Stress-based topology optimization with bi-directional evolutionary tools

José M. Cuevas Zárate[1], Fabiano Bortoluzzi[1], Gustavo C. Rodrigues[1], Fernanda B. Link[1], Walter J. P. Casas[1], Lucas L. Agne[2], Ederval de S. Lisboa[2]

[1]*Department of Mechanical Engineering, Federal University of Rio Grande do Sul*
Rua Sarmento Leite 425, 90050-170, Porto Alegre, RS, Brazil
jmcuev89@gmail.com, fbortoluzzi7@gmail.com, gustavocomerlato@gmail.com, feulink@gmail.com, walter.paucar.casas@ufrgs.br
[2]*Cachoeira do Sul Campus, Federal University of Santa Maria.*
Rua Ernesto Barros 1345, 96506-322, Cachoeira do Sul, RS, Brazil
lucas.l.agne@gmail.com, ederval.lisboa@ufsm.br

Abstract. This work presents a stress-based application of the bi-directional evolutionary structural optimization (BESO) method for topology optimization of structures where the objective is to minimize the stress while subjected to a structural volume constraint. The structure is discretized with a finite element mesh and subjected to a static external load to evaluate the von Mises stress distribution that will be used to obtain an equivalent global stress measure through a modified P-norm approach. The elemental sensitivity numbers are derived from the modified P-norm stress and filtered to update the design. The results were compared to cases found in the literature for problems based on stress and compliance. It was verified that the applied methodology was able to obtain a maximum stress reduction.

Keywords: P-norm approach, stress minimization, evolutionary optimization

1 Introduction

In many industrial applications, it is often desirable to have lightweight components, and one way to accomplish this objective is to employ topology optimization methods during the design process. In this context, the work of Le *et al.* [1] pointed out that most advances in this field have been carried out based on the compliance problem and summarized the principal challenges of stress-based topology optimization. Bruggi and Duysinx [2] showed examples where optimized designs obtained in compliance-based problems can present high-stress concentration.

The BESO method, detailed by Huang and Xie [3], is a topology optimization algorithm based on the gradient of an objective function using discrete design variables, where each design variable represents either presence or absence of material. The optimization procedure is performed with an iterative method where a discretized domain is submitted to a finite element analysis which allows one to identify elements that can be removed or included at each iteration through sensitivity analysis. The main advantages of BESO over other methods are high-quality topology solutions, excellent computational efficiency, easy to understand, simplicity to implement algorithms, and the presence of a well-defined boundary between different materials.

Recently, Xia *et al.* [4] have extended the BESO method to stress minimization problems, where they employed the P-norm stress as a global stress measure and proposed new filtering and stabilization methods to avoid the complications of stress-based topology optimization. Nabaki *et al.* [5] presented a similar method and reported better results than those provided by Xia *et al.* [4]. However, recent publications have considered the stress as a local quantity and, therefore, avoid aggregation techniques.

This work is concerned with the implementation of the methodology presented by Nabaki *et al.* [5] to solve different 2D cases found in the literature and evaluates the efficiency of the obtained results. All programming of the finite element method (pre-processing, solver, and post-processing) besides the optimization algorithm were implemented in the Matlab© software, the results being discussed in some applications.

2 Problem formulation

In this work, the stress-based topology optimization is formulated to minimize the modified P-norm stress for a given volume. In each iteration, a finite element analysis is performed to determine the von Mises stress distribution and the P-norm stress sensitivity of the elements. The topology optimization problem is formulated as shown in eq. (1),

$$\begin{aligned} &\text{minimize: } \sigma_G^{PN}(x) = \left(\frac{1}{N_i} \sum_{i=1}^{N_i} \left(\sigma_i^{vm}(x) \right)^P \right)^{\frac{1}{P}} \\ &\text{subject to: } \begin{cases} V^* - \sum_{i=1}^{N_i} V_i x_i = 0 \\ x_i = x_{min} \text{ or } 1 \end{cases} \end{aligned} \tag{1}$$

where σ_G^{PN} is the modified P-norm stress measure. V^* and V_i are the specified final structural volume and the volume of the i-th element, respectively. In the BESO method, the density of the i-th element is denoted by a discrete design variable x_i, where $x_i = 1$ represents solid elements and $x_i = x_{min} = 0.001$ represents void elements. The exponent P is the P-norm factor, σ_i^{vm} is the von Mises stress in the centroid of the elements and N_i is the number of elements.

