Time dependent reliability: a time series point of view

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Abstract. Engineering problems where material properties deteriorate over time or in cases of random loading modelled as a random process, the evaluation of the probability of structural failure generates a significant computational cost, mainly because it is a time-variant reliability problem. Stochastic problems, in which the probability of failure is time-dependent, shows to be interesting to know about the computational cost and accuracy of the method disponible in the literature to evaluate the reliability. The assessment of time-dependent reliability problems is still a challenging task. Besides the difficulty to characterize a problem from real-world data, most of known solutions rely on approximations suitable only for specific cases or on burdensome simulation approaches. This is due to the difficulty in working with general stochastic processes, particularly for situations of non-ergodicity. A time-series model is a particular case of stochastic process that operates in continuous state space and discrete time set. Such models can be used to represent a wide range of random phenomena that spans through time, usually with simpler formulation. They are also relatively simple to build from data tables, which are usually all the information available about time-dependent behavior of random engineering systems. Thus, this work makes a comparison of the application between the expansion optimal linear estimation (EOLE) method and time-series model (e.g. ARMA), used to evaluate the time-dependent failure probability, presenting the computational cost and the accuracy of the results obtained, and details about the solutions are addressed.

Keywords: Time-dependent Reliability, Time-series models, Structural Reliability

1 Introduction

In deterministic methods, all the values of the parameters involved in the analyzed problem are exactly known. However, due to the uncertainties of the problem and the influence of several external factors, these values must be evaluated as random variables with a non-zero probability of failure.

Thus, over the last few years, research in the field of reliability analysis has been intensified due to the need to take this factor into account in the problems analyzed. It is important to incorporate these uncertainties into the projects, considering that the engineering project must meet a series of criteria and some of these depend on different sources (types of forces and loads, for example) that can be addressed such as independent or time-dependent reliability issues.

From the works of [1] and [2], where the authors showed a method of importance sampling applied to stochastic processes (more precisely applied in dynamic systems), using autoregressive temporal series, AR, ARMA and ARIMA models (already known in the areas of finance, economics, biomedicine) to generate random trajectories, being one of the first works to apply this type of series in stochastic processes aimed at engineering. The technique is also based on first-passage-time problems and, unlike the KL and EOLE expansions that always assumed stationarity and the costly calculation of eigenvalues/vectors, trajectories via AR do not suffer from these limitations. Results showed that, when compared to Monte Carlo, AR required a much smaller value of G simulations.

Although some authors such as [3], [4], [5] and [6] have performed reliability analysis with autoregressive moving average models, such studies were not directly directed to the calculation of the probability of instantaneous or accumulated, mainly for small values, which are the most usual in the field of Engineering, in addition to not having its effectiveness compared to other models in the literature.

In the work of [6], the author shows some studies on the application of ARMA models in structural engineering focused on the area of atmospheric turbulence, waves and earthquakes. However, the author does not perform the time-dependent reliability analysis per se, he only highlights some preliminary studies that were carried out in this field, most of them using experimental data. The author makes clear the intention of using ARMA models in structural reliability problems with the advancement of computational technology, but he does not carry out such studies.

Thus, the objective of this paper is to perform a time-dependent reliability analysis from time series models, adopting here the Autoregressive Moving Average (ARMA) model and compared to the results via EOLE, in order to verify the convergence and efficiency of such a model. Among the possible advantages of using ARMA models in time-dependent reliability analysis, compared to other methods, we can mention:

- Generally, time series models have simpler formulations (basically they are composed by the combination of polynomials and linear combinations) and with little computational cost, resulting in advantages for future research, especially when involve derivation and optimization;
- As it is already known and used in several other areas (eg economics, biology, climatology, etc.), it emphasizes its great efficiency, applicability and robustness, appearing as an alternative use also in the area of structural reliability;
- If the problem already has a history of data, this original data can be used in its entirety to calibrate the time series models, bringing it closer to reality;
- If its efficiency and convergence are shown, the traditional techniques of importance sampling and optimization may also be applied to these models;
- • In the studies found on reliability analysis and application of time series models, for the most part, a major limitation found was the need for a historical series of initial data for creation and application of these models. However, the purpose of this paper is to use only the statistical properties of the problem, without necessarily needing the initial historical data.

In the next sections, a brief theory about the equations involved in time-dependent reliability analysis will be presented (Section 2), where the EOLE and the time series model used will be shown. In Section 3 we present the formulation of the computational algorithm used, followed by numerical examples (Section 4) and conclusions (Section 5).

