A co-rotational model for elastoplastic analysis of planar frames

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Abstract. A co-rotational model is employed to analyze planar frames considering plasticity effects lumped at plastic hinges. The element is locally formulated as the traditional linear element based on the Euler-Bernoulli theory. The hinges effects are introduced into the generalized strain fields as Dirac deltas centered at the element ends, which naturally results in the formulation of the element with plastic hinges. The plastic constrained nonlinear system of equation of the local problems is solved with the Newton-Raphson method running through all plastic possibilities, whereas a classical force-control procedure solves the global nonlinear equilibrium equations. Two examples are presented to demonstrate the robustness of the formulation to deal with geometrical nonlinear elastoplastic analysis of frames.

Keywords: co-rotational model, plastic hinges, nonlinear analysis.

1 Introduction

The non-linearities that occur in the analysis of frame structures are mainly triggered by two sources: geometrical and material. With regard to the nonlinear geometric analysis, it can be efficiently handled by the co-rotational formulation [1–5]. The fundamental idea of such a formulation is to decompose the large motion of the element into rigid body and pure deformation parts, through the use of a local system which continuously rotates and translates with the element. The deformation is captured at the level of the local reference system, whereas the geometric non-linearity induced by the large rigid-body motion, is incorporated in the transformation matrices relating local and global displacements. The main interest is that the pure deformation part can be assumed as small and can be represented by a linear or a low order theory. Avoiding the nonlinear relationship between the strain tensor and the displacement gradient makes the co-rotational approach very attractive to deal with geometrical nonlinearity. With respect to the nonlinear material analysis of frames, it can be distinguishably placed into two branches: the distributed plasticity and the lumped plasticity (plastic hinges) [6–8]. Herein, the latter one is adopted due to its efficiency in engineering practices since fewer elements are used to model frame members and none integration over the discretized cross-sections is necessary to get the internal forces. As a consequence, the element between the plastic hinges is assumed to remain elastic. The elastoplastic hinge assumption is the earliest formulation to be dated back among the others. Quite many research works have adopted some variations of this method to investigate the inelastic behavior of steel or concrete frame structures [9–12].

In this paper, a co-rotational finite element model is developed to analyze planar frames with elastoplastic hinges. The hinges effects are introduced into the generalized strains fields as Dirac deltas centered at the element ends, which naturally results in the formulation of the element with elastoplastic hinges. The plastic constrained nonlinear system of equation of the local problems at the element level is solved with the Newton-Raphson method, whereas a classical force-control procedure solves the global nonlinear equilibrium equations. Two examples with reference solutions are presented to demonstrate the robustness of the proposed formulation.

2 Co-rotational finite element model with plastic hinges

A co-rotational model is adopted to describe the motion of the element from an initial configuration C_0 to the current configuration C_n. The rigid body motion is identified by an intermediate configuration C_{0n} such that the motion between the intermediate configuration and the current configuration involves all the deformation of the element, under small strains assumption [13]. Thus, the equilibrium equations of the element refereed to C_{0n} can be obtained based on the Euler-Bernoulli beam linear theory [14]. The co-rotational model relates the total displacements of the element, i.e., from C_0 to C_n, with the displacements involved in the motion from C_{0n} to C_n. The co-rotational model adopted in the present study allows any planar beam finite element with two nodes and three degrees of freedom per node (two displacements and one rotation) to be used to handle non-linear analyses, considering small strains, but large displacements and rotations.

