Numerical simulations of four-point beam bending test using a macroscopic probabilistic model

Mariane R. Rita[1], Henrique C. C. de Andrade[1], Magno T. Mota[1], Eduardo de M. R. Fairbairn[1], Fernando L. B. Ribeiro[1], Jean-Louis Tailhan[2], Pierre Rossi[2]

[1]*Programa de Engenharia Civil - Universidade Federal do Rio de Janeiro*
Centro de Tecnologia, Ilha do Fundão, CEP: 21941-909, Rio de Janeiro, Brasil
mariane_rita@coc.ufrj.br, henriqueconde@coc.ufrj.br, magnomota@coc.ufrj.br, eduardo@coc.ufrj.br
fernando@coc.ufrj.br
[2]*Departamento de Materiais e Estruturas - Université Gustave Eiffel.*
jean-louis.tailhan@univ-eiffel.fr, pierre.rossi@univ-eiffel.fr

Abstract. This article aims to present the numerical simulations of a four-point bending test of a large plain concrete beam, performed by using a three-dimensional macroscopic probabilistic model. The model is developed in the context of the finite element method and considers that each finite element represents a volume of heterogeneous material, with the mechanical properties of tensile strength and local cracking energy being randomly distributed over the mesh. In this way, that means the cracks are created within the concrete with different energy dissipation depending on the spatial distribution of material constituents and initial defects. Once the model is based on a probabilistic approach, a Monte Carlo procedure is used to give coherent statistical results. In order to verify the efficiency of the developed model related to different levels of refinement, analyzes for two mesh refinements are presented. The results show agreement with the expected physical behavior for this type of beam in terms of global response and macrocrack propagation. Regarding the mesh refinements, the results are quite similar indicating that the model provides a suitable global response in both cases and, therefore, it can be stated that the utilization of the model with the set of adopted parameters furnishes a consistent outcome for the analyzed cases.

Keywords: Numerical Modeling, Four-point Bending Test, Concrete Cracking, Probabilistic Model

1 Introduction

The modeling of several phenomena which happens in concrete has been of great interest of the scientific community, once as it is well known, concrete is a widely used material due to its great applicability as well as because of its accessible cost. Thereby, over the years different models have been developed considering different approaches, from the simplest to the most sofisticated. In that context, there are the probabilistic models, proposed as a way to overcome some of the challenges imposed by the material complexity, which are not completely taken into account by the deterministic models, because of the assumed simplifications. Besides, as stated by Mota et al. [1], studies have shown that probabilistic cracking models that directly taking into account the concrete heterogeneity are promising to simulate the nonlinear behaviour of concrete structures ([2], [3], [4], [5] [6], [7]).

Regarding the probabilistic modeling of concrete cracking, there is the macroscopic model implemented in this work, in which the basic idea follows the results shown by Rossi and Wu [3]; Rossi et al. [8]; Rossi et al. [9] and the subsequent works developed by Tailhan et al. [10]; Tailhan et al. [11] e Tailhan et al. [12]). The main characteristics of the model are related to the consideration of the material intrinsic heterogeneity and its implications in the so-called scale effect, through the random distribution of material mechanical properties over the finite element mesh, through probabilistic distribution functions. The model was implemented within the framework of the finite element method (FEM), on a platform that has been developed and constantly improved by researchers from the civil engineering program at COPPE/UFRJ. The code is written in FORTRAN language and the mentioned platform has already been used as basis for several works in different applications ([13]; [14]; [15],

[16], [17], [18]; [19]; [20]; [21]; [22]; [1]).

Therefore, this work presents a macroscopic probabilistic model for concrete cracking that is applied to the three-dimensional numerical simulations of a four-point bending test of a large plain concrete beam using two mesh refinements. The objective of the performed simulations is to show the model applicability and to analyze its numerical mechanical response. Since it is a probabilistic model, a Monte Carlo procedure is used in order to assure the results accuracy.