2 Time-dependent reliability analysis

Stochastic processes $\{X(t) : t = 0, \pm1, \pm2, \pm3, ...\}$ can be understood as statistics and/or uncertainties that evolve in time or space, for example: dollar value, tide variation, annual temperature, dynamic load, wind load during the year, level of corrosion, among others. There are basically two main ways to do the reliability analysis of these stochastic processes, either through the calculation of the failure rate, for example the PHI2 ([7]), PHI2+ ([8]), JUR/FORM ([9]), TRPD ([10] and [11]) and NEWREL ([12]) methods, and others, or through estimates based on Monte Carlo, for example EOLE-OLE-KL ([13]) and time serie model.

Although it is usually faster and less computationally costly, the methods based on the failure rate have several limitations regarding inaccuracy in the result, because they are approximations often adopting linear conditions for the entire problem and usually have complex mathematical formulas, making it difficult for engineers to understand and apply.

Furthermore, when non-stationary processes are adopted, these methods usually have a significant increase in the computational cost, losing efficiency. Finally, since it involves the concept of crossing rate, the model is most often adopted as a Poisson process, which is still a controversial issue in the scientific community, mainly due to the existing approximations that it entails in the determination only of the maximum values (limits).

Regarding the methods based on the Monte Carlo simulation, although they are usually more computationally expensive [14], they present better results. In the EOLE case, even presenting good results because it is a process based on Monte Carlo estimation, it was found in some cases where the time discretization value is small, that eigenvalues and eigenvectors can lead to numerical inconsistencies, becoming a problem, in addition to its high computational cost.

Thus, time series models, as they are already well known in several other areas (eg economics, natural science, biomedical, meteorology, etc.) are also based on the idea of Monte Carlo and not have the problems and limitations found in EOLE, mainly because they make use of linear combinations and polynomial functions, making them less computationally costly and maintaining efficiency in the convergence of results, it emerges as a more robust alternative to solve time-dependent reliability issues.

Regarding the probability distribution f, this type of process is defined by the distribution of $x(t)$ for all values of t, that is, $f(x(t), t)$. When it comes to the mean and standard deviation, these parameters can also have their values changed (or not) over time. As it is difficult and costly to calculate such process, some models are usually used to simplify calculations and stochastic analyses, mainly by adopting stationary and ergodic processes.

In the case of the probabilistic distribution, if it does not depend on t, that is, when $f(x(t), t) = f(x(t))$, in this case, it is said that the process is stationary, with mean μ constant equal to:

$$E[x(t)] = \int_{\omega} x(t).f_x(x(t)).dx = \mu, for\{t = 0, \pm 1, \pm 2, \pm 3, ...\} \tag{1}$$

Furthermore, analyzing the variation of values in these processes, we have that sometimes the values obtained in two consecutive times t_a and t_b may have a low correlation, sometimes they may have a high correlation. Using the covariance formula $\gamma_{a,b} = Cov[t_a, t_b]$ to help determine the level of correlation, we have:

$$Cov[x(t_a), x(t_b))] = \gamma_{a,b} = E[x(t_a), x(t_b)] - E[x(t_a)].E[x(t_b)] \tag{2}$$

because it is a stationary process, the f doesn't change and the value of Cov will depend on the difference τ (or Δt) of time:

$$\gamma_{t,t+\Delta t} = \gamma_{0,\Delta t} = \gamma_{\tau} = E[x(0), x(\Delta t)] - E[x(0)].E[x(\Delta t)] \tag{3}$$

From Cov value, can also calculate the correlation coefficient $\rho_{a,b}$:

$$\rho_{a,b} = Corr(x(t_a), x(t_b)) = \frac{\gamma_{a,b}}{\sqrt{\gamma_{a,a}\gamma_{b,b}}}, \tag{4}$$

When it comes to mean variation, another way to simplify stochastic problems is to define that they behave as ergodic processes, that is, when the average of a process performance is equivalent to the average (or expected value) of $x(t)$ for all t. More details on the equation of stochastic processes can be found in [15].

To calculate the failure probability, a limit state function $g(\mathbf{X})$ is used, where $\mathbf{X}$ is the set of random variables that are related to the safety of the structure, such that $g(\mathbf{x}) < 0$ indicates the system failure, $g(\mathbf{x}) > 0$ indicates that the system is safe and $g(\mathbf{x}) = 0$ defines the limit state function of the reliability problem.

