Efficient compliance-based topology optimization with many load cases scenarios

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Abstract. In topology optimization setting, we can cast a variety of problems into a weighted-sum of compliances minimization. Robust design, for example, is commonly addressed in the form of a finite sum of deterministic load cases scenarios. Another example is the optimization of structures subjected to dynamic loads using the equivalent static load method, where a finite set of associated loads is defined according to the displacements over time. But when the number of loads involved is high, the solution of these problems becomes extremely expensive from a computational point of view, due to the necessity of solving one finite element problem for each of these loading cases along the steps of the optimization algorithm to evaluate the objective function. In this context, two methods for dimensionality reduction of the problem are presented. First, an equivalent stochastic problem to the original one is determined, which reduces to only a few the number of necessary load cases at each step. The second approach uses the singular-value decomposition of the matrix that gathers the different loading scenarios to reduce the number of linear systems to be solved. The applicability of both methods to different topology optimization scenarios is discussed, and numerical examples are proposed to compare the final topologies obtained and quantify the reduction in the number of necessary finite element solves.

Keywords: Topology optimization, Stochastic sampling, Singular value decomposition, Sample average approach, Equivalent static loads.

1 Introduction

When analysing real engineering structures, it is evident that in service they are not only subjected to one deterministic loading scenario, and the effects off all possible situations must be taken into account for topology optimization. For that, this work studies the weighted-sum compliance approach that involves the analysis of various load cases at each step of the optimization process. However, if the number of possible loading scenarios is elevated, as the traditional approaches require the solution of one finite element problem for each scenario separately, the optimization process becomes extremely expensive from a computational point of view. To reduce the cost of evaluating this sum, Zhang *et al.* [1, 2] proposes one equivalent associated stochastic problem, based on the Sample Average Approximation (SAA) method (Verweij *et al.* [3]), that requires a smaller number of solves at each optimization step. Moreover, other interesting approach is the one based on the Singular Value Decomposition (SVD) of the matrix that represents the loading scenarios, presented in Tarek and Ray [4].

In this work these two existing approaches are investigated, and it is shown their application to different topology optimization situations that can be reduced to a weighted-sum compliance formulation, with a particular focus in the following situations:

Independent load cases. The simplest cases are structures where the different load cases scenarios of interest that the structure can be subjected to are known. In this context, the mean compliance can be calculated directly considering equal weights for all cases.

Time-dependent loads. Other important cases that can be treated by the weighted-sum compliance approach are structures subjected to dynamic loads, as shown by the work of Lavôr and Pereira [5]. In these cases, one time discretization is proposed, the displacements for each step are computed using the Newmark-β integration method (Clough and Penzien [6]) and the Equivalent Static Load (ESL) method (Kang *et al.* [7], Choi and Park [8]) is used to calculate the equivalent forces that cause the same displacements as those obtained from the dynamics, again for each time step. This approach means that for each displacement calculated, one equivalent load is obtained, and the set of equivalent loads can be used as loading scenarios in topology optimization.

Random loads. Moreover, it is also possible to deal with random loads dependent on one known probability distribution. For these problems, many realizations of the random variable associated to the random loads can be done and the results treated as independent loading scenarios, similarly to the first case.

The objective of this work is to show through numerical examples involving different loading possibilities the efficiency of the SAA and SVD-based methods to solve the topology optimization problem, the final topologies obtained with each formulation, and quantify the enormous reduction in the computational cost to perform the optimization process. Improving performance in these situations can not only result in faster algorithms, but also allow the use of more refined meshes, better discretizations of the time-dependent loads acting over the domain, among other changes that can improve the final result.

2 Theoretical background

In topology optimization, the objective is to find a material distribution that minimizes the objective function inside the domain region, satisfying the imposed constraints (Bendsoe and Sigmund [9]). In this paper, the focus is on using weighted-sum compliance based approaches (Zhang *et al.* [1]) to solve the topology optimization, and as discussed previously, it is supposed that the domain is not only subjected to one deterministic load case scenario, but to one of the situations enumerated in Section 1.

The solution of the optimization problem is based on the Solid Isotropic Material with Penalization (SIMP) method (Bendsoe and Sigmund [9]), where one design variable ρ associated to each element of the mesh is defined, with $0 \leq \rho \leq 1$. $\rho = 0$ means that the element has no material, while $\rho = 1$ means that the associated element is solid. The design variable updates are based on the Optimally Criteria (OC) method, as presented in Talischi *et al.* [10].

