Truss topology optimization considering uncertainties in the applied loads and progressive collapse

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Abstract. The progressive collapse is a phenomenon that has received increasing attention from engineers and researchers in recent years. The study of optimal structural design considering load redistribution due to the progressive failure of elements is recent. In this context, this paper deals with the topology optimization of truss structures, considering uncertainties in the applied loads and the progressive collapse of elements. The optimal topologies are determined following the ground structure approach. Uncertainties are included in the optimization problem through the RBDO (Reliability Based Design Optimization) and RO (Risk Optimization) formulations. Progressive collapse is incorporated into the analyses by considering load redistribution after member failure. The PSO algorithm is applied to solve optimization problems. System reliability is evaluated using the Monte Carlo Simulation method with stratified sampling. In RO problems, the costs of hyperstatic and isostatic failures are differentiated and solutions are obtained for different costs. The results suggest that the RBDO formulation leads to isostatic optimal topologies, since there is no incentive for the permanence of hyperstatic elements in the final solution. On the other hand, in the RO formulation, hyperstatic structures are among the optimal solutions, indicating that this formulation is the most appropriate for dealing with problems including progressive collapse.

Keywords: topology optimization, progressive collapse, reliability-based design optimization, risk optimization.

1 Introduction

The occurrence of events that led to large structural collapses, such as the Ronan Point Tower accident (UK, 1968) and the terrorist attack at the World Trade Center (NY, 2001), has raised engineers' and researchers' awareness of the importance of robust design with respect to the progressive collapse phenomenon [1]. This phenomenon can be described as a disproportional propagation of a local damage, which can lead to the total or partial collapse of a structural system [2]. The complex nature of problems that involve progressive collapse arises from features like nonlinearity and the large number of uncertainties involved, such as those related to abnormal loads, load paths, and material post-failure behavior.

Although several studies on progressive collapse have been developed in the last two decades, as reported in Adam *et al.* [1], the study of optimal design under uncertainty with objective consideration of this phenomenon is still recent, dating from the last two years [3-5]. The presence of uncertainty can lead structures to present an undesirable behavior, being more prone to failure. The formulations used to address uncertainties in optimization problems include Reliability-Based Design Optimization (RBDO) and Risk Optimization (RO), which have been the focus of several studies in recent years [6-9].

In this context, this paper addresses the topology optimization (TO) of hyperstatic trusses considering uncertainties in the applied loads and progressive collapse. Topology optimization is a computational tool used to find the best material distribution within the design domain, by removing elements that do not contribute significantly to structural performance [10,11]. Thus, this study aims to obtain optimal topologies that are robust with respect to the progressive collapse by considering this phenomenon in TO problems under uncertainties. For this purpose, the optimal solutions for an 11-bar truss are obtained based on the RBDO and RO formulations. The structural analysis is performed using a finite element model that considers the load redistribution after element failure. System reliability is evaluated through the Monte Carlo method with stratified sampling. Albeit simple, the problems discussed in this paper present important elements of a probabilistic analysis of progressive collapse, providing a basis for the study of larger problems. Furthermore, such problems are extremely relevant for Civil Engineering, considering the current trend of incorporating robustness in projects.

2 Formulation

2.1 Systems Reliability

Let **X** and **d** be, respectively, the vectors of the random and design variables of a structural system. Additionally, let $g_i(\mathbf{X},\mathbf{d})$ be the limit state function associated with the *i*th failure mode. Analyzing the collapse of hyperstatic systems involves the definition of the possible failure sequences, including the load redistribution after the failure of each element. Hence, the probability of failure for a typical structural system is given by:

$$p_{f_{SYS}} = \int_{\Omega_{f_{SYS}}} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{1}$$

where $\Omega_{f_{sys}}(\mathbf{d})$ is the system failure domain, written as:

$$\Omega_{f_{SYS}}(\mathbf{d}) = \left\{ \mathbf{X} \,|\, \bigcup_k \left[\bigcap_{i \in C_k} \left(g_i(\mathbf{X},\mathbf{d}) \le 0 \right) \right] \right\} \tag{2}$$

where C_k is the *k*th failure sequence and $g_i(\mathbf{X},\mathbf{d}) < 0$ the *i*th event that composes the failure sequence.