Although the stochastic process varies over time, it is also possible to calculate the probability of instantaneous failure at a fixed time, making a reliability analysis similar to that with only random variables, represented as follows:

$$P_{f_i}(t) = Prob[g(t, \mathbf{X}(t, \omega)) \leq 0] = \int_{G<0} f_{\mathbf{X}}(\mathbf{x})\mathbf{dx}, \tag{5}$$

However, as the problem must be evaluated at all times $[t_1, t_2]$, it is necessary to calculate the cumulative failure probability $P_{fc}(t_1, t_2)$ according to:

$$P_{f_c}(t_1, t_2) = Prob[\exists \tau \in [t_1, t_2] : g(\tau, \mathbf{X}(\tau, \omega)) \leq 0] \tag{6}$$

For being based on the Monte Carlo simulation, making $i = 1, ..., N$ then evaluates all times $t_i = (i-1).\Delta t$, with $\Delta t = \frac{\tau}{N-1}$.To calculate the number of failures between times $[t_i; t_{1+1}]$, a counter k_{i+1} is used and stores a failure counter that refers to the interval $[t_i; t_{1+1}]$, so that all counters k_n, with $n = i+1, ..., N$ are increased whenever the limit state is violated for the first time (i.e. all the remaining counters after the outcrossing are increased).

Assuming k_0 as the number of failures at time $t = 0$, the cumulative failure probability $P_{fc_{MC}}$ can be calculated by:

$$P_{fc_{MC}}(0, t_i) = \frac{k_i + k_0}{N_{MC}}, \tag{7}$$

2.1 Expansion Optimal Linear Estimation - EOLE

The EOLE, developed by [13], is a method used to generate the $X(t, \omega)$ simulations based on Monte Carlo over time. Selecting P points in the interval $[0, \tau]$, with $t_1 = 0$ and $t_P = \tau$, represents the EOLE through:

$$X(t, \omega) \approx \mu(t) + \sigma(t) \sum_{i=1}^{r} \frac{\xi_i(\omega)}{\sqrt{\lambda_i}} \phi_i^T C_{t,t_i}(t), \tag{8}$$

where $\mu(t)$ is the mean, $\sigma(t)$ the standard deviation, $\xi_i(\omega), i = 1, ..., P$ are independent normal random variables, $\phi_i, \lambda_i, i = 1, ..., r$ are the eigenvectors and eigenvalues of the correlation matrix $C_{ij} = \rho_X(t_i, t_j), i, j = 1, ..., P$, where $\rho_X(t_i, t_j)$ and the autocorrelation coefficient function evaluated at the points $[t_i, t_j]$ and $r \leq P$ is the order of expansion.

2.2 Temporal Series Models

As detailed in the materials of [15] and [16], some characteristics of the time series models will be detailed below.

Time series are important and are already well known in the areas of economics, market and natural sciences, examples of which are in the analysis of the stock market index, temperature values, rainfall etc. In brief, through them we are able to simulate future observations from databases and/or past/present measurements. To estimate values, time series use, in addition to a historical data series, autocorrelation values, that is, the relationship and dependence of a given measure with its previous values.

Among the most common models are the Auto-Regressive (AR), the Moving Average (MA), the Auto-Regressive Moving Average (ARMA) and the Auto-Regressive Integrated Moving Average (ARIMA). As in this paper we adopted stochastic processes as stationary and ergodic, even so that EOLE could be used, only ARMA(p,q) models were adopted, which can be understood as a combination of AR(p) and MA(q) models), with p indicating the AR(p) model order and q the MA(q) model order.

The ARMA(p, q) model can be represented by:

$$X(t,\omega) - \mu = \varphi_1(x_{i-1} - \mu) + \varphi_2(x_{i-2} - \mu) + \cdots + \varphi_p(x_{i-p} - \mu) + \epsilon_t - \theta_1\epsilon_{t-1} - \theta_2\epsilon_{t-2} \cdots - \theta_q\epsilon_{t-q} \quad (9)$$

where μ is the mean, $\epsilon_i \equiv N(0, \sigma_e^2)$ is Gaussian white noise, $\varphi_1, \varphi_2, ..., \varphi_p$ and $\theta_1, \theta_2, ..., \theta_q$ are the parameters to be estimated.

The parameters ϕ and θ are commonly found from the historical data of the series or through only the values of the mean μ, standard deviation σ and the value of the autocorrelation coefficient function $\rho_X(t_i, t_j)$, this being the path adopted in this paper, similar to the same data needed to obtain EOLE.

3 Algorithm formulation for ARMA and EOLE models

In order not to necessarily need a historical series of data to create the $ARMA(p, q)$ model [17], the purpose of this work was to elaborate an ARMA model based only on values that characterize the stochastic process, which are the value of time mean of the process μ, standard deviation σ and value of the autocorrelation coefficient function $\rho_X(t_i, t_j)$, being basically the same parameters necessary to obtain the results via EOLE.