There is one particularity when analyzing the approximation for time dependent loads: the ESL's depends on the stiffness of the strucure. For that, various cycles of optimization should be performed, with the first cycle based on the stiffness of the original structure, and the followings based on the stiffness of the final topology obtained in the last iteration, repeating this process until convergence (Lavôr and Pereira [5], Choi and Park [8]).

2.1 Mathematical formulation

The formulation for the density-based topology optimization approach involving the multiple loading scenarios $\mathbf{f}_i$ and its associated displacements $\mathbf{u}_i$ can be expressed by eq. (1):

$$
\begin{cases}
\text{min} \;\; C\left(\mathbf{u}(\boldsymbol{\rho})\right) = \displaystyle\sum_{i=1}^{n} \mathbf{f}_i^T \mathbf{u}_i\left(\boldsymbol{\rho}\right) = \text{tr}\left(\mathbf{F}^T\mathbf{U}\right) = \text{tr}\left(\mathbf{F}^T\mathbf{K}^{-1}\mathbf{F}\right) \\
\text{s.t.:} \quad \displaystyle\sum_{e=1}^{m} V_e - V_{max} \leq 0 \\
\qquad\quad 0 < \rho_{min} \leq \rho_e \leq \rho_{max}, \quad e = 1, \ldots, m \\
\text{with:} \;\; \mathbf{K}\left(\boldsymbol{\rho}\right)\mathbf{u}_i\left(\boldsymbol{\rho}\right) = \mathbf{f}_i, \quad i = 1, \ldots, n
\end{cases}
\tag{1}
$$

where $C\left(\mathbf{u}(\boldsymbol{\rho})\right)$ is the compliance, V_e and V_{max} represent the element and the maximum allowable volumes, respectively, and ρ_{min} and ρ_{max} the lower and upper bounds for each component ρ_i of the design variables vector $\boldsymbol{\rho}$. $\mathbf{K}$ is the global stiffness matrix and m is the total number of elements in the mesh.

In the first line of eq. (1), $\mathbf{F} = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n]$ represents the load cases scenarios matrix, where each column represents one load case $\mathbf{f}_i$, and $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ is the associated displacements matrix with each column representing the nodal displacements associated to the load in the corespondent column inside $\mathbf{F}$.

In this context, it is necessary to define the scenarios $\mathbf{f}_i$ to be used for the optimization process according to the studied problem. When independent load cases are applied, the mean-compliance approach can be used (Tarek and Ray [4]), and the load cases $\mathbf{f}_i$ defined as:

$$\overline{C} = \frac{1}{n}\sum_{i=1}^{n} \overline{\mathbf{f}}_i^T \mathbf{K}^{-1} \overline{\mathbf{f}}_i \quad \longrightarrow \quad \mathbf{f}_i = \frac{1}{\sqrt{n}} \overline{\mathbf{f}}_i$$

where $\overline{C}$ represents the mean compliance and $\overline{\mathbf{f}}_i$ the i-th loading scenario applied to the structure.

But when the problem is subjected to time-dependent loads, one weighted-sum compliance can be used to approximate the dynamic compliance using the equivalent static loads method (Kang *et. al* [7], Lavôr and Pereira [5]), along with the associated loading scenarios $\mathbf{f}_i$ for the optimization:

$$C(t) = \int_0^{t_f} \overline{\mathbf{f}}^T(t)\, \mathbf{u}(t)\, dt \approx \sum_{i=1}^{n_{\text{Time}}} \omega_i \left(\mathbf{f}_i^{ESL}\right)^T \mathbf{K}^{-1} \mathbf{f}_i^{ESL} \quad \longrightarrow \quad \mathbf{f}_i = \sqrt{\omega_i}\, \mathbf{f}_i^{ESL}$$

where $C(t)$ is the dynamic compliance, $\mathbf{f}(t)$ represents the nodal loads and $\mathbf{u}(t)$ are the associated displacements, $\mathbf{f}_i^{ESL}$ denotes the i-th equivalent static load, n_{Time} the number of steps to discretize the time and ω_i the numerical integration weight.