2.1 Finite element analysis

The structure is discretized in a finite element mesh and submitted to a static load to evaluate the element stresses. The equilibrium is described by eq. (2),

$$\mathbf{F} = \mathbf{KU} \tag{2}$$

where **F** is the vector containing the force components, **K** is the global stiffness matrix and **U** is the displacement vector of the structure. The global stiffness matrix is assembled according to eq. (3),

$$\mathbf{K} = \sum_{i=1}^{N_i} x_i^q \mathbf{K}_i^0 \tag{3}$$

where $\mathbf{K}_i^0$ and q are the stiffness matrix of the solid element and the penalty exponent, respectively; in this work $q = 3$. The plane stress tensor of each element is expressed by eq. (4),

$$\boldsymbol{\sigma}_i = \mathbf{D}_i \mathbf{B}_i \mathbf{u}_i = \{\sigma_{xx}, \sigma_{yy}, \tau_{xy}\} \tag{4}$$

$\mathbf{D}_i$ is the material constitutive matrix, $\mathbf{B}_i$ is the strain-displacement matrix and $\mathbf{u}_i$ is the displacement vector of the i-th element. Employing the solid isotropic material with penalization (SIMP) model to interpolate the material and considering the plane-stress state, $\mathbf{D}_i$ is given as shown in eq. (5),

$$\mathbf{D}_i = \frac{E^0 x_i^{q_\sigma}}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix} \tag{5}$$

where E^0 and υ are the elastic modulus and the Poisson's ratio, respectively. It is adopted $q_\sigma = 1$ in this work. Finally, the von Mises stress can be determined by eq. (6), as follows,

$$\sigma_i^{vm} = \left(\sigma_{ix}^2 - \sigma_{ix}\sigma_{iy} + \sigma_{iy}^2 + 3\tau_{ixy}^2\right)^{0.5} \tag{6}$$

2.2 Sensitivity analysis

The calculation of the sensitivities begins through application of the chain rule in the modified P-norm stress, as shown in eq. (7).

$$\frac{\partial\sigma_G^{PN}(x)}{\partial x_i} = \frac{\partial\sigma_G^{PN}(x)}{\partial\sigma_i^{vm}}\frac{\partial\sigma_i^{vm}(x)}{\partial x_i} = \frac{\partial\sigma_G^{PN}(x)}{\partial\sigma_i^{vm}}\left(\frac{\partial\sigma_i^{vm}(x)}{\partial\boldsymbol{\sigma}_i}\right)\frac{\partial\boldsymbol{\sigma}_i(x)^T}{\partial x_i} \tag{7}$$

The term $\partial\sigma_G^{PN}(x)/\partial\sigma_i^{vm}$ is determined by eq. (8).

$$\frac{\partial\sigma_G^{PN}(x)}{\partial\sigma_i^{vm}} = \left(\frac{1}{N_i}\sum_{i=1}^{N_i}\left(\sigma_i^{vm}(x)\right)^P\right)^{\left(\frac{1}{P}-1\right)} \times \frac{1}{N_i}\left(\sigma_i^{vm}(x)\right)^{P-1} \tag{8}$$

The derivatives of the von Mises stress, given in Eq. (6), with respect to its stress components are obtained as described by eqs. (9), (10) and (11),

$$\frac{\partial\sigma_i^{vm}(x)}{\partial\sigma_{ix}} = \frac{1}{2\sigma_i^{vm}(x)}\left(2\sigma_{ix}(x) - \sigma_{iy}(x)\right) \tag{9}$$

$$\frac{\partial\sigma_i^{vm}(x)}{\partial\sigma_{iy}} = \frac{1}{2\sigma_i^{vm}(x)}\left(2\sigma_{iy}(x) - \sigma_{ix}(x)\right) \tag{10}$$

$$\frac{\partial\sigma_i^{vm}(x)}{\partial\tau_{ixy}} = \frac{3}{\sigma_i^{vm}(x)}\tau_{ixy}(x) \tag{11}$$