2 Macroscopic probabilistic model for concrete cracking

The main principle of the model is to incorporate the concrete heterogeneity, which is a complex material characteristic. With that purpose, the model considers that each finite element is representative of a given volume of heterogeneous material, whose behavior is controlled by its heterogeneity degree, defined as the ratio of the finite element volume over the coarsest aggregate volume ($r_e = V_e/V_a$). Moreover, the material mechanical properties of tensile strength (f_t) and fracture energy (G_c) are distributed over the finite element mesh, according to the Weibull distribution ([23], [24]) and lognormal distribution, respectively.

From the element perspective, the cracking process, that means, the creation and propagation of one crack within the element itself induces some local dissipation of energy. This dissipative process starts when the maximum principal stress at a given Gauss point reaches the material random tensile strength and it ends when the whole amount of energy that this element can consume is reached, at that point, the element is considered damaged (failed) and its elementary stiffness matrix is set to zero. In this way, the evolution of this process is mathematically represented through a probabilistic isotropic damage law ([25]). For sake of simplicity, a bilinear formulation of the stress-strain (σ, ε) relation is considered to simulate the softening behavior of the material. According to Rastiello et al. [6], this constitutive law is completely defined by the following parameters: tensile strength and volumetric density of dissipated energy (g_c). This latter can be evaluated considering the use of an energetic regularization technique ([26]), taking into account the material fracture energy, as follows:

$$g_c = G_c/l_e. \tag{1}$$

Where l_e represents the elementary characteristic length and is evaluated as: $l_e = (V_e)^{1/3}$.

According to Lemaitre's Law, the damage stress-strain relation can be expressed as the undamaged nominal stress-strain relation, as follows described in the Equations 2 and 3.

$$\sigma = \tilde{\mathbf{E}}\varepsilon. \tag{2}$$

$$\tilde{\mathbf{E}} = \mathbf{E_0}(1 - D), \quad (0 \leq D \leq 1). \tag{3}$$

where $\tilde{\mathbf{E}}$ and $\mathbf{E_0}$ are, respectively, the elastic modulus of the damaged and undamaged material; D is the damage variable whose evolution can be given by the Eq. (4), where $\tilde{\varepsilon}_0$ represents the damage initialization strain; $\tilde{\varepsilon}_{fi}$ represents the maximum critical strain and $\tilde{\varepsilon}^k$ represents the equivalent strain.

$$D = 1 - \frac{\tilde{\varepsilon}_0}{\tilde{\varepsilon}^k}\left[1 - \frac{(\tilde{\varepsilon}^k - \tilde{\varepsilon}_0)}{(\tilde{\varepsilon}_{fi} - \tilde{\varepsilon}_0)}\right]. \tag{4}$$

It is necessary to state that, in this modeling approach, the creation and propagation of a crack at a macroscopic level, is the consequence of the elementary failure of successive elements that randomly appear and can coalesce to form the macroscopic cracks. In that context, the model does not deal with crack propagation laws in the sense of fracture mechanics ([5], [6]).

The model is probabilistic regarding the random distribution of material properties and, due to that, the correct parameters estimation of the statistical laws that describe these properties are of great relevance. Furthermore, since it deals with random parameters, the use of Monte Carlo (MC) procedure is needed. The general idea behind the MC method is to provide results based on repeated random sampling and statistical analyzes, besides each simulation is characterized by the solution of a finite element problem and can be seen as an independent sample. For each analysis is generated a new set of random variables according to the probability density functions and, as a result, load-displacement ($P - \delta$) diagrams are obtained.

2.1 Distribution of random material properties

For the random distribution of tensile strength (f_t), the Weibull's law is used ([23],[24]) and it was chosen because is the best distribution to take into account the rupture in tension of a britlle and heterogeneous material, as is the case of concrete. Its probability density function, cumulative distribution function and its corresponding inverse function for a random variable x are written as follows:

$$f(x,b,c) = \frac{b}{c}\left(\frac{x}{c}\right)^{b-1} e^{\left(-\frac{x}{c}\right)^{b}}. \tag{5}$$

$$F(x,b,c) = 1 - e^{-\left(\frac{x}{c}\right)^{b}}. \tag{6}$$

$$F^{-1}(x,b,c) = c\left(\ln(1-x)\right)^{\frac{1}{b}}. \tag{7}$$

where $b > 0$ is the shape parameter and $c > 0$ is the scale parameter of the distribution.