In this study, the probability of failure given in eq. (1) is evaluated using the Monte Carlo method with Stratified Sampling, or simply Stratified Sampling Monte Carlo (SSMC), which is described in the next section.

2.2 Stratified Sampling Monte Carlo

This approach is a modified version of the Monte Carlo simulation (MCS) method and has been proposed by Valentini [12]. The modification is based on the Stratified Sampling technique, described in Shields *et al.* [13]. Valentini [12] has applied the method to topology optimization of continuum structures subjected to uncertainties in the excitation frequency. However, only problems with a single random variable were analyzed. Nevertheless, the formulation presented by the author can be easily generalized to multidimensional problems. Therefore, let $\mathbf{X} = \{X_1, X_2, ..., X_n\} \in \mathbb{R}^n$ be the vector of random variables. Assuming that the histogram corresponding to the variable X_i, generated from n_s samples, is discretized into n^i_{bins} parts, it follows that the domain Ω can be divided into N_{st} mutually exclusive subsets Ω_k, with $k = 1, 2, ..., N_{st}$ and N_{st} given by eq. (3):

$$N_{st} = \prod_{i=1}^{n} n^i_{bins} \tag{3}$$

Each subset Ω_k is called a *stratum*. For each *stratum k*, we determine the mean value of the realizations of **X**, denoted by $\bar{\mathbf{x}}_k$, and the number n_{ek} of realizations of $\mathbf{X} \in \Omega_k$. Hence, a sample weight w_k can be assigned to each *stratum*, according to eq. (4):

$$w_k = \frac{n_{e_k}}{n_s}, \quad k = 1, \ldots, N_{st} \tag{4}$$

Thus, the probability of failure can then be estimated as:

$$p_f = \sum_{k=1}^{N_{st}} I[\bar{\mathbf{x}}_k] w_k = \frac{1}{n_s} \sum_{k=1}^{N_{st}} I[\bar{\mathbf{x}}_k] n_{e_k} \tag{5}$$

where $I[\mathbf{X}]$ is an indicator function, such that $I[\mathbf{X}] = 1$ if $\mathbf{X} \in \Omega_f$ (failure domain) and $I[\mathbf{X}] = 0$ otherwise.

2.3 Truss Topology Optimization: RBDO and RO formulations

The truss topology optimization performed in this study follows the ground structure approach [10]. This method consists in generating a highly interconnected truss (ground structure) containing all (or almost all) possible member connections among all nodes in the structure. Then, the optimizer eliminates the bars that do not contribute significantly to structural performance, and simultaneously optimizes the cross-sectional areas of the remaining elements.

In this work the bars are eliminated from the ground structure following the criteria proposed by Deb and Gulati [14]. The cross-sectional areas of the elements are compared to a small value ε, called critical area. If the element area is smaller than the critical value, the bar is eliminated from the ground structure. Note that the value of ε and the lower ($\mathbf{A}^{\mathbf{min}}$) and upper ($\mathbf{A}^{\mathbf{max}}$) bounds of the cross-sectional areas must be selected so that an unnecessary element has a considerable probability of being removed from the final topology. Besides, in order to prevent the singularity of the stiffness matrices associated with unstable solutions, non-basic nodes connected only to elements of null cross-sectional areas are fixed in all directions.

The influence of uncertainties in the TO problems addressed in this paper is analyzed based on the RBDO and RO formulations. In the RBDO formulation, uncertainties are taken into account by including component or system reliability constraints in the optimization problem. Thus, the RBDO formulation for the minimum weight truss topology optimization problem addressed in this study is given by:

$$\begin{aligned} &\text{find } \mathbf{d^*} \text{ which minimizes } W(\mathbf{d},\mathbf{X}) = \sum_{i=1}^{n} \rho_i A_i L_i \\ &\text{subjected to: } p_{f_{sys}}(\mathbf{d},\mathbf{X}) \le p_{fT_{sys}} \\ &\qquad A_i^{\min} \le A_i \le A_i^{\max}, \quad i = 1,\dots,n \end{aligned} \tag{6}$$

where: $\mathbf{d} = \{A_1, A_2, ..., A_n\}$ is the vector of design variables, which in this case are the cross-sectional areas of the elements; $W(\mathbf{d})$ is the structural weight; ρ_i, A_i, L_i are, respectively, the specific mass, the cross-sectional area and the length of the ith bar; n is the number of bars; A_i^{min} and A_i^{max} are the lower and upper bounds of the cross-sectional area of the ith bar; $r_{Tsys} = (1 - p_{fTsys})$ is the target system reliability.

