Discrete crack model based on nodal duplication for nonlinear analysis of concrete structures

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Abstract. Concrete is classified as a quasi-brittle material and exhibit a gradual decline in response of the stress-strain law in inelastic regime. Upon reaching its strength limit, this material starts to crack. This cracking process critically influences the material's response in its state of stress, which makes crack evaluation an important factor in the analysis of concrete structures. A numerical strategy that can be used to analyze cracks is the Finite Element Method and the cracks can be classified as smeared and discrete. The smeared approach considers that a set of small size cracks are distributed along the finite element. In the discrete approach, which will be used in this work, the crack is considered a geometric discontinuity in the finite element mesh and its analysis involves the following essential keys: a constitutive model for describing the material; a crack propagation criterion; an adequate procedure for remeshing, and an efficient technique for solving a system of nonlinear equations. This work proposes the implementation of a discrete cohesive crack model with mesh redefining based on nodal duplication capable of evaluating the crack behaviour in concrete beams subjected to bending.

Keywords: Discrete crack, Nonlinear analysis, Finite element method

1 Introduction

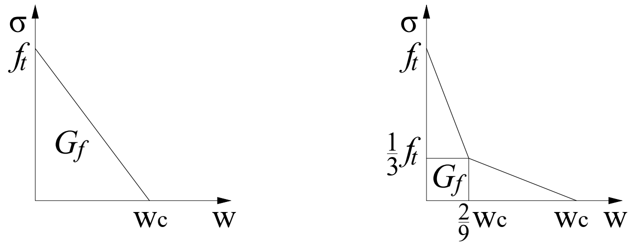
Concrete is a quasi-brittle brittle material that presents a gradual decrease in response to stress-strain. The structures built with this material have a behaviour influenced by the nucleation and propagation of cracks. Thus, the use of appropriate criteria for the crack identification and description of its path are indispensable for the analysis of concrete structures. After crack nucleation, it is necessary to describe the undamaged and damaged regions of the material in different ways, because the cracked region of the material presents a nonlinear behaviour. This description can be accompanied by constitutive models capable of representing the damage process in concrete. Several computational methods have been developed to model the fracture efficiently and accurately [1]. The finite element method is one of those methods that can be used for crack modelling. Using this method, the two most frequent forms of crack representation are the smeared crack model and the discrete crack model [2]. The smeared crack model considers that a set of small cracks are distributed along the finite element and the solid is treated as continuous. Crack propagation in this model is simulated by reducing material stiffness and strength. [3]. Furthermore, according to Yang and Chen [2], the constitutive laws are defined by nonlinear stress-strain relations with strain softening. The discrete model, which is the model used in this work, considers the crack as a geometric discontinuity and is generally preferred in the presence of a finite set of cracks [3]. A model that uses the finite element method to model a discrete crack must have an adequate constitutive model to represent the softening behaviour of the concrete [2]. The cohesive crack model or fictitious crack model developed by Hillerborg et al. [4] is an efficient model for this type of modelling because it considers the action of stresses in the fracture process zone in a narrowly open crack [5]. This zone is responsible for the concrete softening behavior. In addition, the fictitious crack model simulates the gradual opening of cracks, which is generally how crack propagation occurs in a concrete structure. Other factors that must be considered in discrete crack modelling are: the use of an adequate crack propagation criterion, an efficient mesh redefinition technique, an accurate mesh mapping technique to transfer the variables from the old mesh to a new one, and an efficient numerical solution technique to solve systems of nonlinear equations [2]. This work proposes the implementation of a discrete crack model that simulates the cohesion through uniaxial elements and as a crack propagation criterion it can use different classical strength criteria, such as Rankine and Mohr-Coulomb criteria. The proposed model adopts the nodal duplication strategy to create the crack in the finite element mesh and the Newton-Raphson method associated with a control method suited to the problem for solving the nonlinear equation system.