2.1 Equilibrium refereed to C_{0n}

The co-rotational displacement fields of the Euler-Bernoulli linear element with plastic hinges reads

$$\begin{aligned}\bar{u}(\bar{x}) &= \frac{\bar{x}}{L_0}\bar{u}_2^e + (2H(\bar{x})-1)\bar{u}_1^p + 2H(\bar{x}-L_0)\bar{u}_2^p \\ \bar{v}(\bar{x}) &= \left(\frac{\bar{x}^3}{L_0^{\,2}} - \frac{2\bar{x}^2}{L_0} + \bar{x}\right)\bar{\theta}_1^e + \left(\frac{\bar{x}^3}{L_0^{\,2}} - \frac{\bar{x}^2}{L_0}\right)\bar{\theta}_2^e - [2(\bar{x}-L_0)H(\bar{x}-L_0) - \bar{x}]\bar{\theta}_1^p + [2\bar{x}H(\bar{x}) - (\bar{x}+L_0)]\bar{\theta}_2^p\end{aligned} \tag{1}$$

which can be rewritten in matrix notation as

$$\begin{gathered}\begin{Bmatrix}\bar{u}\\ \bar{v}\end{Bmatrix} = \mathbf{U}^e\bar{\mathbf{u}}^e + \mathbf{U}^p\bar{\mathbf{u}}^p \qquad \bar{\mathbf{u}}^e = \lfloor 0 \quad \bar{\theta}_1^e \quad \bar{u}_2^e \quad \bar{\theta}_2^e \rfloor^T \qquad \bar{\mathbf{u}}^p = \lfloor \bar{u}_1^p \quad \bar{\theta}_1^p \quad \bar{u}_2^p \quad \bar{\theta}_2^p \rfloor^T \\ \mathbf{U}^e = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{\bar{x}^3}{L_0^{\,2}} - \dfrac{2\bar{x}^2}{L_0} + \bar{x} \\ \dfrac{\bar{x}}{L_0} & 0 \\ 0 & \dfrac{\bar{x}^3}{L_0^{\,2}} - \dfrac{\bar{x}^2}{L_0} \end{bmatrix}^T \qquad \mathbf{U}^p = 2H(\bar{x})\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \bar{x} \end{bmatrix}^T + 2H(\bar{x}-L_0)\begin{bmatrix} 0 & 0 \\ 0 & -(\bar{x}-L_0) \\ 1 & 0 \\ 0 & 0 \end{bmatrix}^T - \begin{bmatrix} 1 & 0 \\ 0 & -\bar{x} \\ 0 & 0 \\ 0 & \bar{x}+L_0 \end{bmatrix}^T\end{gathered} \tag{2}$$

where $\bar{x}$ is the co-rotational axis from node 1 to 2 in the co-rotated frame, $\bar{\mathbf{u}}^e$ is the elastic nodal displacement vector, $\bar{\mathbf{u}}^p$ is the plastic nodal displacement vector of the plastic hinges, $H(\bar{x})$ is the Heaviside function and L_0 is the initial length of the element. The terms related with $\bar{\mathbf{u}}^p$ appears in the element displacement fields because Dirac deltas are introduced into the generalized strain fields $\varepsilon_0 = d\bar{u}/d\bar{x}$ and $\kappa = d^2\bar{v}/d\bar{x}^2$ at the element ends.

The total co-rotational displacement vector can be written in terms of $\bar{\mathbf{u}}^e$ and $\bar{\mathbf{u}}^p$ in matrix form as

$$\bar{\mathbf{u}} = \begin{bmatrix}\bar{u}_1\\ \bar{\theta}_1\\ \bar{u}_2\\ \bar{\theta}_2\end{bmatrix} = \bar{\mathbf{u}}^e + \bar{\mathbf{M}}\bar{\mathbf{u}}^p \qquad \bar{\mathbf{M}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