Taking the derivative from eq. (4) with regards to x_i, eq. (12) follows,

$$\frac{\partial\boldsymbol{\sigma}_i(x)}{\partial x_i} = \frac{\partial\mathbf{D}(x)}{\partial x_i}\mathbf{Bu} + \mathbf{DB}\frac{\partial\mathbf{u}(x)}{\partial x_i} \tag{12}$$

Applying the chain rule in the equilibrium equation, given by eq. (2) and considering the term $\partial\mathbf{F}/\partial x_i = 0$, the term $\partial\mathbf{u}(x)/\partial x_i$ is calculated as shown in eq. (13),

$$\frac{\partial\mathbf{u}(x)}{\partial x_i} = -\mathbf{K}^{-1}\frac{\partial\mathbf{K}(x)}{\partial x_i}\mathbf{u} \tag{13}$$

Rewriting eq. (7) based on eqs. (13) and (12) results in eq. (14),

$$\frac{\partial\sigma_G^{PN}(x)}{\partial x_i} = \frac{\partial\sigma_G^{PN}(x)}{\partial\sigma_i^{vm}}\left(\frac{\partial\sigma_i^{vm}(x)}{\partial\boldsymbol{\sigma}_i}\right)^T \times \left(\frac{\partial\mathbf{D}(x)}{\partial x_i}\mathbf{Bu} - \mathbf{DBK}^{-1}\frac{\partial\mathbf{K}(x)}{\partial x_i}\mathbf{u}\right) \tag{14}$$

From eq. (14), an adjoint variable λ, can be defined by eq. (15),

$$\boldsymbol{\lambda}^T = \frac{\partial\sigma_G^{PN}(x)}{\partial\sigma_i^{vm}}\left(\frac{\partial\sigma_i^{vm}(x)}{\partial\boldsymbol{\sigma}_i}\right)^T \mathbf{DBK}^{-1} \tag{15}$$

which can be solved by eq. (16), as follows:

$$\mathbf{K}\boldsymbol{\lambda} = \frac{\partial \sigma_G^{PN}(x)}{\partial \sigma_i^{vm}} \mathbf{B}^T \mathbf{D}^T \frac{\partial \sigma_i^{vm}(x)}{\partial \boldsymbol{\sigma}_i} \tag{16}$$

Finally, substituting the adjoint variable into eq. (14), we obtain the element sensitivity number $\alpha_{i,}$. Due to problem's minimization characteristic, α_i is determined by the negative gradient of the P-norm stress, as expressed in eq. (17).

$$\alpha_i = -\frac{\partial \sigma_G^{PN}(x)}{\partial x_i} = -\left[\frac{\partial \sigma_G^{PN}(x)}{\partial \sigma_i^{vm}} \left(\frac{\partial \sigma_i^{vm}(x)}{\partial \boldsymbol{\sigma}_i} \right)^T \frac{\partial \mathbf{D}(x)}{\partial x_i} \mathbf{Bu} - \boldsymbol{\lambda}^T \left(\frac{\partial \mathbf{K}(x)}{\partial x_i} \mathbf{u} \right) \right] \tag{17}$$

2.3 Bi-directional evolutionary structural optimization

This work employs the BESO basic procedure detailed in Huang and Xie [3] and can be summarized in the following steps:

Step 1: Prepare the design domain to finite element analysis;

Step 2: Define the BESO parameters: P-norm factor (P), penalty exponent (q), evolutionary rate (ER), filter radius ($r_{\min}$), and specified final structure volume (V^*);

Step 3: Perform finite element analysis and then determine the sensitivity numbers;

Step 4: Calculate the target volume for the next iteration (V_{k+1}) as described in eq. (18), given below:

$$V_{k+1} = V_k (1 - ER) \tag{18}$$

The current iteration number is represented by k. After the specified volume constraint is reached, the structural volume will be kept constant in the remaining iterations;

Step 5: Filter the sensitivity numbers according to eq. (19);

$$\hat{\alpha}_i = \frac{\sum_{j=1}^{N} w(r_{ij}) \alpha_i}{\sum_{j=1}^{M} w(r_{ij})} \tag{19}$$

where $\hat{\alpha}_i$ is the filtered sensitivity number, r_{ij} is the distance between the center of the element i and the element j and $w(r_{ij})$ is the weight factor defined by eq. (20), as follows:

$$w(r_{ij}) = \begin{cases} r_{\min} - r_{ij} & \text{for } r_{ij} < r_{\min} \\ 0 & \text{for } r_{ij} \ge r_{\min} \end{cases} \tag{20}$$