The expected value μ and the variance σ^2 of the distribution are given, respectively, by Eq. (8) and Eq. (9), where Γ represents the Gamma function given by $\Gamma(\eta) = \int_0^\infty x^{\eta-1}e^{-x}dx$ and when η is a positive integer then, $\Gamma(n+1) = n!$ which means that $\Gamma(n) = (n-1)!$.

$$\mu = c\Gamma\left(1+\frac{1}{b}\right). \tag{8}$$

$$\sigma^2 = c^2\Gamma\left(1+\frac{2}{b}\right) - \mu^2. \tag{9}$$

For the fracture energy (G_c), the lognormal distribution it is used and its probability density function is defined by $f(x,b,c) : x \in (0,\infty] \rightarrow \mathcal{R}$, as follows:

$$f(x,\mu,\sigma) = \frac{1}{\mu\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}. \tag{10}$$

where, μ is the mean and σ the standard deviation of the variable's natural logarithm. The expected value $E(X)$ and variance $Var(X)$ are given by (11) and (12). Since the standard deviation will be considered as the dispersion measure of the distribution, it will be denoted here as $d_{log} = \sqrt{Var(X)}$.

$$E(X) = e^{\mu+\frac{\sigma^2}{2}}. \tag{11}$$

$$Var(x) = \left(e^{\sigma^2} - 1\right)e^{2\mu+\sigma^2}. \tag{12}$$

2.2 Parameters estimation

To use the model consistently is necessary that the parameters of Weibull and lognormal distributions are properly determined to represent correctly the material behavior. Each distribution has two parameters, however, since it is made the assumption that the fracture energy is considered an intrinsic material property, it is defined that its mean value is constant and equal to the value experimentally obtained by Rossi [27], as follows: ($G_c = 1.3141 \times 10^{-4}$ MN/m). Thus, it is necessary to determine three parameters: (b, c) from Weibull and d_{log} from lognormal distribution.

To evaluate the parameters of the Weibull distribution, an iterative numerical procedure was developed in order to solves a non-linear system of equations. This system is formed by combining the equations of mean and standard deviation of the distribution (Eq. (8) and Eq. (9)) with the analytical formulation of scale law proposed by Rossi et al. [9], which estimates the expected values of mean and standard deviation for a given concrete volume, that herein is applied to the scale of the finite element. This formulation was obtained from an experimental investigation aiming to correlate the concrete heterogeneity and scale effect. Thereby, at the end of the procedure, for each element in the mesh, there is a pair of (b, c) obtained as a function of its volume, maximum aggregate size and compressive strength (f_c). More details about the analytical expressions as well as about the description of the iterative procedure implementation can be found in Rita et al. [22].

On the other hand, to estimate the standard deviation of the lognormal distribution, an inverse analysis procedure was carried out. Since a detailed discussion about its performed procedure is beyond the scope of this work, just a brief description of the used strategy will be mentioned. Thereby, for this analysis, were performed several simulations of a macrocrack propagation test on a very large double cantilever beam specimen (DCB), once the specimen dimensions are crucial to provide results with no size effect, and comparing the numerical results with the experimentally obtained by Rossi [27]. This test is used in order to characterizing the concrete fracture under tensile loads and provides information, about the critical stress intensity factor and the critical energy release rate, that allows the evaluation of the material toughness. For the simulations, three mesh refinements were used and from these results, a function was estimated to define the standard deviation of the lognormal distribution according the mesh heterogeneity degree (Eq. (13)), allowing the estimation of the parameter for each finite element of the mesh.

$$d_{log}\left(r_e\right) = \left(A \ln\left(r_e\right) + B\right) \times G_c, \qquad r_e \in [1, 3000]. \tag{13}$$

where, $A = -8.538$ and $B = 70.88$.