The fields $\varepsilon_0(\bar{x})$ and $\kappa(\bar{x})$ can be represented as

$$\begin{Bmatrix}\varepsilon_0\\ \kappa\end{Bmatrix} = \begin{Bmatrix}\varepsilon_0^e\\ \kappa^e\end{Bmatrix} + \begin{Bmatrix}\varepsilon_0^p\\ \kappa^p\end{Bmatrix} = \bar{\mathbf{B}}^e\bar{\mathbf{u}}^e + \bar{\mathbf{B}}^p\bar{\mathbf{u}}^p \quad \bar{\mathbf{B}}^e = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{3\bar{x}^2}{L_0^{\,2}} - \dfrac{4\bar{x}}{L_0} + 1 \\ \dfrac{1}{L_0} & 0 \\ 0 & \dfrac{3\bar{x}^2}{L_0^{\,2}} - \dfrac{2\bar{x}}{L_0} \end{bmatrix}^T \quad \bar{\mathbf{B}}^p = 2\begin{bmatrix} \delta(\bar{x}) & 0 \\ 0 & -\delta(\bar{x}-L_0) \\ \delta(\bar{x}-L_0) & 0 \\ 0 & \delta(\bar{x}) \end{bmatrix}^T. \tag{4}$$

Using the Principle of Virtual Displacements, it is possible to establish the equilibrium equations of the element referred to C_{0n}

$$-\int_0^{L_0} \begin{Bmatrix} \delta\varepsilon_0 \\ \delta\kappa \end{Bmatrix}^T \begin{Bmatrix} N \\ M \end{Bmatrix} d\bar{x} + \delta\bar{\mathbf{u}}^T \bar{\mathbf{r}} + \int_0^{L_0} \begin{Bmatrix} \delta\bar{u} \\ \delta\bar{v} \end{Bmatrix}^T \begin{Bmatrix} q_{\bar{x}} \\ q_{\bar{y}} \end{Bmatrix} d\bar{x} = 0 \qquad \bar{\mathbf{r}} = \lfloor F_{\bar{x}1} \quad M_{\bar{z}1} \quad F_{\bar{x}2} \quad M_{\bar{z}2} \rfloor^T \tag{5}$$

where $N = EA\varepsilon_0^e$, $M = EI\kappa^e$ are the axial force and bending moment, $\bar{\mathbf{r}}$ is the nodal reactions, $q_{\bar{x}}$ and $q_{\bar{y}}$ are the distributed forces on element. The equivalent nodal forces $\bar{\mathbf{p}}$ consistent with the adopted beam theory are derived from the work of the distributed forces

$$\int_0^{L_0} \begin{Bmatrix} \delta\bar{u} \\ \delta\bar{v} \end{Bmatrix}^T \begin{Bmatrix} q_{\bar{x}} \\ q_{\bar{y}} \end{Bmatrix} d\bar{x} = \delta\bar{\mathbf{u}}^T \bar{\mathbf{p}} \qquad \bar{\mathbf{p}} = \lfloor p_{\bar{u}_1} \quad p_{\bar{\theta}_1} \quad p_{\bar{u}_2} \quad p_{\bar{\theta}_2} \rfloor^T. \tag{6}$$

Replacing (4) into (5), one obtains

$$-\int_0^{L_0} \begin{Bmatrix} \delta\varepsilon_0 \\ \delta\kappa \end{Bmatrix}^T \begin{Bmatrix} N \\ M \end{Bmatrix} d\bar{x} = -\delta\bar{\mathbf{u}}^{eT} \bar{\mathbf{f}} - \delta\bar{\mathbf{u}}^{pT} \bar{\mathbf{f}}^p \tag{7}$$

where

$$\bar{\mathbf{f}} = \int_0^{L_0} \bar{\mathbf{B}}^{eT} \begin{Bmatrix} N \\ M \end{Bmatrix} d\bar{x} = \int_0^{L_0} \bar{\mathbf{B}}^{eT} \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix} \bar{\mathbf{B}}^e d\bar{x}\, \bar{\mathbf{u}}^e = \frac{E}{L_0} \begin{bmatrix} A & 0 & -A & 0 \\ & 4I & 0 & 2I \\ & & A & 0 \\ \text{sym} & & & 4I \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{\theta}_1 - \bar{\theta}_1^p \\ \bar{u}_2 - \bar{u}^p \\ \bar{\theta}_2 - \bar{\theta}_2^p \end{Bmatrix} = \begin{Bmatrix} N_1 \\ M_1 \\ N_2 \\ M_2 \end{Bmatrix}$$

