Effect of the load factor on the construction of Kriging surrogate models for structural reliability analysis of redundant systems

Mariana O. Milanez[1], Wellison J. S. Gomes[1]

[1]*Dept. of Civil Engineering, Federal University of Santa Catarina*
Rua João Pio Duarte Silva, 88037-000, Santa Catarina, Brazil
mariana.omilanez@gmail.com, wellison.gomes@ufsc.br

Abstract. The combination of surrogate models with structural reliability methods has helped to reduce computational demands, allowing to perform reliability analysis of more complex problems, such as those related to redundant systems. When surrogate models are employed, the actual value of the limit state function is usually necessary for the construction of the surrogate. Then, for all simulations, the load must be incremented until P_{lim}, when failure of the system occurs, increasing the computational costs associated to the mechanical model evaluations. If the simulation is stopped before reaching P_{lim}, there is a loss of accuracy in the evaluation of the limit state function, which may lead to less accurate surrogate models and, consequently, errors in the estimated failure probabilities. This paper aims to investigate when the simulation may be stopped without significant losses in the accuracy of the surrogate model. Failure probabilities and computational costs are compared for a number of structural reliability problems from the literature. For the examples presented herein, results have shown that when the load factor is larger than 1.1, the metamodel may be capable to estimate the failure probability, although it can be necessary more limit state function evaluations to achieve the convergence.

Keywords: Structural system reliability analysis, Redundant systems, Global structural response, Active learning Kriging

1 Introduction

The estimation of systems failure probability involves modeling the structural system and identify the dominant failure modes having more influence on the failure probability, generally represented by event trees (Srividya et al. [1]). However, if a structural model is built and adequately represents the structural response, failure of the system can be characterized considering the global structural response, as presented by Gomes et al. [2]. Global structural responses are usually obtained by employing mechanical models capable of dealing with material and geometric nonlinearities. When dealing with engineering practice, many evaluations of these models are required to solve the corresponding reliability problems, which may result in considerable computational efforts.

In the last few years, the search for efficient procedures to reduce the computational demands has resulted in several approaches such as Subset Simulation (Au et al. [3], Papadopoulos et al. [4]) and the Weighted Average Simulation Method (Rashki et al. [5], Okasha [6]). Moreover, the use of surrogate models, including neural networks (Chojaczyk et al. [7], Gomes [8], Gomes [9]) and response surface (Roussouly et al. [10], Jiang et al [11]), have been shown promising results for many decades. In this situation, the mechanical model can be replaced by a less demanding surrogate model. Once the accuracy of the surrogate response is related to the sample point used to build the model, active learning methods have largely been applied to combine Kriging and reliability methods, due to the ability of Kriging to provide not only prediction but also their variance (Echard et al. [15], Xiao et al. [16], Yuan et al. [13] Kroetz et al. [14], Yi et al. [12]).

However, when using reliability methods as Monte Carlo simulation (MCS), for example, for each sample i, the load is incrementally applied until the value associated with this sample, P_i, is achieved, or until the limit load of the structure, P_{lim}, is reached. The signal of the limit state function is used to determine if the sample is on the failure domain or not. For the small failure probabilities usually found in practice, only a few simulations are performed until P_{lim} is reached. On the other hand, when surrogate models are employed, the actual value of the limit state function is necessary to build the surrogate to obtain accurate results, avoiding errors in the estimated failure probabilities. The aim of this paper is to verify the loss of accuracy in the estimated failure probability when the response of the mechanical model is obtained before failure is reached for samples located on safe domain. To do so, the maximum load to be applied in each simulation is given as the minimum between αP_i, where P_i is the load associated with each sample i, and P_{lim}, considering different values of α. The reliability analysis is performed combining Kriging and Monte Carlo simulation, and a recently developed active learning function is used to choose the sample points necessary to build the surrogate model. The limit state function is formulated considering the global structural response, obtained from a mechanical model capable to represent the redundancy of the structural system as well the dependencies between failure modes.