2 Cohesive Discrete Crack Model

Since concrete has a behaviour directly influenced by the nucleation and propagation of cracks, the study and adequate modelling of cracks become essential factors for the analysis of these structures. The model proposed in this work for modelling cracks is described in the following sections. The section 2.1 corresponds to the formulation of the cohesive model, 2.2 refers to the propagation criteria, and 2.3 represents the crack accommodation strategy in the finite element mesh.

2.1 Cohesive Crack Model

The cohesive crack model or fictitious crack model can describe the process of energy dissipation in quasi-brittle materials during the cracking process [6]. In this model, it is assumed the existence of a fracture process zone that transfer stresses until the crack opening reaches a critical value. Another consideration made is that the non-cracked region of the concrete has an elastic behaviour, thus there is no energy dissipation in the undamaged region of the material [7]. When the crack opens, the stress does not go to zero immediately, this value decreases according to the increase in the crack width [4]. Thus, this residual resistance is represented by cohesive stresses that tend to close the crack. The cohesion of the discrete crack in this work will be simulated by uniaxial elements, whose only displacement allowed will be in the direction of its axis, which characterizes the cohesion corresponding to mode I of crack opening. These elements will be associated with cohesive laws to represent the material's response to crack opening. Figure 1 shows two cohesive laws as presented by Petersson [8]. In Fig. 1(a) is a linear cohesive law, and the parameters f_t, w_c and G_f respectively represent the tensile strength of the material, the critical crack opening and the fracture energy; and in Fig. 1(b) is a bilinear cohesive law.



(a) Linear law (b) Bilinear law

Figure 1. Cohesive laws

According to Carpinteri et al. [9] and Barpi [10], the cohesive crack model will be described numerically below. The Fig. 2 represents a cracked solid where cohesive stresses act. Applying the principle of virtual work to formulate the problem in terms of finite element approximation, the following expression is obtained:

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \int_V \delta\mathbf{u}^T \mathbf{b} dV + \int_S \delta\mathbf{u}^T \mathbf{p} dS + \delta\mathbf{u}^T \mathbf{F}, \tag{1}$$

where $\delta\mathbf{u}^T$ represents the vector of virtual displacements, $\delta\boldsymbol{\varepsilon}^T$ is the vector of virtual strains, $\boldsymbol{\sigma}$ is the vector of stresses, $\mathbf{b}$ is the vector of body forces, $\mathbf{F}$ is the vector of concentrated forces and $\mathbf{p}$ is the surface force vector.

According to the cohesive crack model, the fracture process zone near to the crack tip can be represented by closing tractions $\mathbf{p_c}$ acting on both sides of the crack [9]. The contribution of the surface forces will then be given by the cohesive force $\mathbf{p_c}$ and by the surface force $\mathbf{p_s}$ acting on the crack surface in which the cohesive forces are

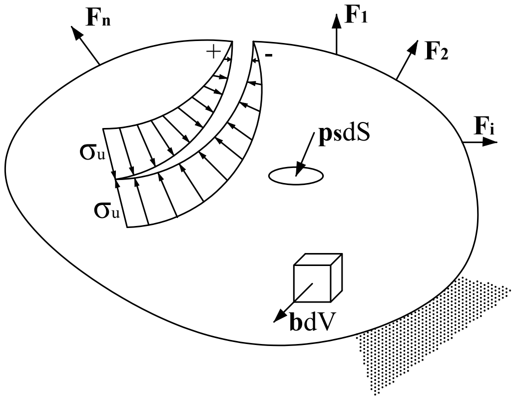


Figure 2. Cracked solid

active S_c [10]. Considering the surface forces divided into $\mathbf{p_c}$ and $\mathbf{p_s}$ and based on the equilibrium across the crack surface, the eq. (1) can be rewritten as follows:

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \int_V \delta\mathbf{u}^T \mathbf{b} dV + \int_{S-S_c} \delta\mathbf{u}^{\mathbf{T}} \mathbf{p_s} dS + \int_{S_c^+} \delta\mathbf{u}^{+^{\mathbf{T}}} \mathbf{p_u}^+ dS + \int_{S_c^-} \delta\mathbf{u}^{-^{\mathbf{T}}} \mathbf{p_u}^- dS +$$

$$\int_{S_c^+} \delta\mathbf{u}^{+^T} \mathbf{T}^T \mathbf{L} \mathbf{T} (\mathbf{u}^+ - \mathbf{u}^-) dS - \int_{S_c^-} \delta\mathbf{u}^{-^T} \mathbf{T}^T \mathbf{L} \mathbf{T} (\mathbf{u}^+ - \mathbf{u}^-) dS + \delta\mathbf{u}^T \mathbf{F}, \tag{2}$$

where $\mathbf{p_u}$ is the ultimate tensile strength vector, $\mathbf{T}$ is the transformation matrix from the global to the local system, $\mathbf{L}$ is the cohesive constitutive matrix in the local system of reference and the + and - signs correspond to the positive and negative sides of the crack as shown in Fig. 2.

The fifth and sixth term after the equality of eq. (2) can be rewritten in matrix form as follows:

$$\int_{S_c/2} \left\{ \begin{array}{c} \delta\mathbf{u}^+ \\ \delta\mathbf{u}^- \end{array} \right\}^T \begin{bmatrix} \mathbf{T}^T & 0 \\ 0 & \mathbf{T}^T \end{bmatrix} \begin{bmatrix} \mathbf{L} & -\mathbf{L} \\ -\mathbf{L} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{T} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{u}^+ \\ \mathbf{u}^- \end{array} \right\} dS. \tag{3}$$

According to the finite element method, the internal displacements can be expressed through shape functions $\mathbf{N}$ and nodal displacements $\mathbf{U}$ by the following expression:

$$\mathbf{u}(x, y, z) = \mathbf{N}(x, y, z)\mathbf{U} \tag{4}$$

The derivative of eq. (4) gives the strain field that can be written as follows:

$$\boldsymbol{\varepsilon} = \mathbf{BU}. \tag{5}$$

Choosing an appropriate constitutive law for the undamaged region of the material, the stress-strain relationship can be obtained as follows:

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon_0}) + \boldsymbol{\sigma_0} = \mathbf{DBU} - \mathbf{D}\boldsymbol{\varepsilon_0} + \boldsymbol{\sigma_0}, \tag{6}$$

where $\mathbf{D}$ represents the constitutive matrix, $\boldsymbol{\varepsilon_0}$ and $\boldsymbol{\sigma_0}$ are the initial states of strain and stress.

Replacing eqs. (4), (5) and (6) in eq. (2), the following expression is obtained:

$$\delta\mathbf{U}^T\left\{\left(\sum_e\int_V \mathbf{B}^T\mathbf{D}\mathbf{B}dV\right)\mathbf{U}-\sum_e\int_V \mathbf{B}^T\mathbf{D}\boldsymbol{\varepsilon}_\mathbf{0}dV+\sum_e\int_V \mathbf{B}^T\boldsymbol{\sigma}_\mathbf{0}dV\right\}=$$

$$\delta\mathbf{U}^T\left\{\sum_e\int_V \mathbf{N}^T\mathbf{b}dV+\sum_e\int_{S-S_c}\mathbf{N}^T\mathbf{p_s}dS\right\}+\delta\mathbf{U}^{+T}\sum_e\int_{S_c^+}\mathbf{N}^T\mathbf{p_u^+}dS+\delta\mathbf{U}^{-T}\sum_e\int_{S_c^-}\mathbf{N}^T\mathbf{p_u^-}dS \quad (7)$$

