Numerical analysis of unbonded prestressed concrete beams under long term service loads by Finite Element Method

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Abstract. This work aims to develop a finite element formulation for a numerical analysis for long-term loads of unbonded prestressed concrete beams. The finite element formulation consists of unidimensional elements of plane frames with models of 7 degrees of freedom per element, based on the Euller-Bernoulli beam theory for the reinforced concrete section and a truss element for the simulation of the unbonded tendons. The formulation was developed for the examples of beam under long term loads with the purpose of evaluating the effects due to creep, shrinkage, and relaxation of the prestressing steel. The analysis used the Age-Adjusted Effective Modulus and normative relationships to obtain long-term effects. The numerical results were evaluated through a comparative study with long-term load-displacement curves from the literature, obtaining excellent approximations.

Keywords: Finite element, Unbonded prestressing, Long-term effects.

1 Introduction

The application of prestressing in civil construction has allowed great advantages in the structural elements of concrete, such as: reduction of cracks, limitation of deflections and increase in the durability of concrete. The dimensioning of prestressed structures with unbonded tendons is carried out observing the prestressing losses over the service life of the structural part, which occur immediately or over time. The long-term prestressing losses can establish a great influence on the behavior of the structures and, consequently, command the criteria of choice in the structural design parameters.

Due to advance of prestressed concrete technologies and their propagation in large-scale works, some researchers have studied the behavior of prestressed concrete members with unbonded tendons, in particular beams . However, there are few studies focused on predicting the behavior of structures in bending under long-term service loads [1-3].

This way, Lou et al [2] proposed a element finite formulation to analyze time-dependent unbonded prestressed concrete continuous beams in which it considered the shrinkage deformation independent of the applied stress, differently from the creep that was associated with the history of applied stresses.

Páez and Sensale [1], on the other hand, present an approach based on the finite element method used to model the time-dependent effects (shrinkage, relaxation and creep), in which it considers the time dependence on the constitutive relations for the concrete using the Age-Adjusted Effective Modulus Method to determine the stiffness matrix over time.

Alves [3] also proposed a simplified finite element formulation that considers the creep and shrinkage effects based on the Age-Adjusted Effective Modulus Method (MMEA), but with the consideration of the creep coefficient by standard descriptions [4].

In this work, similarly as in Páez and Sensale [1] and Alves [3], the simplified model based on MMEA was used to formulate finite elements that considers the effects of creep and shrinkage, in addition to steel relaxation. The model considers materials under service loads and disregards geometric nonlinearities. The formulation consists of one-dimensional plane frame elements with 7 degrees of freedom based on Euller Bernoulli's theory and unbonded tendons element implemented in Matlab software.

2 Finite Element formulation

The finite element formulation was based Euller- Bernoulli beam theory considering hypotheses such as: The cross-section of a beam remains plane after deformation and the cross-section remains normal to the deformed axis of the beam. The displacements field are defined:

$$u(X,Y) = u_o(X) - Y.v_o'(X) \quad (1)$$

$$v(X,Y) = v_o(X) \quad (2)$$

where u e v are the axial and transverse displacements and the subscript O refers to displacements at the beam reference axis.

2.1 Frame element

The finite element of a plane frame with 7 degrees of freedom is shown in Figure 1 with the coordinate system, such element is proposed based on the Total Lagrangian approach. The Green-Lagrange strain is used to determine the longitudinal strain of the element ε_x, expressed as a function of the membrane strain ε_o and curvature k as:

$$\varepsilon_x = u_o' + \frac{1}{2}{v_o'}^2 - Y.v_o'' = \varepsilon_o - Y.k \quad (3)$$

The membrane strain and curvature can be interpreted as generalized strain $\boldsymbol{\varepsilon}$. In vector form :

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_o \\ \kappa \end{bmatrix} \quad (4)$$

