Time-dependent analysis of critical buckling load using optimization technique

Kaique M. M. Magalhães[1], Alexandre M. Wahrhaftig[2], Marcelo A. da Silva[3], Reyolando M. L. R. F. Brasil[4]

[1]*Graduate Program in Civil Engineering, Dept. of Structure and Geotechnic Graduates, University of São Paulo*
Av. Prof. Almeida Prado, trav.2, 83, Cidade Universitária, 05508-900, São Paulo, Brazil
kaiquemagalhaes@usp.br
[2]*Dept. of Construction and Structures, Federal University of Bahia*
Rua Aristides Novís, 02, 5º andar, Federação, 40210-630, Salvador, Brazil
alixa@ufba.br
[3,4]*Center for Engineering, Modeling and Applied Social Sciences, Federal University of ABC*
Rua Arcturus, 03, Jardin Antares, São Bernardo do Campo, 09606-070, São Paulo, Brazil
[3]*marcelo.araujo@ufabc.edu.br;* [4]*reyolando.brasil@ufabc.edu.br*

Abstract. A mathematical and numerical analysis was developed to determine the time-dependent critical buckling load of a slender reinforced concrete column, by applying optimization techniques. In the analysis, the non-linearity due to the slenderness of the system together with the rheological property of the concrete considered through the Eurocode 2 was considered in the calculation. The analytical solution was based on the Rayleigh method and the numerical one was developed using the Finite Element Method (FEM). To compare results, two trial functions of the Rayleigh method were used, a trigonometric and a quadratic. In this context, the quadratic equation was used to apply the optimization technique with its coefficients adjusted to produce results as close as possible to that from FEM, which had been evaluated the found error.

Keywords: critical buckling load, optimization, trial function.

1 Introduction

The structural designs of slender reinforced concrete elements are reason for constant investigation in academia and industry, since such elements present very peculiar behaviors in relation to others structural systems usually adopted in reinforced concrete elements. The slenderness of a structure is measured by a coefficient calculated by the ratio of its buckling length and the gyration radius, which involves the cross-section, therefore, the greater this coefficient, the greater the slenderness of the element and, consequently, more careful analyzes should be done to determine the equilibrium condition.

Slender columns, which according to the definition of Uzny [1], are continuous structural elements subjected to different compressive forces, have a wide variety of applications in the industry, such as mobile phone towers, poles, wind energy turbines, etc. Such elements are dimensioned or assessed considering more appropriately the field of stability than that the strength of materials, i.e., the column may be subject to collapse without having reached its ultimate compressive load. Usually, the ultimate limit state related to slender columns submitted to compression is the buckling, which is characterized by a quick lateral deviation, without structural warning and, therefore, dangerous from the safety perspective.

Elastic buckling of columns was first investigated by Euler [2], but in his first postulations he was not able to incorporate the influence of the columns self-weight, which in some particular cases can lead to significant errors to the problem (Wahrhaftig et al. [3]). In order to improve Euler's formulation, Timoshenko [4] managed to solve the problem adding to the analytical formulations the consideration of the structural self-weight, making the formulation more realistic.

Currently, the use of software for structural analysis is increasingly frequent, due to the ease of computational methods in solving complex problems, such as structures with various degrees of freedom, complex geometries, and high slenderness. However, it is important to emphasize that solutions by numerical methods are based on analytical procedures, so both methods are still important in the solution process.

Regard the use of computer in engineering, Courant was the first researcher to create a numerical model for structural analysis using the principle of stationary potential energy, assuming a linear distribution of functions, whose approximation extends to the Rayleigh-Ritz (RR) model. Rayleigh [4] studied vibration problems and presented his postulates in his book called "Theory of Sound". The main advantage of the Rayleigh method is that a global approximation for spatial and temporal discretization is obtained, adopting a trial function that represents the deformed behavior of the element, in addition to reducing the problem to an algebraic system, which is easier to solve.

The trial functions chosen during the problem-solving process must satisfy the boundary conditions of the structural element, in addition to being differentiable throughout its domain. Notably, for different trial functions, different results will be obtained. In this sense, this paper presents a mathematical analysis based on dynamics and applying the Rayleigh method, and numerical with modeling by the Finite Element Method (FEM), to investigate the fit of trial functions in solving the critical load problem of a slender column of reinforced concrete. Optimization techniques were applied to the trial function seeking to minimize the error of mathematical modeling with FEM.