$$\left\{\begin{array}{c}\delta\mathbf{U}^+\\ \delta\mathbf{U}^-\end{array}\right\}^T\left\{\sum_e\int_{S_c/2}\begin{bmatrix}\mathbf{N}^T\mathbf{T}^T & 0\\ 0 & \mathbf{N}^T\mathbf{T}^T\end{bmatrix}\begin{bmatrix}\mathbf{L} & -\mathbf{L}\\ -\mathbf{L} & \mathbf{L}\end{bmatrix}\begin{bmatrix}\mathbf{TN} & 0\\ 0 & \mathbf{TN}\end{bmatrix}\left\{\begin{array}{c}\mathbf{U}^+\\ \mathbf{U}^-\end{array}\right\}dS\right\},$$

where e indicates a generic element of the mesh. The independent terms of $\mathbf{U}$ from eq. (8) are terms known to the system and represent forces. The terms dependent on $\mathbf{U}$, on the other hand, contribute to the stiffness matrix and therefore we can rewrite eq. (7) as:

$$\mathbf{KU}=\mathbf{R_b}+\mathbf{R_s}+\mathbf{R_{U^+}}+\mathbf{R_{U^-}}+\mathbf{R}_{\sigma_\mathbf{0}}+\mathbf{R}_{\varepsilon_\mathbf{0}}+\mathbf{F}, \quad (8)$$

where $\mathbf{K}$ is the stiffness matrix of the problem that considers the classical stiffness matrix and the stiffness due to cohesion. Before crack nucleation, the portion of the stiffness matrix $\mathbf{K}$ referring to cohesion is null. Therefore, a linear elastic stress-strain relationship is considered throughout the material. The terms $\mathbf{R_b}$ and $\mathbf{R_s}$ from eq. (8) represent, respectively, the contribution of body forces and surface forces and are described by eqs. (9) and (10) below:

$$\mathbf{R_b}=\sum_e\int_V \mathbf{N}^T\mathbf{b}dV, \quad (9)$$

$$\mathbf{R_s}=\sum_e\int_{S-S_c}\mathbf{N}^T\mathbf{p_s}dS. \quad (10)$$

The contribution of cohesive forces are calculated by the expression

$$\mathbf{R_{U^+}}=\sum_e\int_{S_c}\mathbf{N}^T\mathbf{p_{u^+}}dS,\quad \mathbf{R_{U^-}}=\sum_e\int_{S_c}\mathbf{N}^T\mathbf{p_{u^-}}dS. \quad (11)$$

Considering $\mathbf{F}$ proportional to a scalar λ and that there are no surface forces or initial deformations and stresses, $\mathbf{R_s}$, $\mathbf{R}_{\sigma\mathbf{0}}$ and $\mathbf{R}_{\varepsilon\mathbf{0}}$ will be null. Therefore, eq. (8) results in

$$\mathbf{KU}=\lambda\mathbf{F_1}+\mathbf{R_b}+\mathbf{R_{U^+}}+\mathbf{R_{U^-}}=\lambda\mathbf{F_1}+\mathbf{F_2}. \quad (12)$$

To obtain the solution of the problem, the expressions $\mathbf{KU1}=\mathbf{F_1}$ and $\mathbf{KU_2}=\mathbf{F_2}=\mathbf{R_b}+\mathbf{R_{U^+}}+\mathbf{R_{U^-}}$ must be solved. Consequently, the solution to the problem will be:

$$\mathbf{U}=\lambda\mathbf{U_1}+\mathbf{U_2}. \quad (13)$$

Obtained the value of the displacements, it is possible to calculate the stress values and verify if there was nucleation or crack propagation.

2.2 Crack propagation criteria

As a criterion for crack nucleation and propagation, different classical strength criteria were implemented. One of these criteria is the Rankine criterion. As presented by Chen and Han [11], this criterion considers that failure will occur when the maximum principal stress at a point reaches the value of the material's tensile strength, regardless of the normal or shear stress values in the other planes through this point.