RBDO allows one to find a structure which is optimal in mechanical sense, and which does not compromise safety. However, the balance between cost and safety is not addressed by this formulation. When the objective is to find the optimal balance between economy and safety, the costs over the life-cycle of the structure must be taken into account. The RO formulation makes it possible through the definition of the following total expected cost function $C_{ET}(\mathbf{d},\mathbf{X})$:

$$C_{ET}(\mathbf{d},\mathbf{X}) = C_{constrction}(\mathbf{d},\mathbf{X}) + C_{operation}(\mathbf{d},\mathbf{X}) + C_{insp.\&maint.}(\mathbf{d},\mathbf{X}) + C_{disposal}(\mathbf{d},\mathbf{X}) + C_{ef}(\mathbf{d},\mathbf{X}) \tag{7}$$

where $C_{ef}(\mathbf{d},\mathbf{X})$ represents the expected costs of failure.

In this work, only the construction costs and expected costs of failure are considered. In addition, it is investigated the influence of progressive collapse on the optimal topologies. The risk-based formulation allows a distinction between the cost of failure of a hyperstatic element (*hyperstatic failure*) and the cost corresponding to the failure of an isostatic element (*isostatic failure*), whose collapse leads to the global failure. A hyperstatic failure can lead to progressive collapse due to load redistribution. When this failure does not lead to collapse, the corresponding cost refers to the maintenance costs, or the cost of replacing the damaged structural elements.

Thus, for a structural system in which progressive collapse is considered, the expected costs of failure are written as:

$$C_{ef}(\mathbf{d},\mathbf{X}) = C_{HF}\, p_{Hf}(\mathbf{d},\mathbf{X}) + C_{UF}\, p_{Uf}(\mathbf{d},\mathbf{X}) \tag{8}$$

where C_{HF} and p_{Hf} are the cost and probability of hyperstatic failure, which does not necessarily lead to global collapse, and C_{UF} and p_{Uf} are the cost and probability of ultimate failure.

Accordingly, for the problems discussed in this paper, eq. (7) can be rewritten as:

$$C_{ET}(\mathbf{d},\mathbf{X}) = C_{construction}(\mathbf{d},\mathbf{X}) + C_{HF}\, p_{Hf}(\mathbf{d},\mathbf{X}) + C_{UF}\, p_{Uf}(\mathbf{d},\mathbf{X}) \tag{9}$$

The construction cost considered herein corresponds to the structural weight. The reference cost (C_{REF}), used to normalize the other costs, is admitted as the structural weight corresponding to a predefined value of cross-sectional areas. On the other hand, the cost of hyperstatic failure is assumed proportional to the weight of the failed bars, representing the material and labor cost of replacing the damaged elements. For the cost of ultimate failure, the following distinction is made: if the structure undergoes progressive collapse, *i.e.*, if the structure collapses after the failure of one or more hyperstatic elements, $C_{UF} = k\, C_{REF}$; if the structure collapses directly, due to the failure of an isostatic element, $C_{UF} = \alpha\, k\, C_{REF}$, where k and α are positive constants. This distinction is justified by the fact that the failure of a hyperstatic element theoretically provides a warning before the final collapse, allowing the structure to be evacuated and reducing the consequences of the failure. However, it is worth noting that the constant α depends on the failure mode and the material behavior model in question. The higher is the material and/or element plastic capacity, the higher is α.