2 Analytical procedure

The analytical procedure developed in this article was based on the Rayleigh method, applying the principle of virtual work (PVW). The mathematical model, shown in Figure 1, represents a column fixed at the bottom and free at the top end, where the PTW was described in terms of coordinates defined at the free end of the column, considering an undamped free vibration:

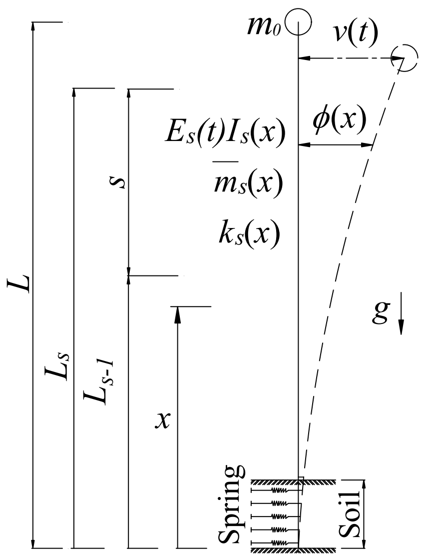


Figure 1. Mathematical model

This model represents a column under axial compressive forces originated by its self-weight and a concentrated mass at the free end. That column may have constant and variable properties along its length. These properties include geometry, elasticity or viscoelasticity, and density, given by $I_s(x)$ e $E_s(t)$, respectively, where the index s represents the considered segment; x is the geometric independent variable with origin at the base of the column. Spring stiffness $k_s(x)$ represents the soil-structure interaction; $\phi(x)$ is the deformation approximate function (trial function); t indicates time; L is the total length of the structure; L_s and L_{s-1} are the height at the upper and lower limits of a given segment, whose length is obtained by the difference between these two positions, and

$v(t)$ is the generalized time-dependent coordinate localized at the free extremity of the column.

For comparison purpose two expressions were used as a trial function: a trigonometric function, Equation (1), and a quadric function, Equation (2).

$$\phi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right), \tag{1}$$

$$\phi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right)\left(a\left(\frac{x}{L}\right)^2 + b\frac{x}{L} + c\right). \tag{2}$$

In Equation (2), the coefficients a, b, and c were adjusted using optimization techniques. Equation (3) calculates the vibration frequency of the structure, in hertz, as a function of time, considered in the portion of the generalized stiffness, $K(m_0,t)$, and the generalized mass concentrated at the free tip, $M(m_0)$.

$$f(m_0,t) = \frac{1}{2\pi}\sqrt{\frac{K(m_0,t)}{M(m_0)}}\,, \tag{3}$$

where the total stiffness of the structure is composed by three terms, as presented in Equation (4):

$$K(m_0,t) = K_0(t) - K_g(m_0) + K_{So}\,, \tag{4}$$

where the final stiffness varies over time and with the lumped mass localized at the tip, being calculated as the sum of the stiffness of each segment defined in the geometry of the analyzed element. $K_0(t)$ is the conventional generalized stiffness, $K_g(m_0)$ is the portion referring to the geometric stiffness, which depends on the effort existing axial, and K_{So} is the elastic stiffness, for consideration of the soil-structure interaction.

Applying the PTW and its derivatives, similarly as in Wahrhaftig et al. [6], the stiffness and the mass of the system can be obtained. The portion of conventional generalized stiffness is defined as:

$$K_0(t) = \sum_{s=1}^{n}\int_{L_{s-1}}^{L_s} E_s(t) I_s(x)\left(\frac{d^2\phi(x)^2}{dx^2}\right)^2 dx\,, \tag{5}$$

and the geometric one as:

$$K_g(m_0) = \sum_{s=1}^{n}\int_{L_{s-1}}^{L_s}\left[N_0(m_0) + \sum_{s+1}^{n} N_s + \bar{m}_s(x)\left(L_s - x\right)g\right]\left(\frac{d\phi(x)}{dx}\right)^2 dx\,, \tag{6}$$