Step 6: Average the sensitivity numbers with their historical information as shown in eq. (21) to improve the convergence of the objective function.

$$\tilde{\alpha}_i = \frac{1}{2}\left(\hat{\alpha}_{i,k} + \hat{\alpha}_{i,k-1} \right) \tag{21}$$

Step 7: Update the design variables based on the sensitivity numbers, employing the optimality criteria.

Step 8: Verify convergence according to the criterion in eq. (22) when the volume constraint is satisfied.

$$error = \frac{\left| \sum_{i=1}^{N} \left(\sigma_{max,k-i+1} - \sigma_{max,k-N-i+1} \right) \right|}{\sum_{i=1}^{Ni} \sigma_{max,k-i+1}} \le 0.01 \tag{22}$$

where σ_{max} is the maximum von Mises stress and N is an integer number, adopted $N = 5$ in this work.

Step 9: Repeat steps 3-8 until the convergence criterion is satisfied.

2.4 Evolutionary topology optimization of continuum structures with smooth boundary representation

Unlike the classical zigzag BESO patterns characterized by the removal/addition of elements, the evolutionary topology optimization (ETO) method proposed by Da *et al.* [6] builds a smooth topology based on a level set function (LSF) which is determined by nodal sensibility numbers. The projection relationship between the domain of interest and the finite element analysis (FEA) model is determined. The analysis of the domain is substituted by evaluating the FEA model with different elemental volume fractions. These fractions are calculated by an auxiliary LSF, and their sensitivities results in intermediate volume elements at the solid-void interface of the FEA model.

We can thus briefly summarize each iteration of the ETO method, as follows: First, a FE mesh is constructed, material properties, boundary and loading conditions are applied, and the variational problem is solved. Then, the elemental sensitivity numbers are calculated and converted into nodal sensitivity numbers. This is performed by the filter and average with their history information. Then, the target volume for the next iteration is determined and the LSF function is constructed. After that, the elemental volume fraction of the FEA model is computed, and the topology of the design model is updated by the level-set value *S*. Finally, this is repeated until the volume constraint and convergence criterion are reached.

3 Numerical examples

This section describes the application of these methods in two examples, considering 2D structures. In the two cases, examples presented by Xia *et al.* [4] were solved and a comparison between both methods was made. All cases are submitted to compliance and stress minimization, subjected to a volume constraint. The domains were meshed with quadrilateral bilinear elements (Q4) considering plane stress state.

3.1 Cantilever beam with distributed load

The cantilever presented in Figure 1**Erro! Fonte de referência não encontrada.**(a) was loaded with a 5 N distributed force over 10 nodes at the top right-hand and was clamped on the left-hand side. The cantilever thickness is 1 mm and the domain was discretized in 20000 elements with 1 mm length. The elastic modulus and Poisson ratio are E = 1 MPa and υ = 0.3, respectively. The filter radius, evolutionary rate, specified volume and P-norm factor are set to $r_{\min}$ = 5 mm, ER = 0.02, V^* = 0.5 and P = 4 (values from 3 to 5 works right).

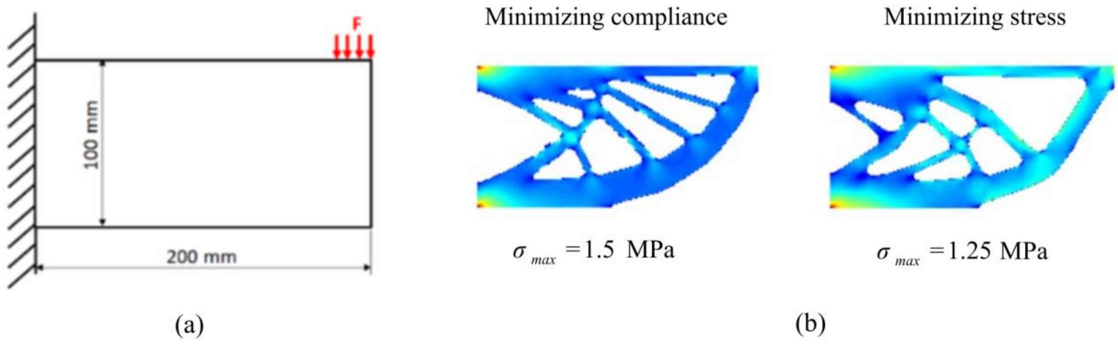

Figure 1. (a) Cantilever beam with distributed load, (b) Results found by (a) Xia *et al.* [4]