$$\bar{\mathbf{f}}^p = \int_0^{L_0} \bar{\mathbf{B}}^{pT} \begin{Bmatrix} N \\ M \end{Bmatrix} d\bar{x} = \begin{Bmatrix} N \\ -M_2 \\ N_2 \\ M_1 \end{Bmatrix} \tag{8}$$

with $\bar{u}^p = \bar{u}_1^p + \bar{u}_2^p$. Note that $\bar{\mathbf{f}}$ and $\bar{\mathbf{f}}^p$ are nodal internal forces that respectively realize work with the elastic virtual displacements $\delta\bar{\mathbf{u}}^e$ and with the plastic virtual displacements $\delta\bar{\mathbf{u}}^p$.

Once, by construction, $\bar{u}_1$ is null in the co-rotated frame, the displacements $\bar{\mathbf{u}}^e$, $\bar{\mathbf{u}}^p$, $\bar{\mathbf{u}}$ can be redefined as

$$\bar{\mathbf{u}}^e \leftarrow \lfloor \bar{\theta}_1^e \quad \bar{u}_2^e \quad \bar{\theta}_2^e \rfloor^T \qquad \bar{\mathbf{u}}^p \leftarrow \lfloor \bar{\theta}_1^p \quad \bar{u}^p \quad \bar{\theta}_2^p \rfloor^T \qquad \bar{\mathbf{u}} \leftarrow \lfloor \bar{\theta}_1 \quad \bar{u}_2 \quad \bar{\theta}_2 \rfloor^T = \bar{\mathbf{u}}^e + \bar{\mathbf{u}}^p. \tag{9}$$

The same applies to the vectors $\bar{\mathbf{r}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{f}}$ and $\bar{\mathbf{f}}^p$:

$$\bar{\mathbf{r}} \leftarrow \lfloor M_{\bar{z}1} \quad F_{\bar{x}2} \quad M_{\bar{z}2} \rfloor^T \qquad \bar{\mathbf{p}} \leftarrow \lfloor p_{\bar{\theta}_1} \quad p_{\bar{u}_2} \quad p_{\bar{\theta}_2} \rfloor^T$$

$$\bar{\mathbf{f}} \leftarrow \lfloor M_1 \quad N_2 \quad M_2 \rfloor^T = \frac{E}{L_0} \begin{bmatrix} 4I & 0 & 2I \\ & A & 0 \\ \text{sym} & & 4I \end{bmatrix} \begin{Bmatrix} \bar{\theta}_1 - \bar{\theta}_1^p \\ \bar{u}_2 - \bar{u}^p \\ \bar{\theta}_2 - \bar{\theta}_2^p \end{Bmatrix} = \bar{\mathbf{k}}^e (\bar{\mathbf{u}} - \bar{\mathbf{u}}^p) \qquad \bar{\mathbf{f}}^p \leftarrow \lfloor -M_2 \quad N_2 \quad M_1 \rfloor^T. \tag{10}$$

Replacing (9) and (10) into (5) results

$$\delta\bar{\mathbf{u}}^{eT} \left(-\bar{\mathbf{f}} + \bar{\mathbf{r}} + \bar{\mathbf{p}}^e \right) + \delta\bar{\mathbf{u}}^{pT} \left(-\bar{\mathbf{f}}^p + \bar{\mathbf{r}} + \bar{\mathbf{p}}^p \right) = 0. \tag{11}$$

Since the virtual displacements $\delta\bar{\mathbf{u}}$ are arbitrary, it is possible to choose a virtual displacement field completely elastic. Supposing that $\delta\bar{\mathbf{u}}^p = \mathbf{0}$ then

$$-\bar{\mathbf{f}} + \bar{\mathbf{r}} + \bar{\mathbf{p}}^e = \mathbf{0} \quad \Rightarrow \quad \bar{\mathbf{k}}^e (\bar{\mathbf{u}} - \bar{\mathbf{u}}^p) = \bar{\mathbf{r}} + \bar{\mathbf{p}}^e \tag{12}$$

which represents the equilibrium of the element refereed to C_{0n}. Notice that the system (22) has more unknown than equations requiring thus additional equations, like those follow provided.