Finally, the RO formulation for the TO problems addressed herein is:

$$\begin{aligned}&\text{find } \mathbf{d^*} \text{ which minimizes } C_{ET}(\mathbf{d}, \mathbf{X})\\&\text{subjected to: } A_i^{\min} \le A_i \le A_i^{\max}, \quad i = 1, \ldots, n\end{aligned} \tag{10}$$

The TO problems considering uncertainties and progressive collapse discussed in this paper are nonlinear, non-convex and discontinuous. Mathematical programming methods (*e.g.* gradient-based) are known to be very efficient for well-behaved problems, finding the global minimum with a small number of iterations. However, for the problems addressed herein, the use of such methods becomes impractical. On the other hand, heuristic algorithms do not require the computation of the function gradients and are suitable for problems involving nonlinear and non-differentiable functions, with multiple local minima. The main drawback of such methods is their computational cost, which grows in proportion to the population size. Nevertheless, for the problems presented in this study, these methods proved to be advantageous. Specifically, the optimal solutions for the problems formulated throughout this section are obtained using the Particle Swarm Optimization (PSO) method. Moreover, in order to better control exploration and exploitation, an inertia weight strategy proposed by Cekus and Skrobek [15] is employed. In the following section, more details about the PSO implementation are presented.

3 Numerical example: 11-bar truss

In this section, the effect of the progressive collapse on the topology optimization of an 11-bar truss is investigated. Nodes and elements are labelled following Fig. 1. The two vertical forces follow a Gumbel distribution with means of 500 kN and standard deviations of 50 kN. The assumed Young's modulus and specific mass are 200 GPa and 7.85 kg/m^3, respectively. The critical area is 0.50 cm^2. The cross-sectional areas are allowed to vary within the range between -100 cm^2 and 100 cm^2. The maximum allowable stresses in tension and compression are admitted deterministic and equal to 250 MPa.

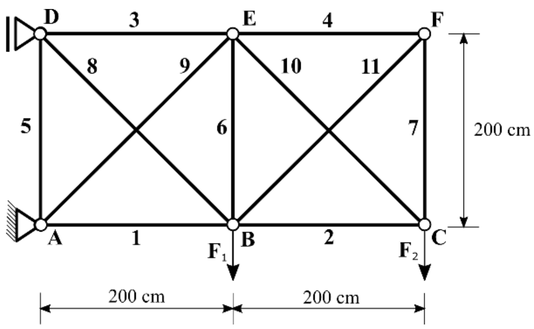

Figure 1. Eleven-bar planar truss example.

The optimization follows the RBDO and RO formulations. The structural analysis is performed using a linear finite element model. Buckling effects are disregarded. The elements are assumed to fail in a brittle manner once they reach the limit stresses. This is done by assigning a zero value to the cross-sectional area of the element with the highest normal force for which the failure occurs. After that, the stiffness matrix is recalculated considering $A_k = 0$, where the index k refers to the damaged element. The stability of equilibrium is then checked by evaluating the determinant of the stiffness matrix. If the determinant is zero, the structure becomes unstable and system failure is assumed. Otherwise, the described procedure is repeated until global collapse is observed or until no elements have failed.

The PSO acceleration parameters c_1 and c_2 are taken equal to 2. A population size of 100 individuals is considered. For each problem, 20 optimization runs are performed, with a maximum of 100 iterations per run. Also, a penalty factor equal to 10^8 is employed. In the RO problems, the reference cost adopted corresponds to the structural weight considering all bars with a cross-sectional area equal to 60 cm^2. In the reliability analyses, the histograms of the random variables are generated from 10^5 Monte Carlo samples. Each histogram is discretized in 80 parts, which gives a total of 6400 sample points. Besides, the same samples are used for each optimization run.

3.1 Results for system RBDO

Results for the RBDO solution of the 11-bar truss are presented in Tab. 1 and Fig. 2. Solutions were obtained for β_T = 2.5, 3.0 e 3.5. In all cases, two optimal isostatic topologies of equivalent weight were found, (Figs. 2 (a) and (b)), as well as a hyperstatic solution with a slightly higher weight than the mentioned ones (Fig. 2 (c)). Tab. 1 shows the cross-sectional areas and the weights corresponding to the best solutions. Also, the system reliability indexes β_{sys} for the optimal solutions (evaluated using MCS with 10^6 samples) are presented in Tab. 1.