where $N_0(m_0)$ is the force concentrated at the tip of the system, $N_0(m_0) = m_0 g$; and N_s is the normal force in the segments above the considered segment, given by:

$$N_s = \int_{L_{s-1}}^{L_s} \bar{m}_s(x)\left(L_s - x\right) g dx\,, \tag{7}$$

with $\bar{m}_s(x)$ being the mass per unit length. The elastic portion of soil K_{So} is found by:

$$K_{So} = \sum_{s=1}^{n}\int_{L_{s-1}}^{L_s} S_s D_s(x)\phi(x)^2 dx\,, \tag{8}$$

where the S_s is the elastic soil parameter, considered constant, in this case, in each layer of soil; and $D_s(x)$ is the diameter along the depth of the foundation, which depends on its geometry, it may be constant or not. To determine the total generalized mass of the system is necessary to do:

$$M(m_0) = m_0 + \sum_{s=1}^{n}\int_{L_{s-1}}^{L_s} A_s(x)\rho_s\left(\phi(x)\right)^2 dx\,, \tag{9}$$

where $A_s(x)$ represents the cross-section area and ρ_s the density of the material, in the respective segment s. Taking the above considerations and using the concentrated mass as the independent variable of the problem, the critical buckling force, N_f, can be found by the concept present in Equation (9), such as:

$$f(m_0, t) = 0 \Rightarrow N_0(m_0)\Big|_{f(m_0,t)=0} = N_f. \tag{10}$$

3 Model of the practical investigated structure

The structure investigated in this paper, is a slender, real, reinforced concrete pole with variable geometry, shown in Figure 2, which is 46 m high, including the 40 m superstructure, with a hollow circular section, and the foundation, which is relatively deep, 6 m long, is a circular full section-type foundation. Its slenderness index is greater than 400. The geometric details of the column are shown in Figure 2, where g is the acceleration due to gravity; S, D, and th are the type of section, the outside diameter, and the wall thickness, respectively; d_b represents the diameter of the steel bars; n_b is the number of bars; and c' is the concrete cover in the respective cross sections.