2.2 Evolution laws and complementary conditions

Evolution laws of the plastic relative displacements $\bar{u}_1^p$, $\bar{u}_2^p$ and plastic rotations $\bar{\theta}_1^p$, $\bar{\theta}_2^p$ of the hinges can be written as a function of nodal internal forces N_1, M_1 and N_2, M_2 [15]. Let the respective yield functions for the hinges 1 and 2 be represented by

$$f_1 = f_1(N_1, M_1) \leq 0 \qquad f_2 = f_2(N_2, M_2) \leq 0. \tag{13}$$

The evolution laws of the plastic relative displacements $\bar{u}_1^p$, $\bar{u}_2^p$ and plastic rotations $\bar{\theta}_1^p$, $\bar{\theta}_2^p$ are given by the (discrete form) associative normality rule

$$\Delta\bar{u}_1^p = \Delta\lambda_1 \frac{\partial f_1}{\partial N_1} \qquad \Delta\bar{u}_2^p = \Delta\lambda_2 \frac{\partial f_2}{\partial N_2} \qquad \Delta\bar{\theta}_1^p = \Delta\lambda_1 \frac{\partial f_1}{\partial M_1} \qquad \Delta\bar{\theta}_2^p = \Delta\lambda_2 \frac{\partial f_2}{\partial M_2}. \tag{14}$$

Recording that $\bar{u}^p = \bar{u}_1^p + \bar{u}_2^p$, one writes

$$\Delta\bar{u}^p = \Delta\bar{u}_1^p + \Delta\bar{u}_2^p = \Delta\lambda_1 \frac{\partial f_1}{\partial N_1} + \Delta\lambda_2 \frac{\partial f_2}{\partial N_2} \tag{15}$$

where λ_1 and λ_2 are the plastic multipliers of plastic hinges 1 and 2. The evolution laws of λ_1 and λ_2 are

$$\begin{aligned} &\Delta\lambda_1 = 0 \ \text{ if } \ f_1(N_1, M_1) < 0 \qquad f_1(N_1, M_1) = 0 \ \text{ if } \ \Delta\lambda_1 > 0 \\ &\Delta\lambda_2 = 0 \ \text{ if } \ f_2(N_2, M_2) < 0 \qquad f_2(N_2, M_2) = 0 \ \text{ if } \ \Delta\lambda_2 > 0. \end{aligned} \tag{16}$$

Analytical expressions for the yielding functions f_1 and f_2 depends on the cross-section geometry and on the constitutive model of the material. Suitable empirical expressions for symmetric cross-sections and elastic perfectly plastic materials are given by

$$f_1(N_1, M_1) = \sqrt{\left(\frac{N_1}{N_y}\right)^2 + \left(\frac{M_1}{M_y}\right)^2} - 1 \le 0 \qquad f_2(N_2, M_2) = \sqrt{\left(\frac{N_2}{N_y}\right)^2 + \left(\frac{M_2}{M_y}\right)^2} - 1 \le 0 \tag{17}$$

where M_y is the yield moment of the cross-section without axial forces and N_y produces the total plasticization of the element when there is no bending moments.

2.3 Equilibrium refereed to C_0

The equilibrium equations of the element referred to C_0 can be obtained returning the rigid body motion that occurs from of C_0 to C_{0n} to the element, i.e., identifying the relationship between $\bar{\mathbf{u}} = \lfloor \bar{\theta}_1 \quad \bar{u}_2 \quad \bar{\theta}_2 \rfloor^T$ and $\mathbf{u} = \lfloor u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad \theta_2 \rfloor^T$. Such relationship can be established in a differential form as [14]