3 Four-point bending test simulation

3.1 Test description

A representation of the test is shown on Fig. 1, ilustrating an un-notched beam specimen geometry as well as the place where loads and constraints are applied. In the figure, P is total force applied to the specimen by two loading pins; d is the distance between the supporting and loading pins; l is the specimen width and h is the specimen height. In the case of the beam analyzed here, the following measurements are considered: length $L = 5\ m$, $d = 1.4\ m$, $l = 0.5\ m$ and height $h = 0.5\ m$.

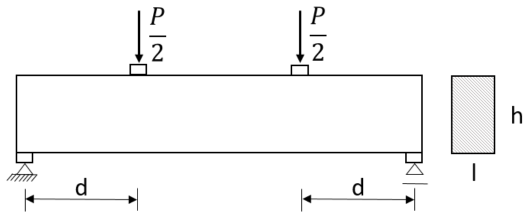


Figure 1. Representation of the four point bending test.

In general, this type of experiment in large beams is carried out on reinforced concrete beams with or without fibers, which is not the case of the beam analyzed here, in which the main characteristics are: great dimensions, plain concrete as mix design composition and no presence of reinforcements. However, as the objective of this work is to simulate the structural behavior of elements whose size is compatible with the scale of a real concrete structure, since it is a macroscopic model, it is not intended to compare the numerical results with any specific experiment, once, due to the circumstances, no experimental data was found in the literature.

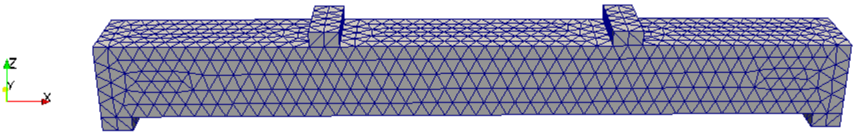
3.2 Numerical simulation description

Therefore, three-dimensional simulations were carried out in order to give an example of the developed probabilistic macroscopic model applicability, comparing the results for two mesh refinements (Mesh 1 and Mesh 2). The meshes are composed of tetrahedral elements approximated by linear interpolation functions and can be seen on Fig. 2. The mainly information about the meshes, such as: number of nodes (nnode), number of elements (numel), mean and standard deviation of heterogeneity degree, as well as its minimum and maximum values are given on Table 1 .

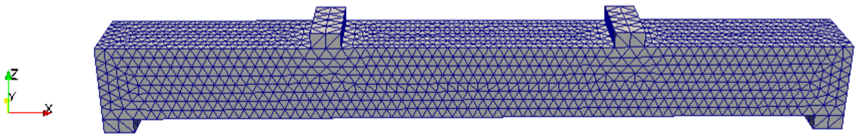
Table 1. Description of the meshes

Reference	nnode	numel	r_e^{mean}	r_e^{sd}	r_e^{min}	r_e^{max}
Mesh 1	2637	12051	118.4	32.8	37.1	323.9
Mesh 2	7853	39666	35.9	14.4	10.4	207.9

The model's input data are corresponding to a plain concrete, which composition was already defined by Tailhan et al. [12], and the mechanical parameters are: Young's Modulus $E = 36\ GPa$; compressive strength $f_c = 50$ MPa; Poisson's ratio $\nu = 0.2$ and volume of the largest aggregate $V_a \approx 9 \times 10^{-4} dm^3$. The total prescribed displacement applied in all the simulations is equal to $0.25 \times 10^{-1}\ dm$, divided into 50 displacement steps of $0.5 \times 10^{-3}\ dm$ and for each Monte Carlo simulation thirty samples were analyzed.



(a) Mesh 1 with $r_e^{mean} \approx 118$ and $r_e \in (37, 324)$.

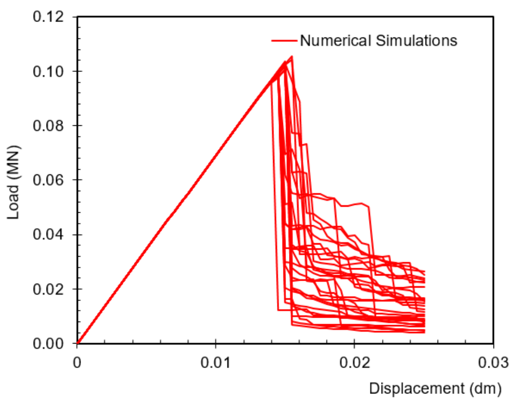


(b) Mesh 2 with $r_e^{mean} \approx 36$ and $r_e \in (10, 208)$.