In the analysis, we adopt the same parameters for EOLE and ARMA models and two different paths were performed, where in first path the ARMA model is created from EOLE, and in second path the EOLE model is created from ARMA , so that it was possible to assess the quality via both possibilities, as shown in Fig. 1.

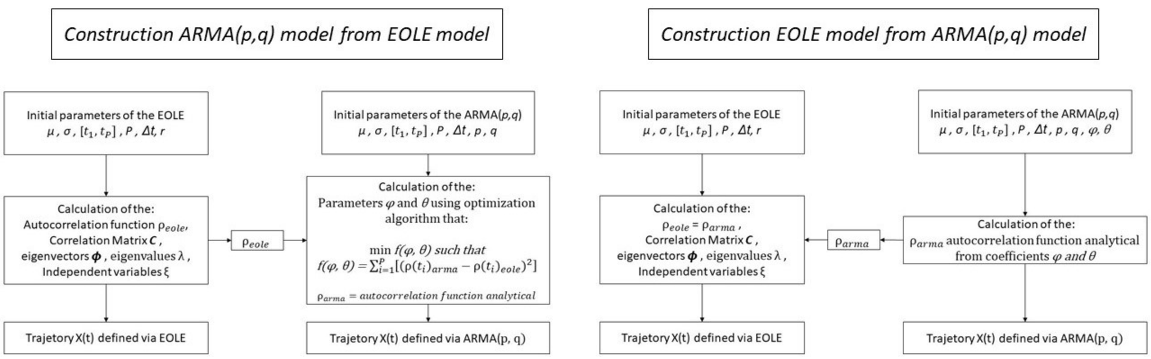

Figure 1. Construction ARMA(p,q) and EOLE

In this paper, adopting α as the autocorrelation length, the autocorrelation function ρ_{eole} used between two points in time x_{i+1} and x_i was of the type:

$$\rho(x,\alpha) = \exp\left[-\left(\frac{x_{i+1}-x_i}{\alpha}\right)^2\right] \tag{10}$$

The autocorrelation function ρ_{arma} used in the ARMA models was calculated analytically from the values of the coefficients $\varphi_1, \varphi_2, ..., \varphi_p$ e $\theta_1, \theta_2, ..., \theta_q$, in addition to the mean μ and standard deviation σ, which define the stochastic process. The developed algorithm was based on [18].

4 Numerical Examples

In order to evaluate the capability of using ARMA models, in comparison to EOLE, in different situations for time-dependent reliability analysis, 1 numerical example was chosen. The chosen example is a benchmark that shows a beam subject to corrosion subjected to a stochastic process and four random variables.

In this example, the ARMA models were created starting from the EOLE models, as shown in Fig. 1.

4.1 Example 1: steel beam subject to corrosion under random loading

The example was adapted from [7], of a beam subject to corrosion and acting with a time-dependent load, as shown in Fig. 2. Here it is a problem of variable barrier, determined by the limit state equation eq.(11). The mean and standard deviation values of the random variables and the stochastic process, as well as the values of the beam properties, are presented in Table 1.

$$g(t) = \frac{(b_0 - 2\kappa t)(h_0 - 2\kappa t)^2 \sigma_y}{4} - \left(\frac{F(t)L}{4} + \frac{\rho_{st} b_0 h_0 L^2}{8}\right), \tag{11}$$

Table 1. Corroded beam random variables and parameters

Parameter	Distribution	Mean	COV
Force F (kN)	Gaussian Process	6000	3%
Steel yield stress σ_y (MPa)	Lognormal	240	10%
Beam initial breadth b_0 (m)	Lognormal	0.20	3%
Beam initial height h_0(m)	Lognormal	0.04	3%
Growth rate κ (mm/year)	Deterministic	0.10	-
Span length L (m)	Deterministic	5.00	-
Steel material mass density ρ_{st} (kN/m^3)	Deterministic	78.50	-

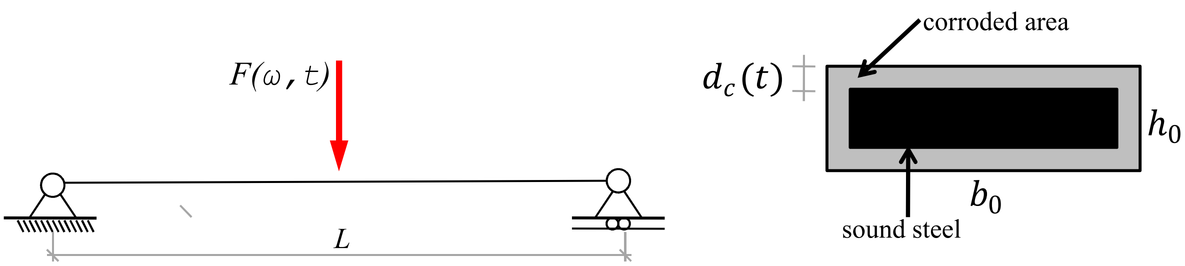

Figure 2. Corroded beam under a midspan load, after [7]