$$\delta\bar{\mathbf{u}} = \frac{\partial\bar{\mathbf{u}}}{\partial\mathbf{u}}\delta\mathbf{u} = \mathbf{T}\delta\mathbf{u} \qquad \mathbf{T}(\mathbf{u}) = \begin{bmatrix} \frac{\partial\bar{\theta}_1}{\partial u_1} & \frac{\partial\bar{\theta}_1}{\partial v_1} & \cdots & \frac{\partial\bar{\theta}_1}{\partial\theta_2} \\ \frac{\partial\bar{\theta}_2}{\partial u_1} & \frac{\partial\bar{\theta}_2}{\partial v_1} & \cdots & \frac{\partial\bar{\theta}_2}{\partial\theta_2} \\ \frac{\partial\bar{u}_2}{\partial u_1} & \frac{\partial\bar{u}_2}{\partial v_1} & \cdots & \frac{\partial\bar{u}_2}{\partial\theta_2} \end{bmatrix}. \tag{18}$$

Since the work realized by nodal reactions $\bar{\mathbf{r}}$, internal nodal forces $\bar{\mathbf{f}}$ and equivalent nodal forces $\bar{\mathbf{p}}^e$ are invariant with respect to changes in the coordinate system, it is possible to obtain the equilibrium equations of the element referred to C_0 by pre-multiplying (12) by $\mathbf{T}^T$

$$\mathbf{T}^T\left(-\bar{\mathbf{f}} + \bar{\mathbf{r}} + \bar{\mathbf{p}}^e\right) = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\psi}(\mathbf{u}) = -\mathbf{f} + \mathbf{r} + \mathbf{p}^e = \mathbf{0}. \tag{19}$$

The previous relation forms a system of nonlinear equations, which can be solved by the Newton-Raphson procedure. The tangent stiffness matrix $\mathbf{k}(\mathbf{u}) = \partial\boldsymbol{\psi}/\partial\mathbf{u}$ can be obtained explicitly. To calculate $\boldsymbol{\psi}(\mathbf{u})$ and $\partial\boldsymbol{\psi}/\partial\mathbf{u}$ it is necessary to obtain $\bar{\mathbf{f}}$ (and $\bar{\mathbf{u}}^p$) from $\mathbf{u}$. The computation of $\bar{\mathbf{f}}$ from $\mathbf{u}$ is denominated local problem.

2.4 Local problem

To solve the local problem, the constitutive equations (10) must be combined with the evolution laws (14) – (16) in a discrete form as discussed in what follows. Let $\bar{\mathbf{u}}_o^p$ be the vector of plastic elongation and plastic rotations of the previous known solution $\bar{\mathbf{u}}_o$, c_k the constant related to a particular hinge k ($k = 1{,}2$) which assumes the values $c_k = 0$ or $c_k = 1$ depending on $f_k^* < 0$ or $f_k^* \ge 0$, where f_k^* is the yield function of the hinge k given by (17) and evaluated for the elastic prediction $\bar{\mathbf{f}}^* = \bar{\mathbf{k}}^e\left(\bar{\mathbf{u}} - \bar{\mathbf{u}}_0^p\right)$. Assuming that the components of the plastic displacement vector $\bar{\mathbf{u}}^p$ are small, one decomposes $\bar{\mathbf{u}}^p$in the additive form

$$\bar{\mathbf{u}}^p = \bar{\mathbf{u}}_0^p + \Delta\bar{\mathbf{u}}^p \tag{20}$$

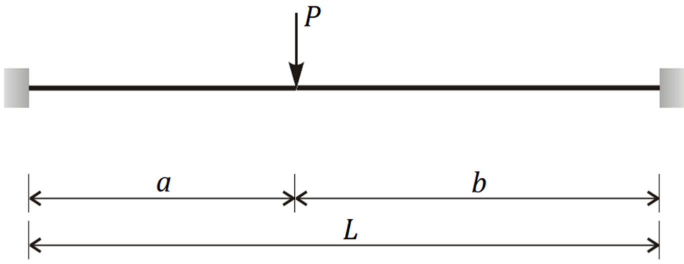

Figure 1: Clamped beam with a concentrated load

where, from (14) – (16)

$$\Delta\bar{\mathbf{u}}^p = \lfloor \Delta\bar{\theta}_1^p \quad \Delta\bar{u}^p \quad \Delta\bar{\theta}_2^p \rfloor^T = \left\lfloor c_1\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial M_1} \quad c_1\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial N_1} + c_2\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial N_2} \quad c_2\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial M_2}\right\rfloor^T. \tag{21}$$