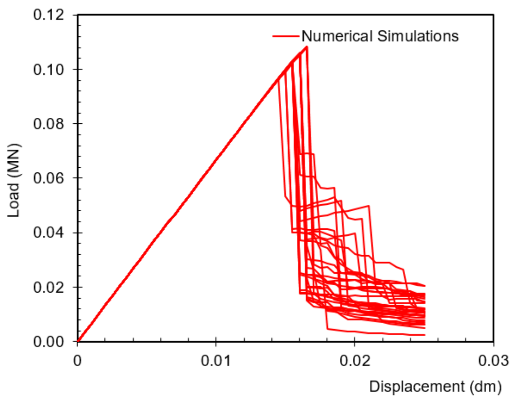
Figure 2. Three-dimensional meshes with height equal $0.5\ m$.

3.3 Numerical simulation results

In Fig. 3 are displayed the results of the Monte Carlo simulations for Mesh 1 and Mesh 2, representing the global mechanical behaviour of the beam through the complete load-displacements curves for the 30 analyzed samples and in Fig. 4 is shown a comparison of the arithmetic mean curve obtained for the two mesh refinements. AS can be observed, the load-displacement curves for the two meshes are similar, indicating that the determination of the statistical parameters is coherent.



(a) Mesh 1 with $r_e^{mean} \approx 118$.



(b) Mesh 2 with $r_e^{mean} \approx 36$.

Figure 3. Results of load-displacement curves of 30 Monte Carlo samples performed for the two mesh refinements of the simple beam.

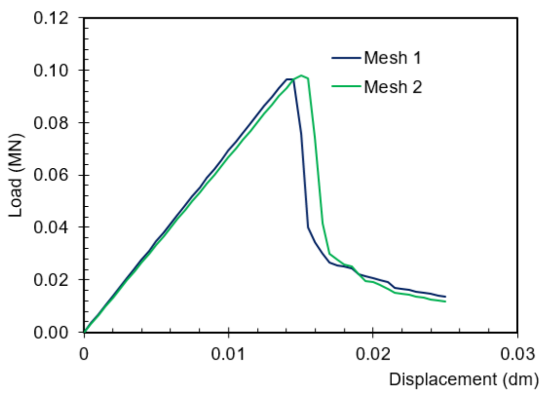


Figure 4. Comparison between the mean structural global responses obtained from the two mesh refinements.

In Fig. 5 one example of cracking pattern obtained for each mesh refinement is given. The images are concerning to the cracking process at the final stage of the simulation and represent the deformed mesh with scale factor equal 100. Besides, the color red indicates cracked elements ($D = 1$), intermediate colors represent elements in damage process ($0 < D < 1$) and the blue are elements that do not suffer any damage ($D = 0$). As can be seen in the figure, the cracking coalescence process was clearly observed, in accordance with the expected structural behavior.

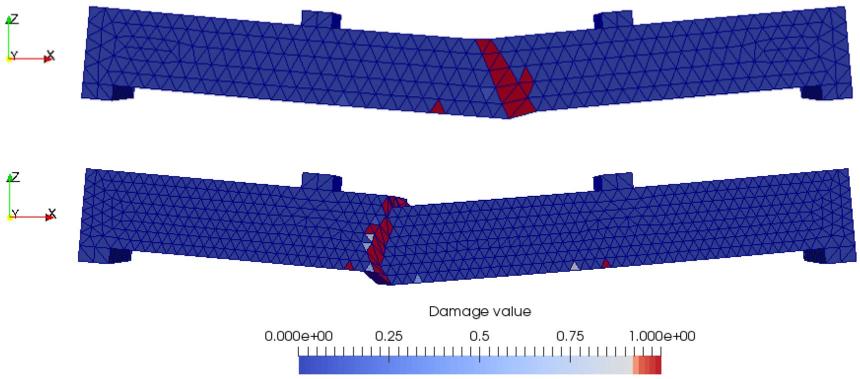


Figure 5. Example of cracking pattern obtained with the model, for Mesh 1 and Mesh 2, respectively.