Finally, when the structure is subjected to random loads, the expected compliance $\mathbb{E}[C]$ can be used to define the load cases $\mathbf{f}_i$ for topology optimization:

$$\mathbb{E}[C] = \frac{1}{n_s}\sum_{i=1}^{n} \overline{\mathbf{f}}_i^T \mathbf{K}^{-1} \overline{\mathbf{f}}_i \quad \longrightarrow \quad \mathbf{f}_i = \frac{1}{\sqrt{n_s}} \overline{\mathbf{f}}_i$$

where $\overline{\mathbf{f}}_i$ represents the load associated to the i-th realization random variable and n_s the number of samples used in the simulation.

2.2 SAA-approach

Zhang *et al.* [1] define an equivalent stochastic problem that reduces the number of required solves at each step of the optimization. For that, it is proposed to use the Hutchinson estimator (Hutchinson [11], Avron and Toledo [12]) to estimate the trace of the objective function in the first line of eq. (1). The application of the Sample Average Approximation (SAA) yields the result shown in eq. (2), where $\boldsymbol{\xi}_k$ is a random vector with entries $+1$ or -1, and each value with probability of occurrence equal to $1/2$, *i.e.*:

$$\text{tr}\left(\mathbf{F}^T\mathbf{K}^{-1}\mathbf{F}\right) = \mathbb{E}\left[\left(\mathbf{F}\boldsymbol{\xi}\right)^T \mathbf{K}^{-1}\left(\mathbf{F}\boldsymbol{\xi}\right)\right] = \lim_{n_s\to\infty} \frac{1}{n_s}\sum_{k=1}^{n_s} \left(\mathbf{F}\boldsymbol{\xi}_k\right)^T \mathbf{K}^{-1}\left(\mathbf{F}\boldsymbol{\xi}_k\right) \tag{2}$$

Then, it is possible to simulate a finite number of random vectors $\boldsymbol{\xi}_k$ to obtain one approximation for the objective function at each optimization step. By this formulation, the number of solves necessary at each step is equal to n_s, the quantity of the random vector simulations. According to Zhang *et al.* [1], $n_s = 6$ is typically a good choice for 2D problems.

For numerical implementation, it is possible to define the matrix $\boldsymbol{\Xi} = \left[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_{n_s}\right]$ of realizations of the random vector $\boldsymbol{\xi}$, define the associated loads calculating $\tilde{\mathbf{F}} = \mathbf{F}\boldsymbol{\Xi}$ and use the result in eq. (3) to estimate the compliance at each step of the algorithm, with $\tilde{\mathbf{f}}_k$ representing the *k-th* column of $\tilde{\mathbf{F}}$:

$$\hat{C}\left(\mathbf{u}(\boldsymbol{\rho})\right) = \frac{1}{n_s}\sum_{k=1}^{n_s} \tilde{\mathbf{f}}_k^T\, \mathbf{K}^{-1}\left(\boldsymbol{\rho}\right)\, \tilde{\mathbf{f}}_k. \tag{3}$$

To implement this approach in the `PolyTop` software (Talischi *et al.* [10]), the one used in the numerical studies of this work, at each step of the optimization algorithm a new matrix $\boldsymbol{\Xi}$ is determined, and the associated loading scenarios $\tilde{\mathbf{F}}$ defined before the objective function estimation.

2.3 SVD-approach

More recently, Tarek and Ray [4] proposed one approach based on the SVD decomposition of the loading scenarios matrix $\mathbf{F}$, defined in Section 2.1. This formulation yields the result in eq. (4), where $\mathbf{F} = \boldsymbol{\mathcal{U}}\boldsymbol{\Sigma}\mathbf{V}^T$ is the SVD decomposition of matrix $\mathbf{F}$, with $\boldsymbol{\Sigma}$ representing the singular values diagonal matrix, $\boldsymbol{\mathcal{U}}$ the left and $\mathbf{V}$ the right-singular vectors matrix, *i.e.*:

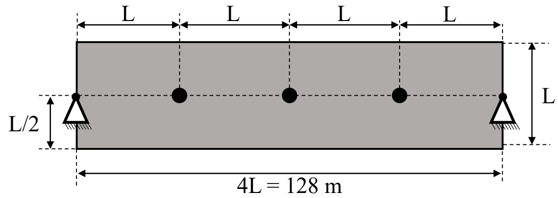
$$C\left(\mathbf{u}(\boldsymbol{\rho})\right) = \mathrm{tr}\left(\mathbf{F}^T\mathbf{K}^{-1}\mathbf{F}\right) = \mathrm{tr}\left(\mathbf{V}\boldsymbol{\Sigma}\boldsymbol{\mathcal{U}}^T\mathbf{K}^{-1}\boldsymbol{\mathcal{U}}\boldsymbol{\Sigma}\mathbf{V}^T\right) = \mathrm{tr}\left(\boldsymbol{\Sigma}\boldsymbol{\mathcal{U}}^T\mathbf{K}^{-1}\boldsymbol{\mathcal{U}}\boldsymbol{\Sigma}\right) \tag{4}$$

In problems where a few number of mesh nodes are loaded, or when the loading scenarios are very correlated, some singular values are much greater than others, in a way that the singular vectors associated to the non-dominant singular values can be neglected for compliance calculation. Consequently, the implementation involves the computation of the matrices $\boldsymbol{\Sigma}$ and $\boldsymbol{\mathcal{U}}$, the definition of the dominant singular values, and the definition of the loading scenarios $\tilde{\mathbf{F}} = \boldsymbol{\mathcal{U}}\boldsymbol{\Sigma}$ associated to the dominant singular values. As a result, the number of necessary solves at each step is reduced to the number of dominant singular values of $\tilde{\mathbf{F}}$.

3 Numerical examples

In this section, one 2D box domain as schematized in Fig. 1, similar to the one analysed by Zhang *et al.* [1], is considered. Three points are highlighted in its center, where different loading conditions can be applied. All the results obtained in this work are based on the open source topology optimization software `PolyTop` (Talischi *et al.* [13]), with the changes done by Lavôr and Pereira [5] to include the structure mass and damping matrices and one Newmark-β integration routine in the original program, to calculate the dynamic displacements along time if the problem is subjected to dynamic loads.

In all simulations, the mesh is composed of 320×80 equally-distributed Q4 elements, the numerical results from the SAA-approach are averaged over five trials, the filter radius is set as $R = 0.90\,m$ and the fraction of volume is $V_{max} = 0.30$. The continuation of the penalty parameter is run from $p = 1$ to 3 in steps of 0.5, and a maximum number of 500 iterations for each value. The optimization tolerance is $\tau_{opt} = 10^{-2}$ and the parameters values for the damping scheme in the SAA-approach, necessary to achieve convergence in this case, are equal to the suggested values in Zhang *et al.* [1].



Parameters	Symbol	Value	Unit
Young's modulus	E	2×10^{11}	Pa
Poisson's Ratio	ν	0.33	–
Domain thickness	t	0.01	m

Figure 1. Schematic illustration of the considered 2D box domain, with the loaded points in the center of the image and the parameters used in the simulations

3.1 Independent load cases

In the first analysis, 108 independent load cases of intensity $f = 500\,N$ are applied, equally distributed along the three highlighted nodes in Fig. 1. In this scenario, the objective is to reproduce the topologies from Zhang *et al.* [1], where it was just used the Standard and SAA-approaches, and establish the result using SVD-based approach. Figure 2 illustrates the applied loads and the final results obtained by each method, and Table 1 gathers all results for this simulation.

For the standard approach, 108 solves are necessary at each step of the optimization, number that reduces to 6 solves when the SAA-approach is applied, that is number of realizations of the Rademacher vector at each step. In the SVD case, the load matrix $\mathbf{F}$ of the problem has 6 dominant singular values, *i.e.*, with numerical value much greater than the others. Due to this fact, only the singular vectors associated to these singular values must be analysed, consequently only 6 solves are necessary at each step in this case, as discussed in Section 2.3.