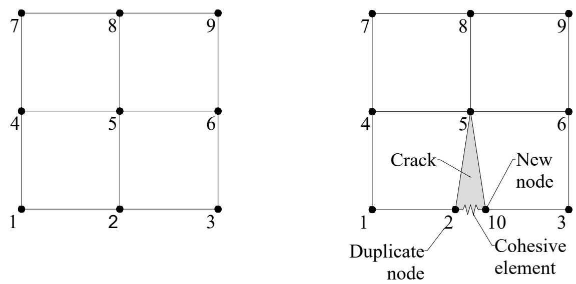
Another criterion is the Mohr-Coulomb criterion. This second criterion considers the maximum shear stress as a decisive factor for the occurrence of the failure [11]. Due to the main stresses ($\sigma_1 \geq \sigma_2 \geq \sigma_3$) the Mohr-Coulomb criterion can be written as:

$$\sigma_1 \frac{1 + \sin\phi}{2c\cos\phi} - \sigma_3 \frac{1 - sen\phi}{2c\cos\phi} = 1, \tag{14}$$

where c represents cohesion and ϕ is angle of internal friction. The variables c and ϕ are experimentally determined.

2.3 Nodal Duplication Strategy

After verifying that there was crack nucleation or propagation, the strategy used to accommodate the crack in the finite element mesh is the nodal duplication, which is shown in Fig. 3. This is a simple strategy that alters the mesh only in the region where crack nucleation and propagation occurred. Nodal duplication will occur according to the following steps: in the phase of updating variables from one incremental step to the next within the technique of solving nonlinear equations, it will be verified at which points of the mesh the resistance criterion was reached; from that point, the node that reached the limit stress, according to the adopted criterion, will be duplicated with the transfer of the state variables to the new node; once this is done, the incidence of the elements around the duplicated node will be redefined as a function of the crack orientation, which leads to the insertion in the finite element mesh of a discontinuity that represents the crack; finally, the element representing cohesion will be inserted between the new node and the existing one in the direction perpendicular to the crack propagation connecting the two altered nodes of the mesh.

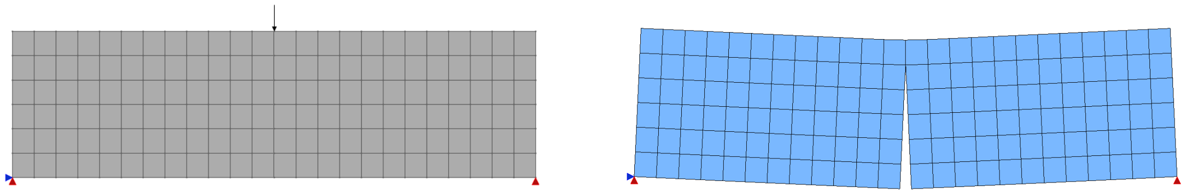


(a) Original mesh (b) Redefined mesh

Figure 3. Nodal duplication strategy

3 Numerical Examples

The results obtained in this section are derived from the implementation of the proposed discrete cohesive crack model in the open access and open source computational system INSANE (INteractive Structural ANalysis Environment) developed at the Structural Engineering Department of the Federal University of Minas Gerais. In the current phase of the work, the insertion of the crack in the mesh of finite elements and cohesive elements were implemented and the influence of the cohesive forces represented by eq. (11) is in the implementation stage. Therefore, the results obtained and presented here are of a brittle crack propagation. The first numerical example is represented by Fig. 4 and it is a simply supported beam subjected to a load concentrated in the middle of the span of 15 kN. The beam is 0.80 m high, 1.80 m long, and 0.10 m thick. The material has Young's modulus E = 20 GPa, Poisson's ratio ν = 0.2 and tensile strength f_t = 2.5 MPa. As shown in Fig. 4(a), quadrangular finite elements were used to model the example. Analyzing Fig. 5, which corresponds to the equilibrium path of the vertical displacement of the node loaded in Fig. 4, it can be seen the brittle propagation of the discrete crack, whose final deformed state is represented in Fig. 4(b).