4 Conclusions

The response obtained with the macroscopic probabilistic model, regarding the numerical simulation of four point bending test of a large plain concrete beam, can be considered satisfactory and promising. It is observed that the load-displacement curves show agreement with the expected physical behavior for this type of beam, in terms of global response, presenting mostly elastic-brittle behavior. Besides, the cracking pattern obtained in both cases is compatible with reality. The results regarding the model's applicability for different levels of mesh refinement, present very similar $P - \delta$ mean curves, indicating that the simulations provide a suitable global response. In this context, it can be said that the used strategy to estimate the parameters of the statistical distributions based in inverse analysis, for lognormal distribution, and from the iterative procedure with input data provided by scale effect laws formulated based in empirical results, for Weibull distribution, can be considered quite accurate. Therefore, given the above, it may be concluded that use of the implemented model associated with the set of estimated parameters furnishes a consistent outcome for the analyzed cases.

Acknowledgements. The authors acknowledge the financial support of the Brazilian Scientific Agencies CNPq, CAPES and FAPERJ.

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] M. Mota, E. Fairbairn, F. Ribeiro, P. Rossi, J.-L. Tailahn, H. Andrade, and M. Rita. A 3d probabilistic model for explicit cracking of concrete. *Computers and Concrete*, vol. 27, pp. 549–562, 2021.
[2] P. Rossi and S. Richer. Numerical modelling of concrete based on a stochastic approach. *Material and Structures*, vol. 20, pp. 334–337, 1987.
[3] P. Rossi and X. Wu. Probabilistic model for material behavior analysis and appraisement of concrete structures. *Magazine of Concrete Research*, vol. 44, pp. 271–280, 1992.
[4] E. Fairbairn, G. Q., and J. Ulm. An inverse problem analysis for the determination of probalistic parameters of concrete behavior modeled by a statistical approach. *Materials and Structures*, vol. 32, pp. 9–13, 1999.
[5] J.-L. Tailhan, S. Dal Pont, and P. Rossi. From local to global probabilistic modeling of concrete cracking. *Annals of Solid and Structural Mechanics*, vol. 1, pp. 103–115, 2010.

[6] G. Rastiello, J.-L. Tailhan, P. Rossi, and S. Dal Pont. Macroscopic probabilistic cracking approach for the numerical modelling of fluid leakage in concrete. *Annals of Solid and Structural Mechanics*, vol. 7, pp. 1–16, 2015.

[7] C. Nader. *A Multi-Scale Strategy for the Probabilistic Modeling of Reinforced Concrete Structures*. PhD thesis, Université Paris-Est, IFSTTAR, Paris, France, 2017.

[8] P. Rossi, X. Wu, F. Le Maou, and A. Belloc. Effet d'échelle sur le comportement du béton en traction. *BLPC*, vol. 182, pp. 11–20, 1992.

[9] P. Rossi, X. Wu, F. Le Maou, and A. Belloc. Scale effect on concrete in tension. *Materials and Structures*, vol. 27, pp. 437–444, 1994.

[10] J.-L. Tailhan, P. Rossi, and S. Dal Pont. Macroscopic probabilistic modelling of cracking processes in concrete structures. In *Proceedings of World Conference on Computational Mechanics - European Congress on Computational Methods in Applied Science and Engineering (WCCM-ECCOMAS)*, Venice, Italy, 2008.

[11] J.-L. Tailhan, P. Rossi, T. Phan, and J. Foulliaron. Probabilistic modelling of crack creation and propagation in concrete structures: some numerical and mechanical considerations. In *Proceedings of Numerical Modeling Strategies for Sustainable Concrete Structures (SSCS)*, Aix-en-Provence, France, 2012.

