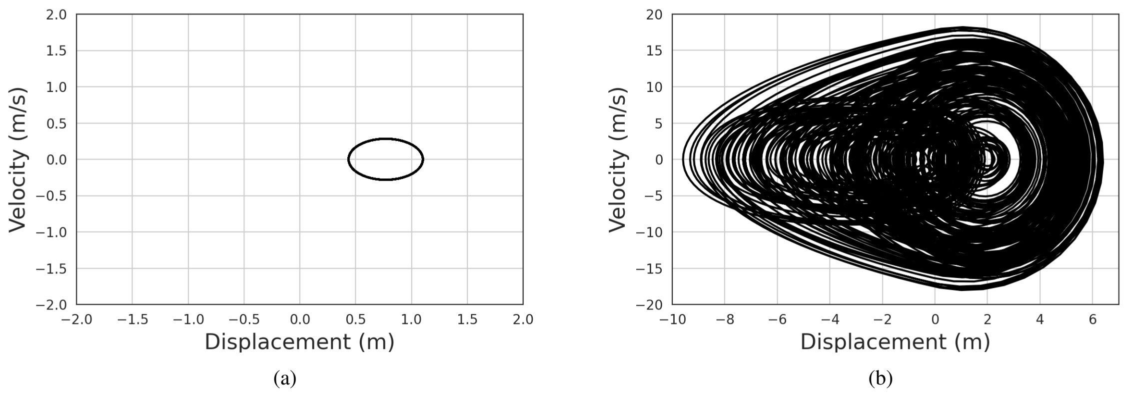

Figure 4. a) Phase Portrait of linear case, b) Phase Portrait of nonlinear case.

In the FFT analysis (Figs.5a and 5b), which is adequate for stationary signals, the signal with λ=4 presents only one frequency and in the signal with λ=15.9 multiples are identified.

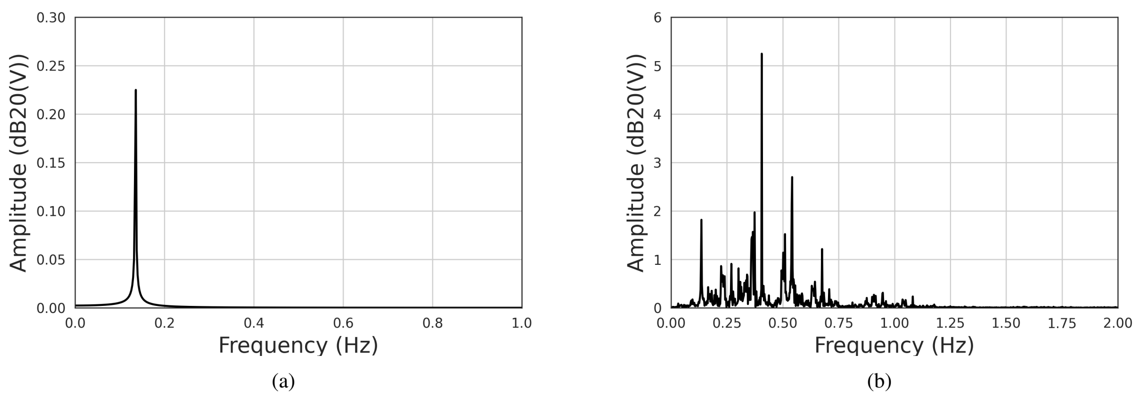

Figure 5. a)FFT of linear case and b) FFT of nonlinear case

Using tools for non-stationary signals, such as the STFT (Fig.6a and 6b), which presents a more superficial analysis due to its great optimization and its recurrent use in signal analysis, one can notice a stationary signal for λ=4, but for λ=15.9 can see a spreading in the frequencies. Due to some limitations and its use not recommended for nonlinear systems, it was chosen to use a CWT (Fig.7a and 7b), being the most suitable method, thus a more accurate visualization in the frequency domain. In the first case, the frequency presented is close to $0.135[Hz]$ (frequency of the external force), and in the second case, the frequency spread, and its non-periodic behavior is more distinct.