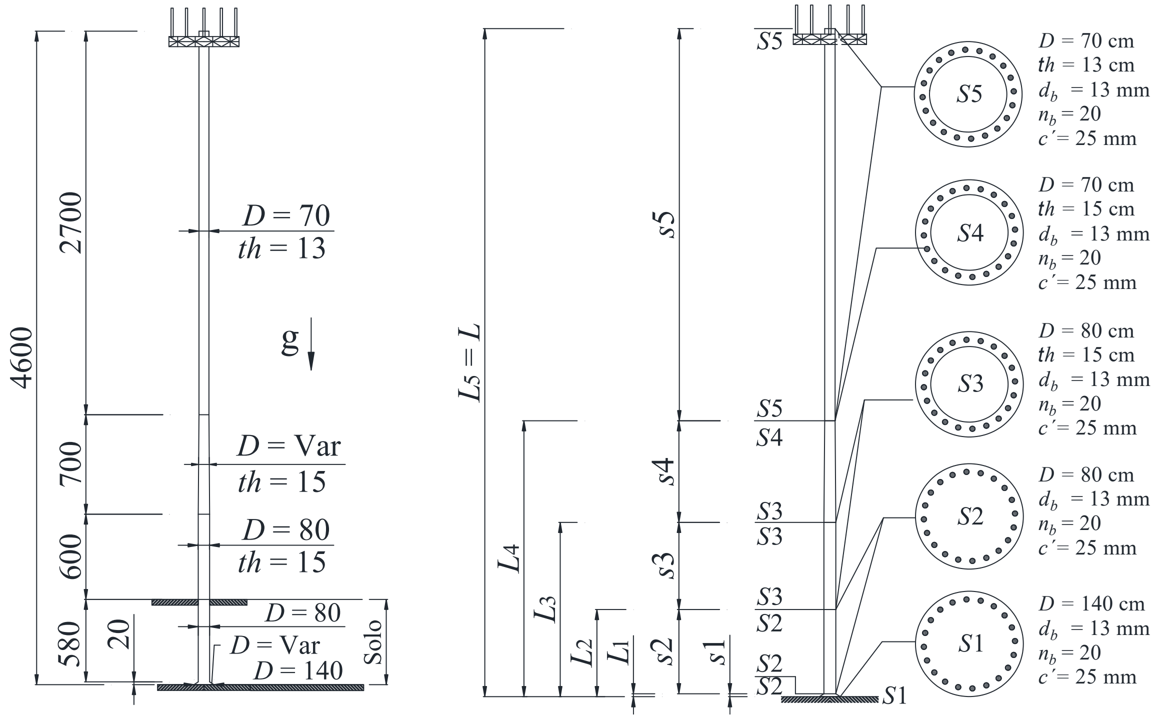


Figure 2. Details of the structure

The lateral action of the soil was represented by an elastic parameter equal to 2669 kN/m^3. The rheological property of the concrete was taking account by Eurocode 2 [7]. The modulus of elasticity adopted for the superstructure and foundation were 37,566 MPa and 25,044 MPa, and their density was 2,600 kg/m^3 and 2,500 kg/m^3, respectively. Modal analysis was performed using the finite element method (FEM), through SAP2000 [8] a commercial software package. Modal shapes for the structures were obtained nonlinearly. The structures were modeled using bar elements with constant and variable cross-sections, as appropriate. The discretization of the structure in FEM was 51 frame elements.

4 Optimization technique

The main objective of optimization is to maximize or minimize functions subjected to restrictions, the problem can be presented as follows:

minimize f(x)

subjected to h,

where f is the objective function and x is the variable vector of the objective function and h is the vector corresponding to the constraints.

In this analysis, optimization techniques were used to approximate the coefficients of Equation (2) with the results generated by the numerical model developed using FEM. The optimization problem took as reference an objective configuration, in this case, consider if a curve obtained in finite element modeling. The coefficients a, $b,$ and c of Equation (2) were determined by solving an optimization problem following the procedure described below:

Determine the vector **b**:

$$\mathbf{b}^T = \begin{bmatrix} a & b & c \end{bmatrix}, \tag{11}$$

that minimize the objective function,

$$f(\mathbf{b}) = \frac{1}{2} r \sum_{i=1}^{q} \left[\phi(\mathbf{b}, x_i) - \phi_{FEM}(x_i) \right], \tag{12}$$

subjected to,

$$\phi(\mathbf{b}, L) = 1 \,. \tag{13}$$

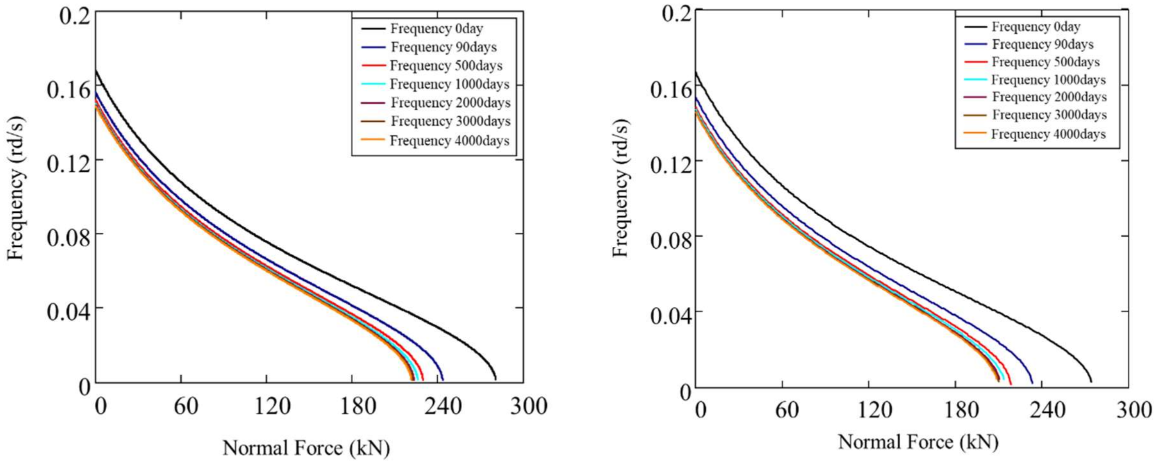
5 Results and discussions

To develop the method, two trial functions were used, a trigonometric and a polynomial function. The trigonometric equation is considered to be the exact solution of the problem even for tapered structures (Wahrhaftig [9]). The polynomial function coefficients were determined through optimization techniques and the results are presented in Table 1, where r is the penalty parameter; a, b, and c are the applied coefficients to optimize the equations.