2 Reliability analysis of redundant structural systems

2.1 Active learning reliability method combining Kriging and Monte Carlo Simulation

Kriging is an interpolation method that aims to predict the response of a given function at any point based on the information obtained at a set of locations known as sample points. This model is based on the idea that a function $g(\mathbf{x})$ may be seen as the realization of a Gaussian process, given by (Echard et al. [15], Xiao et al. [16]):

$$g(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\beta + z(\mathbf{x}) \tag{1}$$

where $\mathbf{f}^T(\mathbf{x})\beta$ is the mean value of the Gaussian process and $z(\mathbf{x})$ is a Gaussian process with zero mean. The covariance, COV, between two points $\mathbf{x}_i$ and $\mathbf{x}_j$, is formulated as (Forrester et al. [17]):

$$COV(\mathbf{x}_i, \mathbf{x}_j) = \sigma^2 R_\theta(\mathbf{x}_i, \mathbf{x}_j) \tag{2}$$

where σ^2 is the process variance and R_θ is the correlation function which depends on a set of parameters θ. There are several correlation functions which can be employed, in this paper the Gaussian correlation function is chosen:

$$R_\theta(\mathbf{x}_i, \mathbf{x}_j) = \exp\left[-\sum_{k=1}^{n} \theta_k \left|x_k^i - x_k^j\right|^2\right] \tag{3}$$

where n is the number of variables and θ_k are the unknown correlation parameters. Based on the design of experiments $\left[\mathbf{x}^{(1)}, ..., \mathbf{x}^{(p)}\right]$ and the corresponding responses $\left[g(\mathbf{x}^{(1)}), ..., g(\mathbf{x}^{(p)})\right]$, on p sample points, the unknown parameters can be determined from maximum likelihood estimation as presented in, for example, in Forrester et al. [17]. The prediction of $g(\mathbf{x})$ at a training point $\mathbf{x}$, follows a normal distribution, with mean $\mu_{\hat{g}}(\mathbf{x})$ and variance $\sigma^2(\mathbf{x})$ expressed as:

$$\begin{aligned} \mu_{\hat{g}}(\mathbf{x}) &= \mathbf{f}^T(\mathbf{x})\hat{\beta} + \mathbf{r}^T(\mathbf{x})\mathbf{R}_\theta^{-1}(\mathbf{g} - \mathbf{F}\hat{\beta}) \\ \sigma_{\hat{g}}^2(\mathbf{x}) &= \hat{\sigma}^2\left(1 + \mathbf{u}^T(\mathbf{x})(\mathbf{F}^T\mathbf{R}_\theta^{-1}\mathbf{F})^{-1}\mathbf{u}(\mathbf{x}) - \mathbf{r}^T(\mathbf{x})\mathbf{R}_\theta^{-1}\mathbf{r}(\mathbf{x})\right) \end{aligned} \tag{4}$$

where $\mathbf{r}^T(\mathbf{x})$ is the correlation between the point $\mathbf{x}$ and each of the p training points, $\mathbf{g}$ is the true performance function response of each one of the p training points, and $\mathbf{u}(\mathbf{x}) = \mathbf{F}R_\theta^{-1}\mathbf{r}(\mathbf{x}) - \beta(\mathbf{x})$.