Erro! Fonte de referência não encontrada.(b) shows the final stresses obtained with compliance and stress minimization objectives, presented by Xia *et al.* [4] and the results obtained in this work can be observed in Figure 2 and Figure 3. The results reported by Xia *et al.* [4] reduced the maximum stress through the stress minimization, relative to the compliance minimization results, by 16.7%. However, in the current work, a reduction of the maximum stress of around 21.4% is observed using BESO and 7.1% using ETO (both showin properely computational performance).

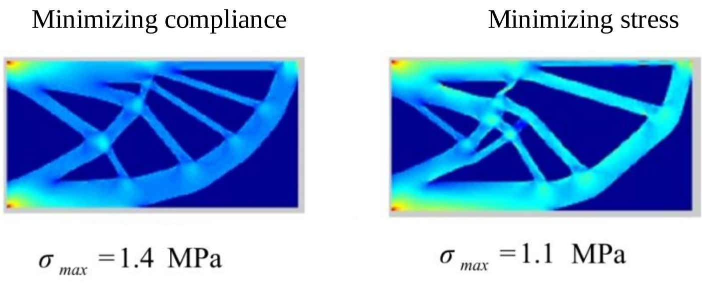

Figure 2. Results of cantilever beam using BESO

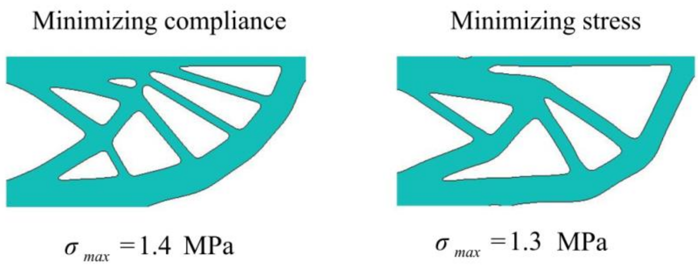

Figure 3. Results of cantilever beam using ETO

3.2 Simply supported pre-cracked beam

In this case, the simply supported beam presented in Figure 4(a) was loaded with a 10 N distributed force over 11 nodes in the middle of the top edge. The structural thickness is 1 mm and the domain was discretized in 10800 elements with 1 mm length. The elastic modulus and Poisson ratio are E = 1 MPa and υ = 0.3, respectively. The filter radius, evolutionary rate, specified volume and P-norm factor are set to r_{min} = 4 mm, ER = 0.02, V^* = 0.5 and P=4. The eight elements around the supports are suppressed in the optimization problem to avoid stress concentration.

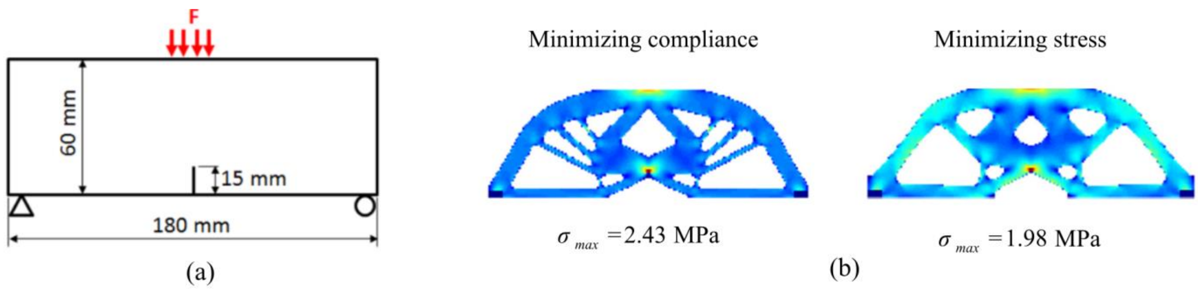

Figure 4. a) Simply supported pre-cracked beam, b) Results found by (a) Xia *et* al. [4]

Figure 4(b) shows the results presented by Xia et al. [4] where the maximum stress was reduced by 18.5%, after comparing the maximum von Mises stress of the stress minimization relative to the compliance minimization; while in the current work a reduction of around 17.9% was obtained both using BESO and ETO (Figure 5 and Figure 6).