The element proposed presents internal degree of freedom not associated with the mesh nodes of the plane frame member, but the axial displacements $u_0(x)$ inside the element are determined by quadratic Lagrange interpolation functions with three degrees of freedom, while the transverse displacements $v_0(x)$ are represent by Hermite interpolation polynomials. The use of the internal node for the axial displacement avoids a membrane locking problem in the order of the degrees of the polynomials

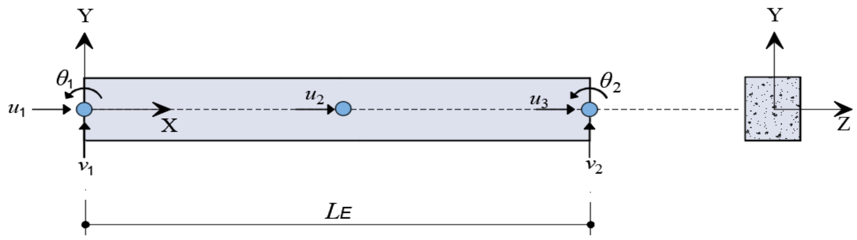


Figure 1. Plane frame element with 7 degree of freedom with the coordinate system

The generalized strain vector $\boldsymbol{\varepsilon}$ is represented as a function of the nodal displacements vector $\boldsymbol{u_e}$ and the strain-displacement matrix $\boldsymbol{B}$.

$$\boldsymbol{\varepsilon} = \boldsymbol{B}.\boldsymbol{u_e} \quad (5)$$

2.2 Shrinkage and creep

Shrinkage and creep are considered using the Age-Adjusted Effective Modulus Method (AAEM) in the formulation for long-term service loads less than 50% of the strength of the concrete beam. The strain in concrete over time $\varepsilon(t)$ that employs the effects of creep and shrinkage by the adjusted modulus of elasticity is defined by [5]:

$$\varepsilon(t) = \sigma_c(t_o)\frac{1 + \emptyset(t,t_0)}{E_c(t_0)} + \frac{\Delta\sigma_c}{E_c(t_0)}\left[1 + \chi(t,t_0)\emptyset(t,t_0)\right] + \varepsilon_{sh}(t) \quad (6)$$

where $E_c(t_0)$ is the longitudinal modulus of elasticity at the date of application of the external load, $\chi(t,t_0)$ is the age coefficient of the concrete, $\emptyset(t,t_0)$ is the creep coefficient and $\varepsilon_{sh}(t)$ is the shrinkage strain. The expression of Eq. 6 can be rewritten using the formulation of the adjusted effective modulus of elasticity $\overline{E}(t,t_0)$, thus:

$$\varepsilon(t) = \sigma_c(t_o)\frac{1+\emptyset(t,t_0)}{E_c(t_0)} + \frac{\Delta\sigma_c}{\overline{E}(t,t_0)} + \varepsilon_{sh}(t) \tag{7}$$

The internal virtual work can be evaluated in the analysis of concrete strain as a function of time for long-term loads and is expressed as a function of the generalized strain vector $\boldsymbol{\varepsilon}(t)$ and the generalized stresses vector $\boldsymbol{\sigma}(t)$:

$$\delta U_e(t) = \int_V \delta\boldsymbol{\varepsilon}(t).\boldsymbol{\sigma}(t).dV \tag{8}$$

where the generalized stresses vector $\boldsymbol{\sigma}(t)$ is composed of normal force *N(t)* and bending moment *M(t)* as a function of time.