Table 1. Errors and coefficients obtained for the optimization problem.

Polynomial function (Equation (2))
r = 100,000
a = -0.26161
b = 0.435296
c = 0.826314
Error = 3.70

Figure 3 presents the results obtained for the critical buckling load by the analytical procedure, using the trigonometric and polynomial equation.

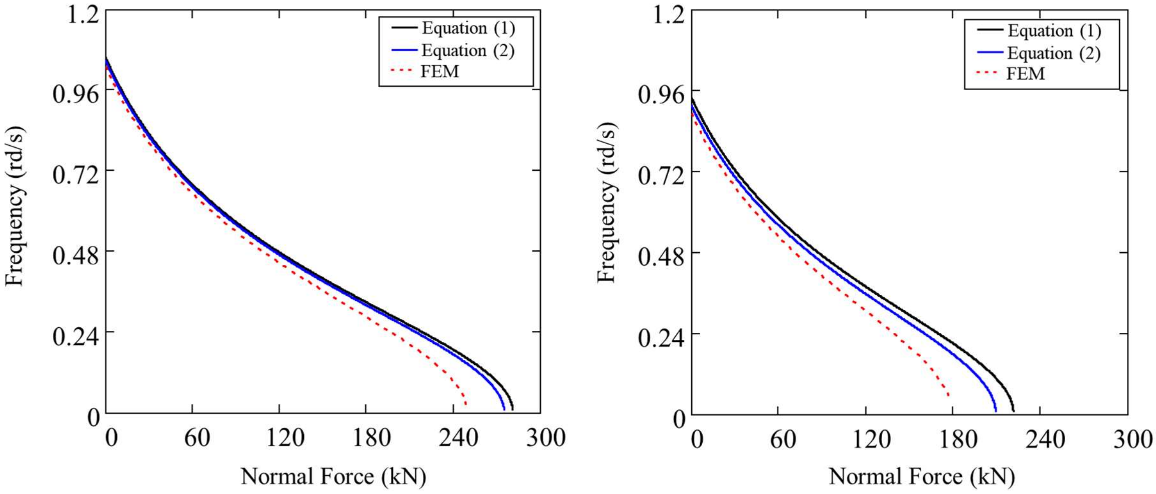


(a) Trigonometric function

(b) Polynomial function

Figure 3. Results for the frequency by Rayleigh method.

Figure 4 shows the comparison between analytical results and MEF for elapsed times equal to 0 and 4000 days. In Table 2 are presented values to the critical buckling load for both analytical and computational procedures.



(a) $t = 0$

(b) $t = 4000$ days

Figure 4. Comparison of the FEM and analytical results.

Table 2. Critical buckling load

Eq.	Classification	Error	Buckling load (kN)			
			$t = 0$	Dif. FEM	$t = 4000$ days	Dif. FEM
FEM	Reference	-	249.6	-	179.5	-
Eq. (1)	Trigonometric	42.2	281.1	12.9%	221.8	23.6%
Eq. (2)	Quadratic optimized	3.7	274.9	10.4%	209.7	16.8%

Eq. = equation; Dif. = difference to FEM.

6 Conclusions

In this article, a mathematical and numerical analysis was developed that determined the time-dependent critical load of a slender reinforced concrete column. The analytical method was based on the concepts of dynamics by applying the Rayleigh method and the numerical by finite element method (FEM). It is possible to conclude that for $t = 0$, the instant at which the structure is loaded, the lowest critical load for the first optimization problem (249.6 kN) was obtained using the FEM, while the largest critical load of 281.1 kN was provided by the trigonometric function given by Equation (1). It was observed that, even by obeying the boundary conditions of the problem, the difference Equation (3) and the FEM solution was 10.4%. It was noted, with the optimization techniques, that was possible to reduce the error between the result obtained by Equation (1) in relation to the FEM, for the time 0 from 12.9% to 10.4% (2.5% reduction), and for 4000 days from 23.6 % to 16.8% (6.8% reduction). It is also concluded that the results obtained by the FEM are more conservative for determining the critical buckling load of the structure.

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