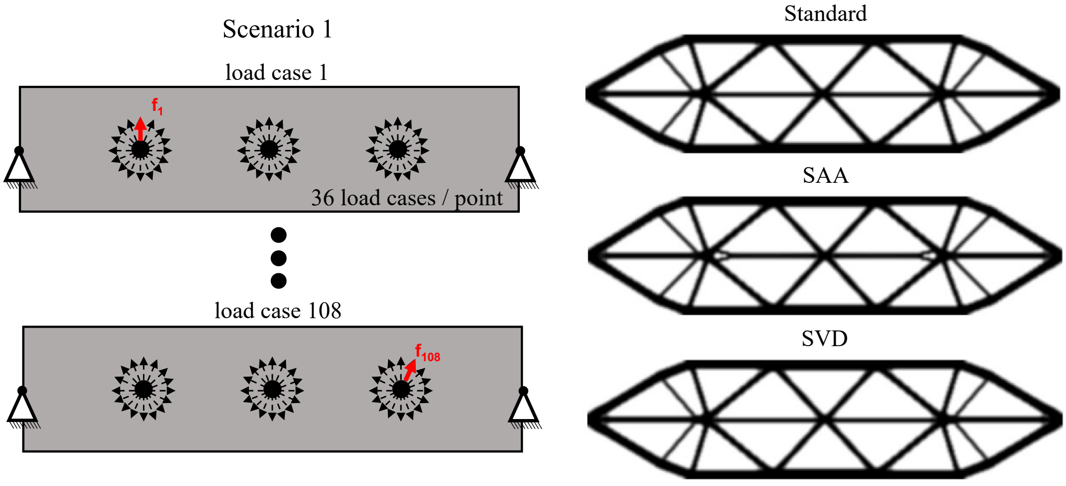


Figure 2. Illustration of the 108 independent load cases in the box domain in the left, and final topologies obtained by each method in the right.

Table 1. Comparison between the simulations for the independent load cases

Method	n	C	Difference	OC iterations	Linear solutions	Reduction
Standard	108	4.74×10^{-3}	–	1192	128736	–
SAA-approach	6	4.84×10^{-3}	2.11%	1875	11250	91.3%
SVD-approach	6	4.74×10^{-3}	0.00%	1192	7152	94.4%

3.2 Time-dependent loads

The second study involves three time-dependent loads applied on the three highlighted points in Fig. 1, as illustrated by Fig. 3.

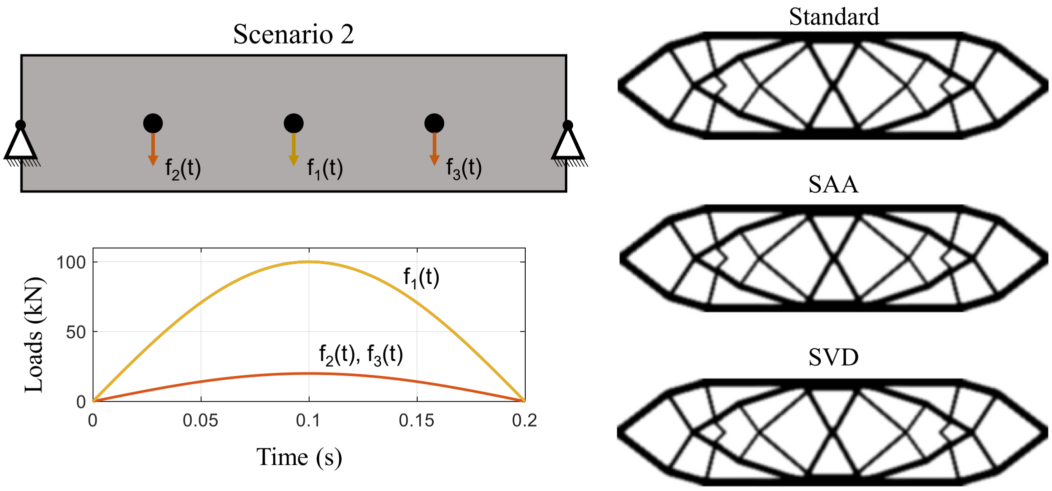


Figure 3. Illustration of the box domain subjected to three time-dependent loads in the left and the final topology obtained by all methods in the right

In this scenario, the time is discretized in steps of $0.01\ s$, implying in 21 ESLs to be taken into account during the optimization process, and the weights for the ESL's are based on the Simpson rule. Here, because of the number of time steps, the standard approach needs to solve 21 systems to evaluate the objective function, while employing SAA this number is reduced to 6. When analyzing the singular values of the load matrix $\mathbf{F}$, only the first 2 are dominant, implying in 2 solves at each step by this method. Table 2 illustrates the performance of all methods, and final topologies are also shown in Fig. 3.