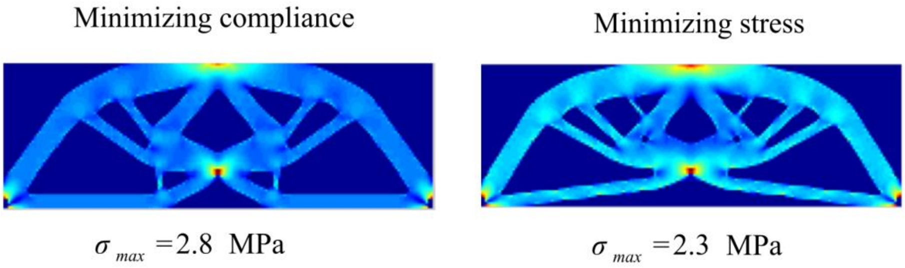

Figure 5. Results of pre-cracked beam using BESO

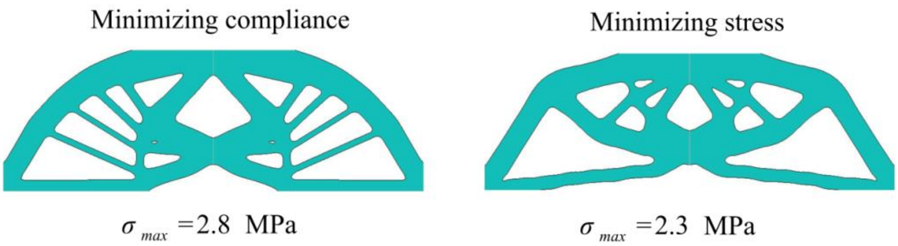

Figure 6. Results of pre-cracked beam using ETO

4 Conclusions

Both BESO and ETO produce comparable results for structural topology optimization. In the first problem, the cantilever beam with distributed load, the reduction of the maximum stress was around 21.4% using BESO and 7.1% using ETO, compared with the compliance results, showing similarity with the final optimized topology and results from Xia *et al*. [4]. In the second problem, the simply supported pre-cracked beam, the reduction of the maximum stress was around 17.9% both using BESO and ETO, again compared with the compliance results, and showing also certain similarities in the final topologies and stresses with Xia *et al*. [4].

Acknowledgments. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. It is also acknowledged the support of the FIPE/UFSM and the following Brazilian research agencies: CNPq and FAPERGS.

Authorship statement. The authors hereby confirm that they are the solely liable persons responsible for the authorship of this work and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] C. Le, J. Norato, T. Bruns, C. Ha and D. Tortorelli, “Stress-based topology optimization for continua”. *Structural and Multidisciplinary Optimization*, vol. 41, pp. 605–620, 2010.
[2] M. Bruggi and P. Duysinx, “Topology optimization for minimum weight with compliance and stress constraints”. *Structural and Multidisciplinary Optimization*, vol. 46, pp. 369–384. 2012.
[3] X. Huang and Y. M. Xie, *Evolutionary Topology Optimization of Continuum Structures: Methods and Applications*. John Wiley & Sons. Chichester. 2010
[4] L. Xia, L. Zhang, Q. Xia and T. Shi, “Stress-based topology optimization using bi-directional evolutionary structural optimization method”. *Computer Methods in Applied Mechanics and Engineering*, vol. 333, pp. 356–370, 2018.
[5] K. Nabaki, J. Shen and X. Huang, “Stress minimization of structures based on bidirectional evolutionary procedure”. *Journal of Structural Engineering*, vol. 145, n. 2, pp. 605–620, 2019.
[6] D. Da, L. Xia, G. Li and X. Huang, *“Evolutionary topology optimization of continuum structures with smooth boundary representation*”, *Structural and Multidisciplinary Optimization*, vol. 57, pp 2143-2159, 2018.