Topologies presented in Fig. 2 are very similar to the optimal solutions obtained by Deb and Gulati [14] for the deterministic version of the same 11-bar problem. From Tab. 1, it is observed that the target reliability indexes variation resulted only in an increase in the cross-sectional areas of the bars. Even considering load redistribution in the mechanical model, the best RBDO solutions for the different values of β_T were isostatic. This indicates that it is more worthwhile to increase the structural safety by adding more material to the bars than by making the system redundant. This is not surprising, since the RBDO formulation does not offer any advantages of having a hypothetical warning after the first element collapse. The same was also observed in Beck, Tessari and Kroetz [3].

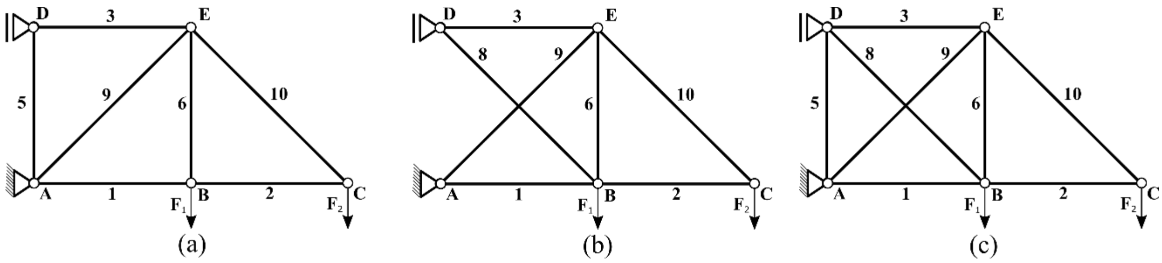

Figure 2. Optimal RBDO topologies for the 11-bar problem: (a) Sol. A; (b) Sol. B; (c) Sol. C.

Table 1. Results for the 11-bar problem: RBDO version.

Design variables	β_T = 2,5			β_T = 3,0			β_T = 3,5		
	Sol. A	Sol. B	Sol. C	Sol. A	Sol. B	Sol. C	Sol. A	Sol. B	Sol. C
A_1 (cm^2)	27.74	27.73	40.67	30.74	30.41	43.57	32.84	32.65	46.41
A_2 (cm^2)	27.98	27.77	28.18	30.76	30.47	30.42	32.79	32.66	33.30
A_3 (cm^2)	76.85	76.34	63.11	82.40	81.64	66.63	86.46	86.97	72.03
A_4 (cm^2)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A_5 (cm^2)	0.54	0.00	15.96	0.55	0.00	16.92	0.52	0.00	17.77
A_6 (cm^2)	29.33	29.87	14.37	31.64	32.29	16.55	35.18	34.43	19.11
A_7 (cm^2)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A_8 (cm^2)	0.00	0.74	21.44	0.00	0.59	24.66	0.00	0.60	25.27
A_9 (cm^2)	71.42	71.23	51.30	74.57	74.70	53.30	80.17	80.07	56.99
A_{10} (cm^2)	39.34	39.17	40.11	43.53	43.33	43.12	47.51	46.22	47.14
A_{11} (cm^2)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Obj. fun. (kg)	500.96	500.67	505.35	538.67	537.84	542.18	578.33	574.88	583.42
Type of system	Isostatic	Isostatic	Hyperstatic	Isostatic	Isostatic	Hyperstatic	Isostatic	Isostatic	Hyperstatic
β_{sys}	2.48	2.46	2.49	3.01	2.99	3.00	3.51	3.48	3.48

3.2 Results for risk optimization

Results for risk optimization with k = 1 and α = 5, 10, 15 and 20 are presented in Tab. 2. In all cases, the minimum cost solution resulted in an 8-bar hyperstatic structure identical to the topology shown in Fig. 2 (c). However, for α = 5 and α = 15, some optimization runs have converged to isostatic solutions of higher total expected costs that were similar to the structures shown in Fig. 2 (a) and (b). In order to simplify the following discussions, the minimum-cost hyperstatic structure is referred to as the best solution, while the isostatic solutions are referred to as local optimal. The best results are presented for each case. The optimal system reliability was evaluated using MCS with 10^6 samples.

Table 2. Results for variation of α parameter in RO solution of the 11-bar problem.