Table 2. Comparison between the simulations for the time-dependent loads study

Method	n	C	Difference	OC iterations	Linear solutions	Reduction
Standard	21	2.467	–	715	15015	–
SAA-approach	6	2.471	0.16%	729	4374	70.9%
SVD-approach	2	2.467	0.00%	715	1430	90.5%

3.3 Random loads

In this last case, it is supposed that each one of the three highlighted nodes in Fig. 1 are subjected to 1000 independent loading scenarios of intensity $f = 500\,N$, and with angle between the load and the horizontal direction following an uniform distribution between 0 and 2π radians, as illustrated by Fig. 4, where it is also shown the final topology obtained by each method for the studied domain.

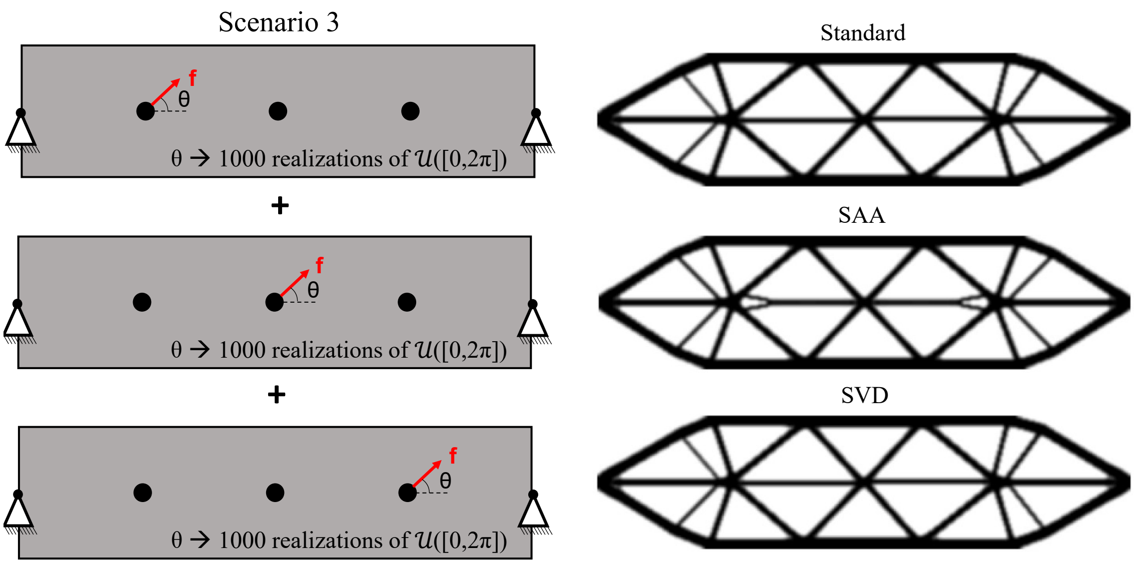


Figure 4. Illustration of the box domain subjected to random loads in the left and final topologies obtained by each method studied in this paper in the right

This study implies in 3000 solves at each step when the standard approach is used, which is extremely expensive. This number can be reduced to 6 when applying either the SVD and SAA-approaches, because this is the suggested number of Rademacher vector samples in the SAA method and also the number of dominant singular values in the SVD decomposition of matrix $\mathbf{F}$. Table 3 summarizes the improvements for this case study when applying the methods studied in this work.

Table 3. Comparison between the simulations for the random loads study

Method	n	C	Difference	OC iterations	Linear solutions	Reduction
Standard	3000	4.71×10^{-3}	–	1498	4494000	–
SAA-approach	6	4.79×10^{-3}	1.70%	1923	11538	99.7%
SVD-approach	6	4.71×10^{-3}	0.00%	1498	8988	99.8%

4 Conclusions and perspectives

This article shows the possibility of using dimensionality reduction methods for solving topology optimization problems with multiple loading cases. This idea is widely used in other fields of science, such as in data analysis and signal processing. The examples presented in Section 3 show reductions from 70% to 99% in the number of finite element problems to be solved until reach the final topologies, which is extremely interesting in the studied context and necessary to increase computational performance of the algorithms.

Although the SVD-decomposition reduced significantly the number of necessary load cases at each step in the studied context, according to Tarek and Ray [4], if the loading scenarios are not highly correlated, the existence of only a few dominant singular values of matrix $\mathbf{F}$ may not happen. For that, it would be interesting in a future work to investigate the combination of both approaches to simplify the cost to perform topology optimization: first, use the SVD-approach to reduce the number of load cases, and then apply the SAA-approach to the resulting loading scenarios from SVD.

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