Based on the definitions (12) and (20), one writes

$$\bar{\mathbf{f}} + \bar{\mathbf{k}}^e\Delta\bar{\mathbf{u}}^p = \bar{\mathbf{f}}^* \tag{22}$$

resulting in the nonlinear system

$$\boldsymbol{\psi}^l(\bar{\mathbf{f}}, \Delta\bar{\lambda}_1, \Delta\bar{\lambda}_2) = \bar{\mathbf{f}} + \bar{\mathbf{k}}^e\Delta\bar{\mathbf{u}}^p - \bar{\mathbf{f}}^* = \begin{Bmatrix} M_1 + \frac{4EI}{L_0}c_1\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial M_1} + \frac{2EI}{L_0}c_2\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial M_2} \\ N_2 + \frac{EA}{L_0}c_1\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial N_1} + \frac{EA}{L_0}c_2\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial N_2} \\ M_2 + \frac{2EI}{L_0}c_1\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial M_1} + \frac{4EI}{L_0}c_2\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial M_2} \end{Bmatrix} - \bar{\mathbf{f}}^* = \mathbf{0}. \tag{23}$$

In order to make (23) solvable, the two following equations $c_1 f_1 = 0$ and $c_2 f_2 = 0$ are added:

$$\boldsymbol{\psi}^l(\bar{\mathbf{f}}, \Delta\bar{\lambda}_1, \Delta\bar{\lambda}_2) \leftarrow \begin{Bmatrix} \boldsymbol{\psi}^l(\bar{\mathbf{f}}, \Delta\bar{\lambda}_1, \Delta\bar{\lambda}_2) \\ c_1 f_1(N_1, M_1) \\ c_2 f_2(N_2, M_2) \end{Bmatrix} = \begin{Bmatrix} M_1 + c_1\frac{4EI}{L_0}\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial M_1} + c_2\frac{2EI}{L_0}\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial M_2} \\ N_2 + c_1\frac{EA}{L_0}\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial N_1} + c_2\frac{EA}{L_0}\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial N_2} \\ M_2 + c_1\frac{2EI}{L_0}\Delta\bar{\lambda}_1\frac{\partial f_1}{\partial M_1} + c_2\frac{4EI}{L_0}\Delta\bar{\lambda}_2\frac{\partial f_2}{\partial M_2} \\ c_1 f_1(N_1, M_1) \\ c_2 f_2(N_2, M_2) \end{Bmatrix} - \begin{Bmatrix} \bar{\mathbf{f}}^* \\ 0 \\ 0 \end{Bmatrix}. \tag{24}$$

Once assumed values for c_k, the above non-linear system of equations can be solved numerically by the Newton-Raphson method. The converged solution obtained for the provisional values of c_k must satisfy the following constrains to be consistent with the elastic perfectly plastic constitutive model: (i) if $c_1 = c_2 = 1$ then $\Delta\lambda_1 \geq 0$ and $\Delta\lambda_2 \geq 0$; (ii) if $c_1 = 1$ and $c_2 = 0$ then $\Delta\lambda_1 \geq 0$ and $f_2(N_2, M_2) \leq 0$; (iii) if $c_1 = 0$ and $c_2 = 1$ then $f_1(N_1, M_1) \leq 0$ and $\Delta\lambda_2 \geq 0$; (iv) $c_1 = c_2 = 0$ then $f_1(N_1, M_1) \leq 0$ and $f_2(N_2, M_2) \leq 0$. Notice that there are only four different combinations for the constants c_k. If these constraints are not satisfied, a new prevision must be made for the constants c_k, and the local problem (24) must be solved again for the new constants.