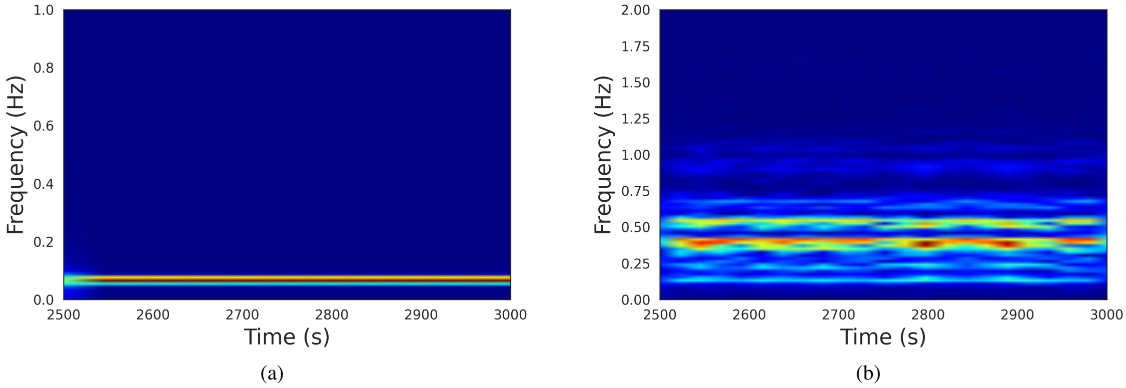

Figure 6. a) STFT of linear case and b) STFT of nonlinear case

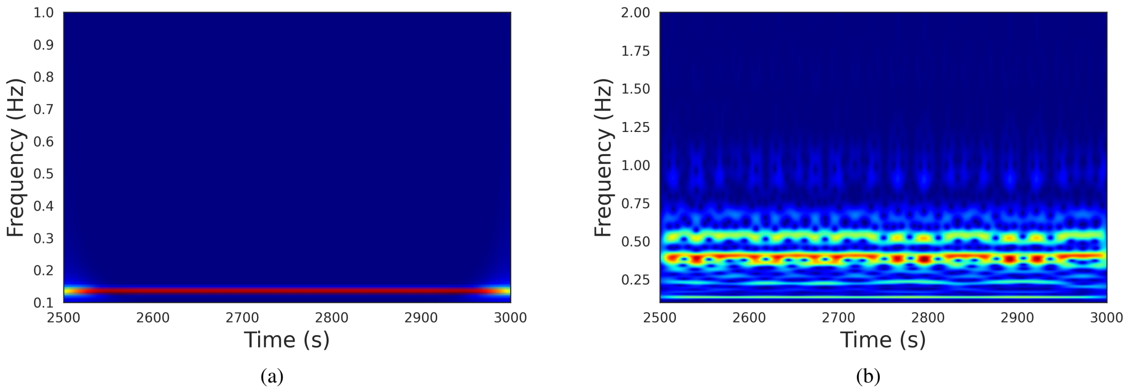

Figure 7. a) CWT of linear case and b) CWT of nonlinear case

In Poincaré Map (Fig.8a) with λ=4 it is possible to see only one point, which represents a single period, but when observing Fig.8b with λ=15.9, this pattern is not followed, presenting many points, evidencing a nonperiodic behavior as seen in the CWT analysis (Fig.7b).

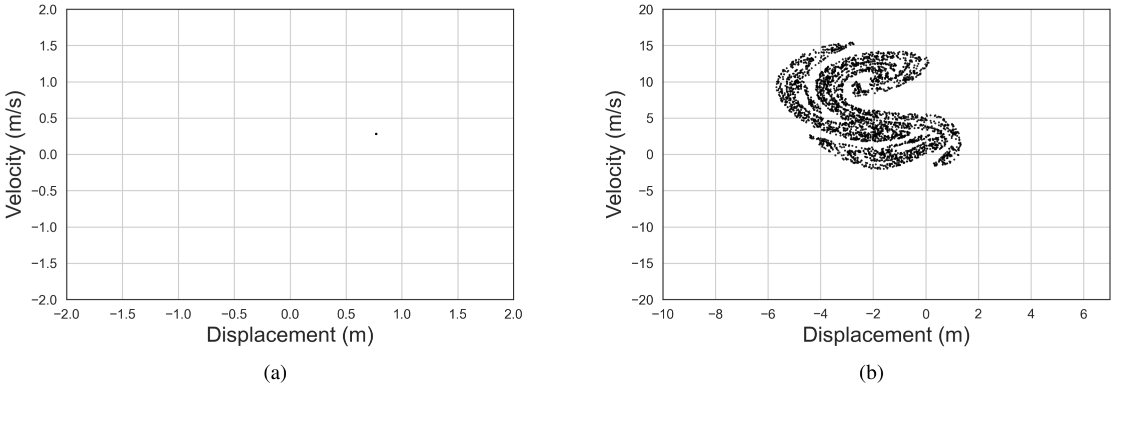

Figure 8. a) Poincaré Map of linear case and b) Poincaré Map of nonlinear case

4 Conclusion

The simulated results presented by the analyzed system may or may not exhibit periodicity. For the periodic case, a single frequency was observed, according to the linear theory, also observed in the frequency domain through analysis with FFT, STFT e CWT, around 0.135 $[Hz]$, in addition, in the Poincaré Map a single frequency was found. In the second case, nonlinear responses are found with frequency-time graph analyses. In addition to the multiple frequencies presents in FFT, STFT e CWT, and the Poincaré Map with multiple periods (which cannot be quantified), evidence that shows a behavior founds in chaotic systems, characteristics of strongly nonlinear systems.

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