The accuracy of the metamodel is directly related to the design of experiments (DoE) used in its construction. In this paper, the selection of the sample points is performed in an iterative process, as presented in Echard [15], combining Kriging and Monte Carlo simulation (MCS). The sample points are chosen considering the learning function U_{DS} and the stopping criterion proposed by Xiao et al. [16]. The main advantages of this function are the tendency of reduction of the variance in each iteration and the guarantee of a small value for the variance of the probability of failure at the end of the learning process. The next best sample point $\mathbf{x}^*$ to be added to the DoE is given by:

$$\mathbf{x}^* = \max(U_{DS}); \qquad \text{where: } U_{DS}^i = \Phi\left(\frac{-\mu_{\hat{g}}(\mathbf{x}_i)}{\sigma_{\hat{g}}(\mathbf{x}_i)}\right)\left[1 - \Phi\left(\frac{-\mu_{\hat{g}}(\mathbf{x}_i)}{\sigma_{\hat{g}}(\mathbf{x}_i)}\right)\right] \quad \text{for } i = 1, ..., N_{MC} \tag{5}$$

where $\mu_{\hat{g}}$ is the mean of the prediction and $\sigma_{\hat{g}}$ its standard deviation.

The stopping criterion considered, related to the failure probability, $\hat{P}_f$, and its variance, $VAR_{\hat{P}_f}$, is computed as:

$$\frac{\sqrt{\mathrm{VAR}_{\hat{P}_f}}}{E_{\hat{P}_f}} < \left|\frac{\epsilon}{\Phi^{-1}(\alpha/2)}\right|; \quad \text{where:} E_{\hat{P}_f} = \frac{1}{N_{MC}}\sum_{i=1}^{N_{MC}} \hat{I}_s(\mathbf{x}_i); \qquad \mathrm{VAR}_{\hat{P}_f} = \frac{1}{N_{MC}^2}\sum_{i=1}^{N_{MC}} U_{DS} \tag{6}$$

where $\epsilon \in [2 \times 10^{-2}, 5 \times 10^{-4}]$ and $\alpha \in [2 \times 10^{-2}, 1 \times 10^{-3}]$.

The number p of initial sample points used to build the initial DoE depends on the number of random variables n. According to Xiao et al. [16], if $n < 10$, $p = 12$ is recommended. Otherwise, $p = n + 5$. A coefficient of variation, $COV_{\hat{P}_f}$, less than 5% is usually acceptable in reliability problems [15].

2.2 Limit state function formulation

In this paper, the finite element models are build by using the software MASTAN2, assuming the material behavior as elastic-perfectly plastic, to aim the possibility of progressive collapse. Considering the mechanical analysis, the applied load P_i, associated to each sample point i is divided into increments j. The load applied at a given increment j of the structural analysis is defined by the load ratio λ, defined as a ratio between the applied load at j and the load P_i. The load ratio, λ, is iteratively increased until a value α, defined in each case, and then the response of the mechanical model is computed. The value of α varies between 1.0, in which the maximum applied load is equal to P_i and the value α_{lim} where the applied load is increased until the structural failure (P_{lim}), where the actual value of the limit state function is obtained for all samples. The limit state function is defined as:

$$g(\mathbf{X}) = \lambda(\mathbf{X}) - 1 \tag{7}$$

where $\mathbf{X}$ is the vector of random variables and λ is the maximum load ratio obtained from the mechanical model considering the value α. The procedure of the method is described on the flowchart presented in Fig. 1.