The time interval analyzed was $t = [0, 10]$ years, discretized into 1200 points, using a length and correlation of one month ($\alpha = 1/12$ year) and an expansion of 300 terms in EOLE. For ARMA, the ARMA(8,4), ARMA(16,4) and ARMA(32,4) models were created. The total number of simulations in each analysis was 10^7 simulations and the beam was subjected to a corrosion rate calculated by $d_c(t) = \kappa t$.

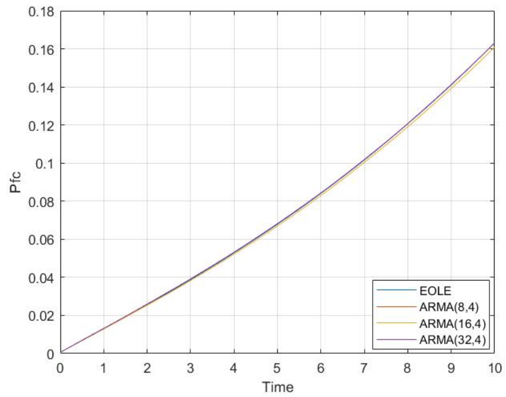

Figure 3. Evolution of the cumulative failure probability in the time

The evolution of P_{fc} over the period of [0,10] years is shown in Figure 3.

The P_{fi} results were analyzed at times t = 0, 5 and 10 years, and the P_{fc} value was analyzed at the final time, as shown in Tables 2 e 3.

Table 2. Values of instantaneous failure probability P_{fi}

	EOLE	ARMA(8,4)	Difference $(\%)$
$P_{fi}(0)(\%)$	0.0638	0.0638	0.47
$P_{fi}(5)(\%)$	0.1701	0.1705	0.24
$P_{fi}(10)(\%)$	0.4296	0.4267	0.68

Table 3. Values of cumulative failure probability $P_{fc}(10)$

	$P_{fc}(\%)$	Difference $(\%)$	95% Confidence Interval
EOLE	16.307	-	[16.284 , 16.330]
ARMA(8,4)	16.094	1.32	[16.071 , 16.117]
ARMA(16,4)	16.133	1.08	[16.110 , 16.156]
ARMA(32,4)	16.315	0.04	[16.292 , 16.337]

According to the results presented, it can be seen that the maximum difference in the P_{fi} values between the EOLE and ARMA models for all times analyzed was of the order of 0.68% and that of the P_{fc} value at the end of the analysis was 1.32%. From the results found, it is possible to notice again a convergence of the results obtained via EOLE and via the ARMA models generated here, confirming the proposal of this paper to use autoregressive mobile-media models in the reliability analysis.

5 Conclusions

In this paper, time-dependent reliability analysis using time-series models was presented, with four numerical examples and comparisons of the results with another method in the literature. From the results, it can be concluded that the $ARMA$ model can be used in the analysis of time-dependent structural reliability, being proven by the convergence of the results when compared to $EOLE$.

Regarding the computational cost, the $ARMA$ models proved to be more efficient than the $EOLE$, mainly due to its simple mathematical formulation basically composed of linear combinations of polynomial functions, unlike the EOLE model. It is $EOLE$ that needs the calculation of eigenvalues and eigenvectors, which increases the computational cost.

Regarding the robustness of the method proposed here, the $ARMA$ models were always robust and convergent, unlike the $EOLE$ which, depending on the values adopted for the time discretization, mainly for small values (optimal values to be adopted were not found in the literature), presented numerical inconsistency in some cases, even appearing complex numbers. Another important point that should be highlighted is that the proposal presented in this paper is the creation of $ARMA$ models starting from the statistical properties of the problem, as shown in Fig. 1, without necessarily needing a data history to creation of the model, as it is normally done, which differentiates it and makes it more applicable to any reliability analysis problem as long as the statistical properties of the problem are known, without necessarily needing the historical data of the series.

Thus, time series models emerge as a robust proposal for the calculation of time-dependent reliability problems, being promising, from the point of view of applicability of the methods, even in cases where it will be necessary, for example, to perform techniques of importance sampling or optimization.

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