The strain increment vector as a function of time is defined using the strain-displacement matrix, obtaining:

$$\delta\boldsymbol{\varepsilon}(\mathrm{t}) = \boldsymbol{B}.\delta\boldsymbol{u_e}(t) \tag{9}$$

Developing the generalized stresses vector by the constitutive relations and the use of the Adjusted effective module, obtaining the expression of the internal virtual work [3]:

$$\delta U_e(t) = \boldsymbol{\delta u_e}(t)^{\boldsymbol{T}} \int_{\boldsymbol{Le}} \boldsymbol{B^T}\left[\,\boldsymbol{C}(t,t_o).\boldsymbol{B}.\boldsymbol{u_e}(t) - \boldsymbol{C_{sh}}.\boldsymbol{\varepsilon_{sh}}(t) + \boldsymbol{\sigma_0}(\boldsymbol{t_0}).\overline{\emptyset}(t,t_o)\right].dX \tag{10}$$

where, $\boldsymbol{C}(t,t_o)$ is the time-dependent constitutive matrix, $\boldsymbol{C_{sh}}$ is the constitutive vector for the shrinkage strain, $\boldsymbol{\sigma_0}$ is the generalized stresses vector at loading time t_0 e $\overline{\emptyset}(t,t_o)$ is an expression as a function of the aging coefficient and creep coefficient:

$$\boldsymbol{C}(t,t_o) = \begin{bmatrix} \overline{E}(t,t_0)A_c & -\overline{E}(t,t_0)S_c \\ -\overline{E}(t,t_0)S_c & \overline{E}(t,t_0)I_c \end{bmatrix} + \begin{bmatrix} E_s.A_s & -E_s.S_s \\ -E_s.S_s & E_s.I_s \end{bmatrix} \tag{11}$$

$$\boldsymbol{C_{sh}} = \begin{bmatrix} \overline{\mathrm{E}}(\mathrm{t},\mathrm{t}_0)\mathrm{A_c} \\ \overline{\mathrm{E}}(\mathrm{t},\mathrm{t}_0)\mathrm{S_c} \end{bmatrix} \tag{12}$$

$$\boldsymbol{\sigma_0}(t_0) = \begin{bmatrix} \mathrm{N_c}(\mathrm{t_0}) \\ \mathrm{M_c}(\mathrm{t_0}) \end{bmatrix} \tag{13}$$

$$\overline{\emptyset} = \frac{\emptyset(\mathrm{t},\mathrm{t}_0).[\chi(\mathrm{t},\mathrm{t}_0)-1]}{1+\chi(\mathrm{t},\mathrm{t}_0)\emptyset(\mathrm{t},\mathrm{t}_0)} \tag{14}$$

the components $\mathrm{A_c}$, $\mathrm{A_s}$, $\mathrm{I_c}$, $\mathrm{I_s}$, $\mathrm{S_c}$ e $\mathrm{S_s}$ are refer to the areas, moments of inertia and moment of area of concrete section and the steel bars of section, respectively. On the other hand, $\mathrm{N_c}(\mathrm{t_0})$ and $\mathrm{M_c}(\mathrm{t_0})$ refer to the normal force and bending moment in the concrete at the loading time $(\mathrm{t_0})$.

The internal virtual work can be considered, as:

$$\delta U_e = \int_L \delta\boldsymbol{\varepsilon}^T\,\boldsymbol{\sigma}\,dX = \delta\boldsymbol{u_e}^T\mathbf{g}_{\boldsymbol{e}} \tag{15}$$

Therefore, replacing the Eq.10 an Eq.15, it is possible to write the element internal force vector $\mathbf{g}_{\boldsymbol{e}}(t)$:

$$\boldsymbol{g_e}(t) = \int_{\boldsymbol{Le}} \boldsymbol{B^T C}(t,t_o)\boldsymbol{B}.\boldsymbol{u_e}(t)\boldsymbol{dx} - \int_{\boldsymbol{Le}} \boldsymbol{B^T C_{sh}\varepsilon_{sh}}(t)\boldsymbol{dx} + \int_{\boldsymbol{Le}} \boldsymbol{B^T\sigma_0}(t_0).\overline{\emptyset}(t,t_o)dx \tag{16}$$