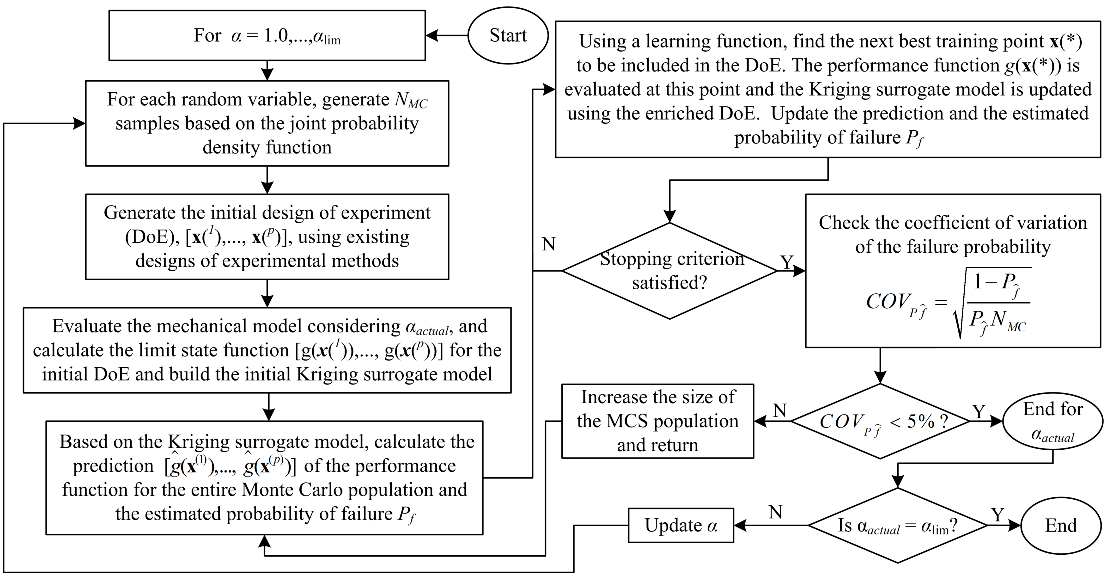


Figure 1. Flowchart of the adopted approach

3 Numerical examples

3.1 Six-bar truss structure

The first example involves the reliability analysis of an indeterminate six-bar truss with one degree of redundancy, shown in Fig 2. The cross-sectional area of each element is equal to 2.30×10^{-4} m^2. The statistically independent random variables are the yield stress $\sigma_{yi}(i = 1, ..., 6)$ and the applied loads $F_i(i = 1, ..., 5)$, with parameters summarized in Table 1. This example has been studied in Kim et al. [18] and Jiang et al. [11].

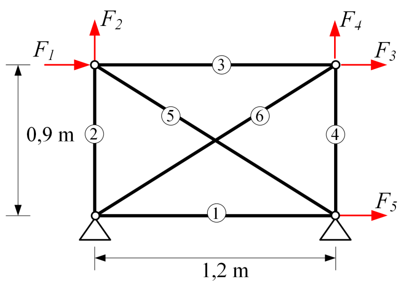


Figure 2. Statically indeterminate six-bar truss

Table 1. Random variables for statically indeterminate six-bar truss

Variable	Distribution	Mean	Coefficient of variation
F_1	Normal	50 kN	0.1
F_2	Normal	30 kN	0.1
F_3	Normal	20 kN	0.1
F_4	Normal	30 kN	0.1
F_5	Lognormal	20 kN	0.1
$\sigma_{yi}(i = 1, ..., 6)$	Normal	276 MPa	0.05

Considering the number of random variables presented in this example, the number of initial training samples p, used to build the initial metamodel, is selected as 16. The initial MCS population and the U_{DS} parameters are set to 1.0×10^5 e 2.0×10^{-2}, respectively. Figure 3 compares the results obtained for the failure probability and the stopping criterion along the adaptive process. Table 2 compares the results obtained after 254 limit state evaluations (16+238), when converge for P_{lim} (Reference) is achieved.

As can be seen, the accuracy of the estimated failure probability increases proportionally to the approximation of the actual value of the limit state function, which means the increase of the applied load ratio until failure of the system occurs (P_{lim}), used as reference. Considering the number of limit state function evaluations necessary for achieving the convergence when used its actual value (254), it can be noticed that when load factors close to 1.0 are used, although approximated results of the failure probability are obtained, the convergence criterion is not achieved. It indicates that a larger number of limit state function evaluations may be necessary in these cases.