[12] J.-L. Tailhan, P. Rossi, T. Phan, G. Rastiello, and J. Foulliaron. Multiscale probabilistic approaches and strategies for the modelling of concrete. In *Proceedings of FRAMCOS-8*, Toledo, Spain, 2013.

[13] F. L. B. Ribeiro and I. A. Ferreira. Parallel implementation of the finite element method using compressed data structures. *Computational Mechanics*, vol. 41, pp. 31–48, 2007.

[14] I. Ferreira. *Solução em paralelo de um modelo termo-químico-mecânico para concreto jovem*. PhD thesis, Programa de Engenharia Civil, COPPE-UFRJ, Rio de Janeiro, Brasil, 2008.

[15] G. Ainsworth, F. Ribeiro, and C. Magluta. A parallel subdomain by subdomain implementation of the implicitly restarted arnoldi/lanczos method. *Computational Mechanics*, vol. 48, pp. 563–577, 2011.

[16] A. B. Silva, J. C. F. Telles, E. M. R. Fairbairn, and F. . Ribeiro. A general tangent operator applied to concrete using a multi-surface plasticity model. *Computers and Concrete, an International Journal (Print)*, vol. 16, pp. 329–342, 2015.

[17] A. B. C. G. Silva, J. Laszczyk, L. C. Wrobel, F. L. B. Ribeiro, and A. J. Nowak. A thermoregulation model for hypothermic treatment of neonates. *MEDICAL ENGINEERING & PHYSICS*, vol. 38, pp. 988–998, 2016.

[18] M. Rita, E. Fairbairn, F. Ribeiro, H. Andrade, and H. Barbosa. Optimization of mass concrete construction using a twofold parallel genetic algorithm. *Applied Sciences*, vol. 8, 2018a.

[19] M. R. Rita, M. T. Mota, E. Fairbairn, and F. Ribeiro. Verificação de efeito escala em concreto sob tração utilizando um modelo probabilístico. In *Anais do XIII SIMMEC 2018 - Simpósio de Mecânica Computacional*, Vitória, Brasil, 2018b.

[20] A. B. C. G. Silva, L. C. Wrobel, and F. L. B. Ribeiro. A thermoregulation model for whole body cooling hypothermia. *JOURNAL OF THERMAL BIOLOGY*, vol. 78, pp. 122–130, 2018.

[21] M. R. Rita, M. T. Mota, H. C. C. Andrade, E. M. Fairbairn, F. L. B. Ribeiro, J.-L. Tailan, and P. Rossi. Modelagem numérica do ensaio de viga em duplo balanço utilizando um modelo probabilístico. In *Proceedings of the XL Ibero-Latin-American Congress on Computational Methods in Engineering*, Natal, Brasil. ABMEC, 2019.

[22] M. R. Rita, H. C. C. Andrade, M. T. Mota, E. M. Fairbairn, F. L. B. Ribeiro, J.-L. Tailan, and P. Rossi. Simulação numérica de viga de concreto protendido em duplo balanço utilizando um modelo probabilístico. In *Proceedings of the XLI Ibero-Latin-American Congress on Computational Methods in Engineering*, Foz do Iguaçu, Brasil. ABMEC, 2020.

[23] W. Weibull. A statistical theory of the strengh of materials. In *Proceedings of The Royal Swedish Institute for Engineering Research*, 1939.

[24] W. Weibull. A statistical distribution function of wide applicability. *Journal of applied mechanics*, vol. 18, pp. 293–297, 1951.

[25] G. Rastiello. *Influence de la fissuration sur le transfert de fluides dans les structures en béton. Stratégies de modélisation probabiliste et étude expérimentale*. PhD thesis, Université Paris-Est, IFSTTAR, Paris, France, 2013.

[26] Z. Bazant and B. Oh. Crack band theory for fracture of concrete. *Material and Structures*, vol. 3, pp. 155–177, 1983.

[27] P. Rossi. *Fissuration du béton: du matériau à la structure-application de la mécanique lineaire de la rupture*. PhD thesis, L'Ecole Nationale des Ponts et Chaussées, Paris, France, 1988.