The components of Eq. 16 can be rewrite as:

$$\boldsymbol{g_e}(t) = \boldsymbol{K_e}(t,t_o).\boldsymbol{u_e}(t) - \boldsymbol{f_{she}}(t,t_o) + \boldsymbol{f_{cre}}(t,t_o) \tag{17}$$

where $\boldsymbol{K_e}(t,t_o)$ is the element stiffness matrix, $\boldsymbol{f_{she}}(t,t_o)$ is the element shrinkage force vector and $\boldsymbol{f_{cre}}(t,t_o)$ is the element creep force vector:

$$\boldsymbol{K_e}(t,t_o) = \int_{\mathrm{Le}} \boldsymbol{B^T C}(t,t_o).\boldsymbol{B}.dx \tag{18}$$

$$\boldsymbol{f_{she}}(t,t_o) = \int_{Le} \boldsymbol{B^T C_{sh}}.\varepsilon_{sh}(t)dx \tag{19}$$

$$\boldsymbol{f_{cre}}(t,t_o) = \overline{\emptyset}(t,t_o)\int_{Le} \boldsymbol{B^T \sigma_0}(t_0)dx \tag{20}$$

After the assembly of the global stiffness matrix of the structure (**K**) and the global external force vectors due to creep and shrinkage $\boldsymbol{f_c}$ and $\boldsymbol{f_{sh}}$ respectively, the equilibrium of the internal force vector $\boldsymbol{g}(t)$ with the external force vector global $\boldsymbol{f}(t)$ is performed by the expression:

$$\boldsymbol{K}(t,t_o).\boldsymbol{u}(t) = \boldsymbol{f}(t) + \boldsymbol{f_{sh}}(t,t_o) - \boldsymbol{f_{cr}}(t,t_o) \tag{21}$$

2.3 Unbonded tendon element

Moreira et al [6], describe that in unbonded prestressed concrete beams, there is no strain compatibility between the concrete and the tendon at a given cross-section. Therefore, they consider that there is only compatibility at the ends of the beam or at anchorage points. This way, the tendon strain depends on the displacements of the tendon as a whole.

The model proposed is based on the consideration of the tendon as a finite element discretized into straight segments that contribute to the internal force vector and to the stiffness matrix of the structure. It is easy evaluate that due to the lack of bond between the plastic sheath and the prestressing steel, it can be considered that the stress and strain in the tendon is constant along the cable length [3-6].

The displacements of the straight tendon segment are obtained from the displacement of the embedding frame element through classical beam theory, see Figure 2, express by [6 -7]:

$$u_{p1}(X_{p1},Y_{p1}) = u_1(X_1) - Y.v_1'(X_1) = u_1(X_1) - Y.\theta_1(X_1) \tag{22}$$

$$v_{p1}(X_1,Y_1) = v_1(X_1) \tag{23}$$

$$u_{p2}(X_{p2},Y_{p2}) = u_2(X_2) - Y.v_2'(X_2) = u_2(X_2) - Y.\theta_2(X_2) \tag{24}$$

$$v_{p2}(X_2,Y_2) = v_2(X_2) \tag{25}$$

where u_{p1} and u_{p2} are axial displacements e v_{p1} e v_{p2} are transverse displacements of the straight segments nodes.

In Matrix form:

$$\boldsymbol{u_{pe}} = \begin{bmatrix} u_{p1} \\ v_{p1} \\ u_{p2} \\ v_{p2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -Y_{p1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -Y_{p2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix} = \boldsymbol{T_e u_e} \tag{26}$$

where $\boldsymbol{u_{pe}}$ is the nodal displacements vector of the tendon segment, $\boldsymbol{u_e}$ is the nodal displacements vector of the frame element and $\boldsymbol{T_e}$ the transformation matrix.