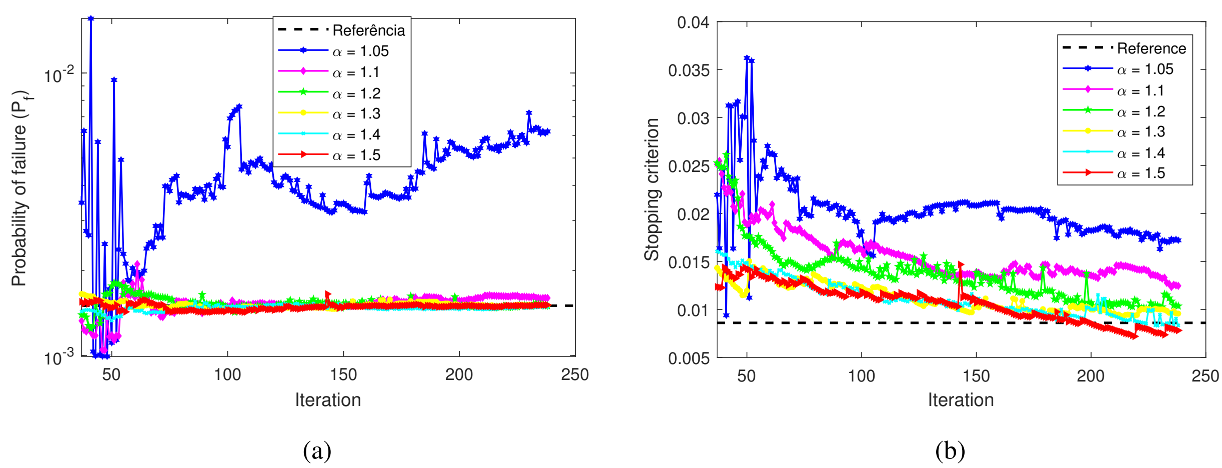


Figure 3. (a) Failure probability and (b) stopping criterion over the adaptive process

Table 2. Probability of failure for 254 limit state function evaluations

Load factor (α)	Probability of failure (P_f)	Stopping criterion
1.0 (P_i)	5.13×10^{-1}	5.78×10^{-4}
1.05	6.20×10^{-3}	1.72×10^{-2}
1.1	1.59×10^{-3}	1.25×10^{-2}
1.2	1.49×10^{-3}	1.03×10^{-2}
1.3	1.50×10^{-3}	9.56×10^{-3}
1.4	1.48×10^{-3}	8.33×10^{-3}
1.5	1.51×10^{-3}	7.81×10^{-3}
Reference (P_{lim})	1.50×10^{-3}	8.59×10^{-3}

3.2 Truss bridge structure

The second example consists in the truss bridge shown in Fig 4. This problem has been investigated previously in Kim et al. [18] and Jiang et al. [11]. The cross-sectional areas of each member are presented in Table 3. The yield stress $\sigma_{yi}(i = 1, ..., 25)$ and the applied loads $F_i(i = 1, ..., 2)$ are statistically independent random variables with parameters summarized in Table 4.