(a) Original configuration of the beam

(b) Deformed beam configuration

Figure 4. Quadrangular mesh beam subjected to three-point bending

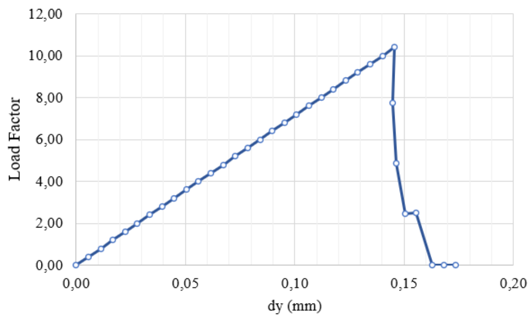
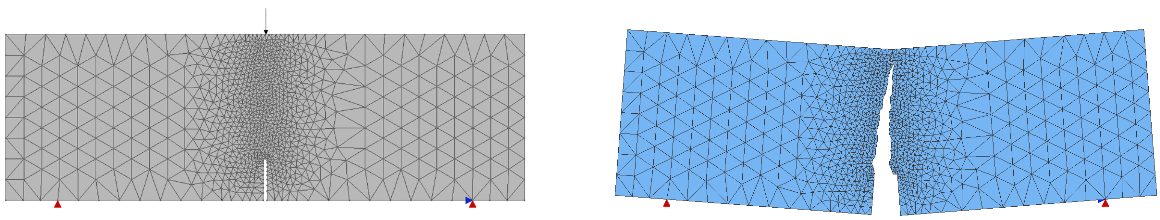


Figure 5. Equilibrium path of the vertical displacement of the loaded node

The second example is a beam with an initial crack subjected to a concentrated load in the middle of the span of 1 kN. This beam was modelled using a three-node triangular finite element mesh and the original configuration of the mesh is shown in Fig. 6(a). The dimensions of this example are: length of 2.50 m, height of 0.80 m, and thickness of 0.50 m. The material has Young's modulus E = 33.8 GPa, Poisson's ratio ν = 0.2 and tensile strength ft = 3.5 MPa.



(a) Original configuration of the beam

(b) Deformed beam configuration

Figure 6. Random mesh beam with initial crack subjected to three-point bending

Observing Fig. 6(b), it can be seen that the crack has propagated in the middle of the span and went towards loaded node. This propagation was a brittle propagation, as shown in Fig. 7, which corresponds to the equilibrium path of the load application node.

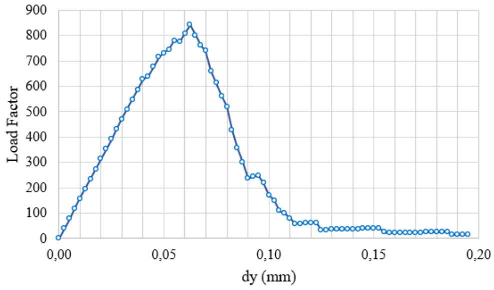


Figure 7. Equilibrium path of the vertical displacement of the loaded node

4 Conclusions

This paper has presented a discrete cohesive crack model based on the finite element method that is so far able to describe the brittle crack propagation in concrete beams subjected to bending. Cohesion modelling was made from the insertion of uniaxial elements that simulate the crack opening mode I. The proposed model also allows the use of different classic strength criteria to verify crack propagation. One of the essential factors for modelling discrete cracks is the redefinition of the finite element mesh for crack insertion. In this work, a simple redefinition strategy, called nodal duplication, was adopted, which redefines the mesh in the region only where the crack was inserted. From the observed results, it can be concluded that the proposed model can efficiently simulate the brittle propagation of the discrete crack so far, and the crack propagation results considering the cohesion will be possible after the completion of the current stage of implementation.

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