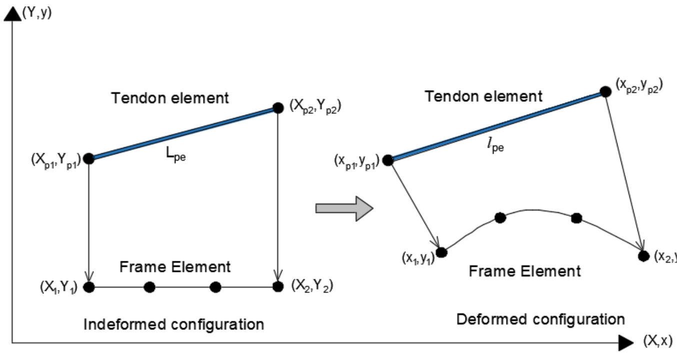


Figure 2. Association of coordinates of plane frame element and tendon element

The internal virtual work of the prestressing tendon element associated with its constant strain and stress, Eq.(15), is given by:

$$\delta U = \int_{Lp}\int_{Ap} \delta\varepsilon_p \sigma_p \, dAdx = \delta\varepsilon_p . \sigma_p . A_p . L_p \tag{27}$$

The strain variation ($\delta\varepsilon_p$) can be written:

$$\delta\varepsilon_p = \frac{\delta l_p}{L_p} \tag{28}$$

where δl_p can be obtained from:

$$\delta l_p = \sum_{\boldsymbol{e=1}}^{\boldsymbol{np}} \delta l_{pe} = \sum_{\boldsymbol{e=1}}^{\boldsymbol{np}} \boldsymbol{r_e}^T \delta \boldsymbol{u_{pe}} = \sum_{e=1}^{np} \boldsymbol{r_e}^{\boldsymbol{T}} \boldsymbol{T_e} \boldsymbol{\delta u_e} \tag{29}$$

$\boldsymbol{r_e}$ represents the vector that is related to the slope (β) of the cable segments with the horizontal axis in the deformed configuration.

$$\boldsymbol{r_e} = \begin{bmatrix} -cos\beta \\ -sen\beta \\ cos\beta \\ sen\beta \end{bmatrix} \tag{30}$$

2.4 Relaxation of prestressed steel

The analysis of the unbonded tendon element under long term loads is proposed considering the behavior of the tendon in the linear elastic regime. Thus, the stress σ_p(t) is determined by [3]:

$$\sigma_p(t) = E_p . \left[\varepsilon_p(t) - \varepsilon_{p,rel}(t, t_o)\right] \tag{31}$$

where, E_p is the elastic modulus of the prestressing steel, $\varepsilon_p(t)$ is the tendon strain at the date *t* e $\varepsilon_{p,rel}(t, t_o)$ is the tendon relaxation strain.

The measurement of the tendon relaxation strain is performed through the expression of stress as a function of time proposed by Magura et al [8], thus:

$$\varepsilon_{p,rel}(t, t_o) = \varepsilon_{po}(t_o) . \left[\frac{\log(t - t_o)}{10}\left(\frac{\sigma_p(t_o)}{\sigma_{py}} - 0{,}55\right)\right] \tag{32}$$

Replacing the tendon relaxation strain in the Eq. (31):

$$\sigma_p(t) = E_p . \left[\Delta\varepsilon_p(t) + \varepsilon_{po}(t_o) . \emptyset_{p,rel}(t, t_o)\right] \tag{33}$$

where:

$$\emptyset_{p,rel}(t, t_o) = \left[1 - \frac{\log(t - t_o)}{10}\left(\frac{\sigma_p(t_o)}{\sigma_{py}} - 0{,}55\right)\right] \tag{34}$$

and $\Delta\varepsilon_p(t)$ is the incremental strain and $\varepsilon_{po}(\mathrm{t_o})$ is the initial strain at the date of application of the external load t_o. Using Eq. (27), (28) and (29) and replacing the stress in the tendon σ_p by the expression of Eq. (33), applied as a function of time *t*, it is possible write the equation in form:

$$\delta U(t) = \sum_{\boldsymbol{e=1}}^{\boldsymbol{np}} \boldsymbol{r_e}^{\boldsymbol{T}} \boldsymbol{T_e} \, \boldsymbol{\delta u_e} . A_p . E_p . \left[\Delta\boldsymbol{\varepsilon_p}(t) + \boldsymbol{\varepsilon_{po}}(t_o) . \emptyset_{\boldsymbol{p,rel}}(t, t_o)\right] \tag{35}$$