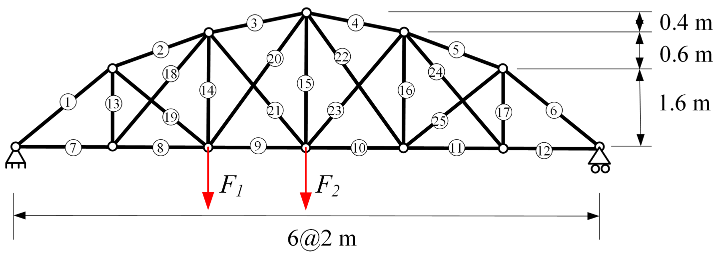


Figure 4. Truss bridge structure

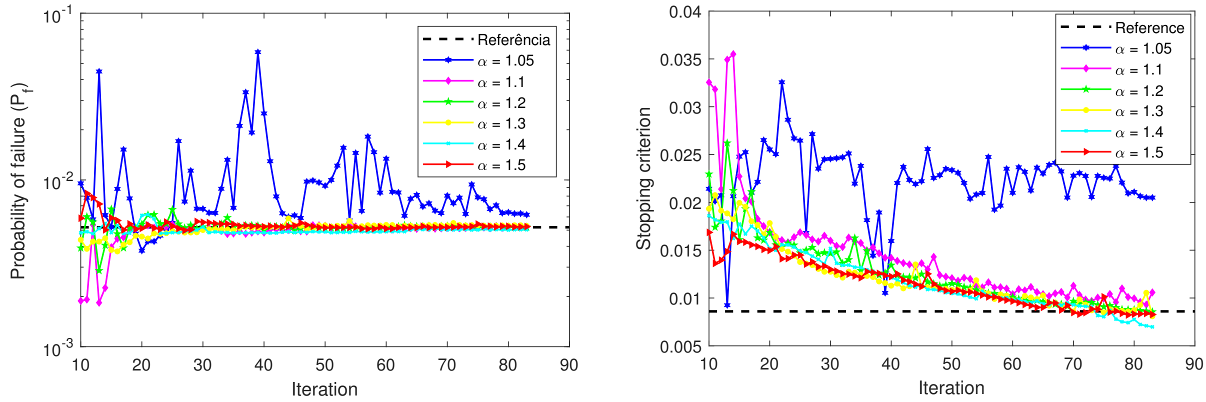
Table 3. Cross section areas of elements

Elements	Cross section areas (m^2)
1 - 6	1.5×10^{-3}
7 - 12	1.4×10^{-3}
13 - 17	1.2×10^{-3}
18 - 25	1.3×10^{-3}

Table 4. Random variables for the truss bridge structure

Variable	Distribution	Mean	Coefficient of variation
F_1	Lognormal	160 kN	0.1
F_2	Lognormal	160 kN	0.1
$\sigma_{yi}(i = 1, ..., 25)$	Normal	276 MPa	0.05

For this example, the number of initial training samples p, used to build the initial metamodel, is selected as 32. The initial MCS population and the U_{DS} parameters are set to 4.0×10^5 e 2.0×10^{-2}, respectively. Results obtained for the failure probability and the stopping criterion along the adaptive process are shown in Fig.5. Table 2 compares the results obtained after 115 limit state evaluations. As shown, the accuracy of the failure probability increases, and the convergence criterion is achieved, for higher values of α.



(a) (b)

Figure 5. (a) Failure probability and (b) stopping criterion over the adaptive process

Table 5. Probability of failure for 115 limit state function evaluations

Load factor (α)	Probability of failure (P_f)	Stopping criterion
1.0 (P_i)	5.88×10^{-1}	1.71×10^{-3}
1.05	6.18×10^{-3}	2.04×10^{-2}
1.1	5.23×10^{-3}	1.05×10^{-2}
1.2	5.21×10^{-3}	8.55×10^{-3}
1.3	5.29×10^{-3}	8.09×10^{-3}
1.4	5.03×10^{-3}	6.98×10^{-3}
1.5	5.25×10^{-3}	8.27×10^{-3}
Reference (P_{lim})	5.23×10^{-3}	8.59×10^{-3}

4 Conclusions

In this paper, the reliability analysis of redundant systems is considered using the global structural response as a limit state function. The computational efficiency is increased by using Kriging surrogate models and the effect of the load factor, applied in the mechanical model to obtain the value of limit state function at the sample points, is evaluated. Results indicate that a good accuracy for the failure probability and the stopping criterion are achieved when the value of the limit state function is as close as possible to its actual value, which means α values close to the reference. For the examples presented herein, values of load factor larger than 1.1 were capable of estimating the failure probability. However, more limit state evaluations can be necessary to achieve the convergence when compared with values close to the one in which failure of the system occurs.

Acknowledgements. The authors acknowledge the Federal University of Santa Catarina and the support financial from Coordination of Superior Level Staff Improvement (CAPES), financial code 001.