Design variables	α = 5		α = 10	α = 15		α = 20
	Best	Local	Best	Best	Local	Best
A_1 (cm^2)	35.80	29.09	35.71	35.14	30.89	31.86
A_2 (cm^2)	26.49	29.09	26.41	25.73	30.82	26.33
A_3 (cm^2)	63.44	78.30	63.75	62.62	82.43	68.04
A_4 (cm^2)	0.00	0.00	0.00	0.00	0.00	0.00
A_5 (cm^2)	11.02	0.50	10.87	10.98	0.00	6.21
A_6 (cm^2)	17.04	30.68	17.54	17.45	32.78	22.02
A_7 (cm^2)	0.00	0.00	0.00	0.00	0.00	0.00
A_8 (cm^2)	15.47	0.00	15.35	15.51	0.52	8.79
A_9 (cm^2)	53.61	71.69	54.31	53.67	76.50	60.23
A_{10} (cm^2)	37.44	41.15	37.35	36.38	43.58	37.24
A_{11} (cm^2)	0.00	0.00	0.00	0.00	0.00	0.00
Weight (kg)	477.96	513.78	479.80	472.88	545.52	478.44
Obj. Fun. (C_{ET})	499.78	534.07	499.70	499.20	559.76	499.66
Type of system	Hyperstatic	Isostatic	Hyperstatic	Hyperstatic	Isostatic	Hyperstatic
β_{sys}	2.11	2.65	2.14	2.03	3.08	2.13

From Tab. 2, unlike the RBDO results, the redundancy in the optimal RO solutions led to a decrease in the total expected costs in comparison to isostatic solutions. Furthermore, the variation of α, for a given k, has not demonstrated a significant influence on the global optimal, which have presented equivalent total expected costs for all cases. On the other hand, for the local minima, it can be noted that the change in α has resulted in an increase in the cross-sectional areas of the bars. Isostatic structures are series systems, so the only way to increase their reliability is by making the elements stronger. This is represented by the increase in the reliability indexes of the isostatic optimal solutions. It can also be noted that the values of β_{sys} for these topologies were higher than those for the corresponding hyperstatic solutions. This is due to the fact that isostatic failure consequences are more severe compared to hyperstatic failure consequences, and therefore require a higher level of safety from the solutions.

Table 3. Results for variation of k parameter in RO solution of the 11-bar problem.

Design variables	k = 1	k = 5	k = 10
A_1 (cm^2)	35.71	37.77	39.85
A_2 (cm^2)	26.41	28.81	29.55
A_3 (cm^2)	63.75	68.78	68.85
A_4 (cm^2)	0.00	0.00	0.00
A_5 (cm^2)	10.87	10.72	12.74
A_6 (cm^2)	17.54	19.74	19.33
A_7 (cm^2)	0.00	0.00	0.00
A_8 (cm^2)	15.35	15.16	18.14
A_9 (cm^2)	54.31	58.19	57.51
A_{10} (cm^2)	37.35	40.74	41.74
A_{11} (cm^2)	0.00	0.00	0.00
Weight (kg)	479.80	513.65	528.07
Obj. Fun. (C_{ET})	499.70	535.71	550.38
Type of system	Hyperstatic	Hyperstatic	Hyperstatic
β_{sys}	2.14	2.65	2.86

Tab. 3 presents the results for k = 1, 5 and 10 with α = 10. The same hyperstatic topology shown in Fig. 2 (c) was found in all cases. However, the increase in k parameter had the same effect as the variation of β_T in the RBDO formulation, resulting in an increase in the cross-sectional areas of the bars and, as consequence, in the reliability indexes of the optimal solutions.

4 Conclusions

This paper addressed the study of the influence of progressive collapse and uncertainties in applied loads on topology optimization of trusses. From the presented numerical example, it was observed that the RBDO solution resulted in isostatic optimal topologies, even changing the target reliability indexes. On the other hand, the best solutions derived from use of the risk-based formulation resulted in hyperstatic structures in all cases. This can be explained by the differentiation in failure costs provided by the RO formulation, which indicates that this is the most appropriate formulation to obtain robust structures with respect to the progressive collapse phenomenon. Finally, the small differences observed between the reliability indexes of the optimal solutions and the corresponding target indexes (due to sampling and simulation errors) show that the PSO-SSMC algorithm was capable of achieving good results for the present problem.

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