The Eq. (35) can be written as:

$$\delta U(t) = \boldsymbol{\delta u_e} . \sum_{\boldsymbol{e=1}}^{\boldsymbol{np}} \boldsymbol{r_e}^{\boldsymbol{T}} \boldsymbol{T_e} . \left[\Delta F_{\boldsymbol{p}}(t) + F_{\boldsymbol{po}}(t_o) . \emptyset_{\boldsymbol{p,rel}}(t, t_o)\right] \tag{36}$$

where:

$$\Delta F_p(t) = A_p . E_p . \Delta\varepsilon_p(t) \tag{37}$$

$$F_{po}(t_o) = A_p . E_p . \varepsilon_{po}(t_o) \tag{38}$$

The internal force vector of the tendon segment can be evaluated as:

$$\boldsymbol{g_{pe}} = \boldsymbol{T_e}^{\boldsymbol{T}} \boldsymbol{r_e}. \left[\Delta F_p(t) + F_{po}(t_o). \emptyset_{p,rel}(t, t_o)\right] \tag{39}$$

Separating into two terms the Eq. (39):

$$\boldsymbol{g_{pe}} = \boldsymbol{T_e}^{\boldsymbol{T}} \boldsymbol{r_e}. \Delta F_p(t) + \boldsymbol{f_{pe,rel}}(t, t_o) \tag{40}$$

where $\boldsymbol{f_{pe,rel}}$ is vector of the tendon segment due steel relaxation:

$$\boldsymbol{f_{pe,rel}}(t, t_o) = \boldsymbol{T_e}^{\boldsymbol{T}} \boldsymbol{r_e}. F_{po}(t_o). \emptyset_{p,rel}(t, t_o) \tag{41}$$

Adding up to the Eq. (21) the contribution of the unbonded tendon and the influence of the steel relaxation on the beam, the general expression for the structure is obtained by:

$$\boldsymbol{K}(t, t_o). \boldsymbol{u}(t) = \boldsymbol{f}(t) + \boldsymbol{f_{sh}}(t, t_o) - \boldsymbol{f_{cr}}(t, t_o) + \boldsymbol{f_{p,rel}}(t, t_o) \tag{42}$$

3 Numerical analysis

The implementation of the finite element formulation was performed using MATLAB software in order to simulate the behavior of the unbonded prestressed concrete beams under long term service loads . The models implemented in this study are continuous beams shown in Figure 3, whose parameters are defined in Table 1. The validation of the implemented models occurred by comparing the results obtained by Lou et al [2] through load-displacement curves. The beams evaluated by Lou et al [2] are designated as, YLA2, YLB2, and YLC1 with dimensions of 15 x 30 x 1000 cm and supported by three supports.