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] A. Srividya and R. Ranganathan. Automatic generation of stochastically dominant failure modes in frame structures for reliability studies. *Reliability Engineering and System Safety*, vol. 37, pp. 15–23, 1992.

[2] W. J. S. Gomes and A. T. Beck. Global structural optimization considering expected consequences of failure and using ann surrogates. *Computers and Structures*, vol. 126, pp. 56–68, 2013.
[3] S. K. Au and J. Beck. Estimation of small failure probability in high dimensions by subset simulation. *Probabilistic Engineering Mechanics*, vol. 16, pp. 263–277, 2001.
[4] V. Papadopoulos, D. G. Giovanis, N. D. Lagaros, and M. Papadrakakis. Accelerated subset simulation with neural networks for reliability analysis. *Computer methods in applied mechanics and engineering*, vol. 223–224, pp. 70–80, 2012.
[5] M. Rashki and M. A. M. nad Moghaddam. Miri. A new efficient simulation method to approximate the probability of failure and most probable point. *Structural Safety*, vol. 39, pp. 22–29, 2012.
[6] N. M. Okasha. An improved weighted average simulation method approach for solving reliability-based analysis and design optimization. *Structural Safety*, vol. 60, pp. 47–55, 2016.
[7] A. A. Chojaczyk, A. P. Teixeira, L. C. Neves, J. B. Cardoso, and C. G. Soares. Review and application of artificial neural networks models in reliability analysis of steel structures. *Structural Safety*, vol. 52, pp. 78–89, 2015.
[8] W. Gomes. Structural reliability analysis using adaptive artificial neural networks. *ASME J. Risk Uncertainty Part B.*, vol. 5, pp. 041004, 2019.
[9] W. Gomes. Shallow and deep artificial neural networks for structural reliability analysis. *ASME J. Risk Uncertainty Part B.*, vol. 6, pp. 041006, 2020.
[10] N. Roussouly, F. Petitjean, and M. Salaun. A new adaptive response surface method for reliability analysis. *Probabilistic Engineering Mechanics*, vol. 32, pp. 103–115, 2013.
[11] Y. Jiang, L. Zhao, M. Beer, L. Wang, and J. Zhang. Dominant failure mode analysis using representative samples obtained by multiple response surfaces method. *Probabilistic Engineering Mechanics*, vol. 59, pp. 103005, 2020.
[12] J. Yi, Q. F.and Zhou. Wu, Y. Cheng, H. Ling, and J. Liu. An active-learning method based on multifidelity kriging model for structural reliability analysis. *Structural and Multidisciplinary Optimization*, vol. 63, pp. 173–195, 2021.
[13] K. Yuan, N. Xiao, Z. Wang, and K. Shang. System reliability analysis by combining structure function and active learning kriging model. *Reliability Engineering and System Safety*, vol. 195, pp. 106734, 2020.
[14] H. Kroetz, M. Moustapha, A. Beck, and B. Sudret. A two-level kriging-based approach with active learning for solving time-variant risk optimization problems. *Reliability Engineering and System Safety*, vol. 203, pp. 107033, 2020.
[15] B. Echard, N. Gayton, and M. Lemaire. AK-MCS: An active learning reliability method combining Kriging and Monte Carlo Simulation. *Structural Safety*, vol. 33, pp. 145–154, 2011.
[16] N. C. Xiao, K. Yuan, and C. Zhou. Adaptive kriging-based efficient reliability method for structural systems with multiple failure modes and mixed variables. *Computer methods in applied mechanics and engineering*, vol. 359, pp. 1–26, 2020.
[17] A. I. J. Forrester, A. Sóbester, and A. Keane. *Engineering design via surrogate modelling: A practical guide*. John Wiley & Sons Ltd, 2008.
[18] D.-S. Kim, S.-Y. Ok, J. Song, and H.-M. Koh. System reliability analysis using dominant failure modes identified by selective searching technique. *Reliability Engineering and System Safety*, vol. 119, pp. 316–331, 2013.