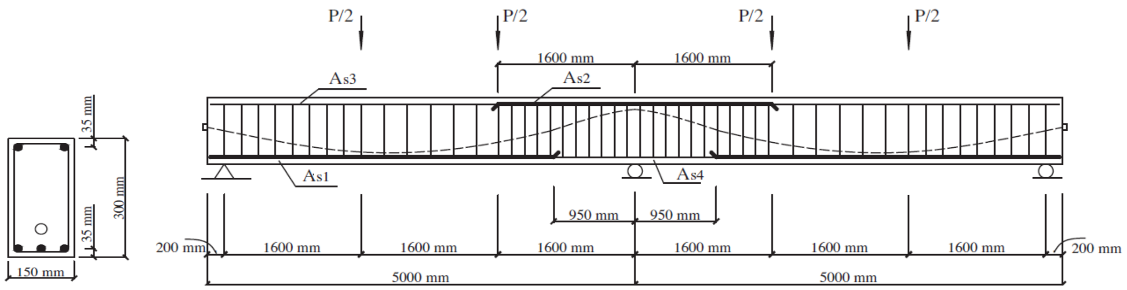


Figure 3. Continuos beams

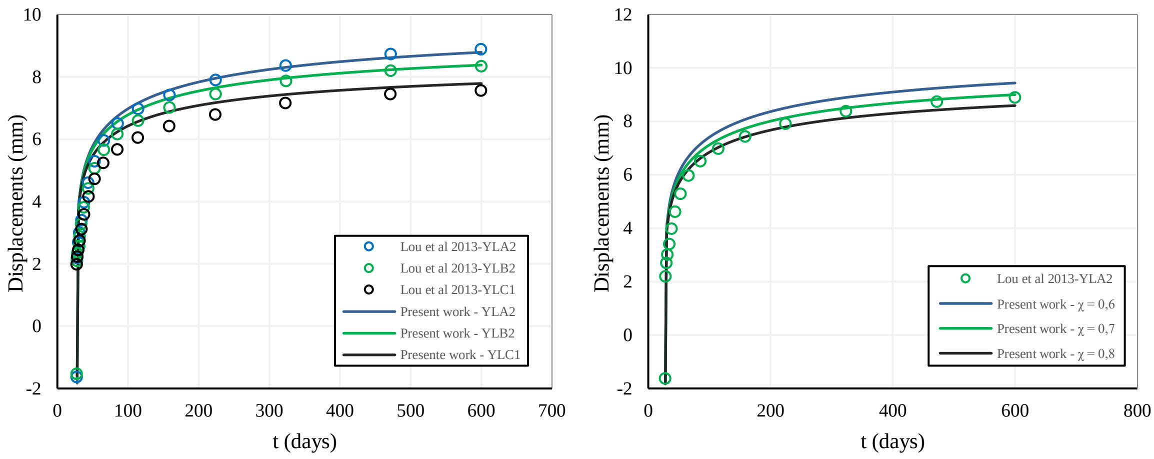
In the implementation of the models, the service load applied to the beam of 30 kN was considered, with a start and end date for loading the structure from 28 to 600 days, respectively. The shrinkage strain $\varepsilon_{sh}(t)$ was determined by the model proposed by CEB-FIP [4]. The relative humidity of the air was considered to be 60%.

Table 1. Material Parameters

Beams	A_{s1} (cm²)	A_{s2} (cm²)	A_{s3} (cm²)	A_{s4} (cm²)	f_{yk} (MPa)	f_c (MPa)
YLA2	4,524	2,262	2,262	2,262	361	36,7
YLB2	6,032	5,089	2,262	6,032	361	33,0
YLC1	7,634	7,634	2,262	7,634	361	37,1

The Figure 4a represent the results obtained for the load-displacement curves for beams YLA2, YLB2 e YLC1. Note that the results obtained present a very good agreement with the data developed by Lou et al [2].

In the Figure 4b there is the presentation of load-displacement curves for beam YLA2 with age coefficients (χ) ranging from 0,6 to 0,8. The curves compared with the results of Lou et al [2] indicate an influence of the age coefficient inversely proportional to the results of the literature presented, signaling the best responses for the age coefficient with a value of 0,7.



(a) (b)

Figure 4. Results displacement-load curves

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4 Conclusions

In this paper, a finite element formulation for the numerical simulation of unbonded prestressed concrete beam under long-term service loads was developed and implemented. The model considered materials in linear regime and disregarded geometric nonlinearities. A plane frame element with 7 degrees of freedom was used to simulate the reinforced concrete section and a finite truss element to simulate the unbonded tendon.

The implementation of the model used the Adjusted Effective Modulus Method to evaluate the effects of creep and shrinkage where were obtained excellents results compared with literature results. The analysis also showed that the aging coeficiente presented best results with values about 0,7.

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