3 Numerical results

The effectiveness of the proposed co-rotational finite element was verified by two examples. The examples are both a beam clamped at both ends and subjected to a concentrated load, as illustrated in Figure 1. The validation was performed by comparing the developed finite element formulation presented in this paper and programed in FORTRAN language with available results of reference solutions from others similar finite element formulations. The examples were evaluated with 2 elements. The tolerance for the convergence of the global and local Newton-Raphson procedures was set 10^{-3}.

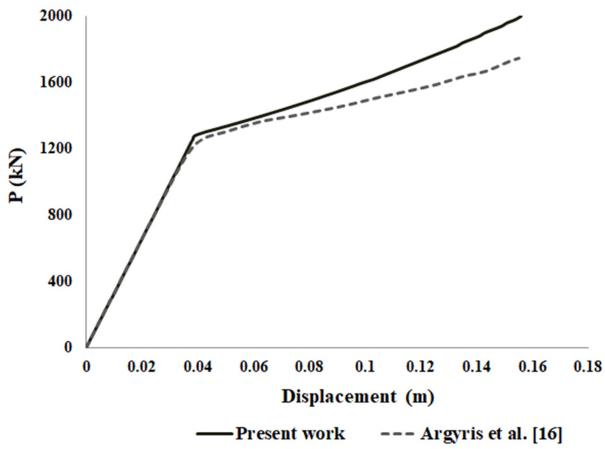

Figure 2: Clamped beam with a centered load

3.1 Clamped beam with a centered load

The first example was analyzed by Argyris et al. [16] with a distributed plasticity model. The authors used 50 finites elements and adopted $L = 500$ cm, $a = b = L/2$, a rectangular cross-section of 20 cm x 40 cm and an elastic perfectly plastic material with Young modulus $E = 20$ GPa and yield stress $\sigma_y = 100$ MPa. A concentrated load $P = 2000$ kN was applied in 100 steps. The load displacement results at the point of application of the load are shown in Figure 1. 2. The transitions between elastic and elastoplastic behavior obtained by the distributed plasticity model by Argyris is smother than the plastic hinge model presented in this work. The differences in the elastoplastic part of the load displacement curve may be due to adopted yield functions.

3.2 Clamped beam with an asymmetrical load

The second example was analyzed by Alhasawi et al. [17] with an elastic perfectly plastic hinges model. The authors used 2 finite elements with plastic hinges at its ends and adopted $L = 720$ cm, $a = L/3$ and $b = 2L/3$. The beam cross-section is of type HEB 220, with a plastic modulus of $z_p = 827.19$ cm^3. The elastic perfectly plastic material has a Young modulus $E = 210$ GPa and a yield stress $\sigma_y = 355$ MPa. It is worth to emphasize that the authors adopted the yield functions defined in (21). A concentrated load $P = 1200$ kN was applied in 100 steps. The load displacement results at the point of application of the load are shown in Fig. 3. Notice that both elastic and elastoplastic load-displacement responses are very similar, which is expected since both works deals with elastic perfectly plastic hinges with the same yield functions. Even though there are some differences in the formulation presented in Alhasawi et al. [17] and the one presented in this work, they seem to give the same initial elastic and final elastoplastic tangent stiffness.

4 Conclusions

In this paper, a co-rotational model with elastic perfectly plastic hinges was developed to analyze planar frames considering lumped plasticity effects. The hinges effects were introduced into the generalized strain fields as Dirac deltas centered at the element ends, which naturally results in the formulation of the element. Both the plastic constrained nonlinear system of equation of the local problems and the global equilibrium nonlinear equations are solved by a full Newton-Raphson procedure. Two bi-clamped beam examples with a concentrated load were presented to validate the developed formulation. The results were consistent with the expected, as discussed in the numerical results section.

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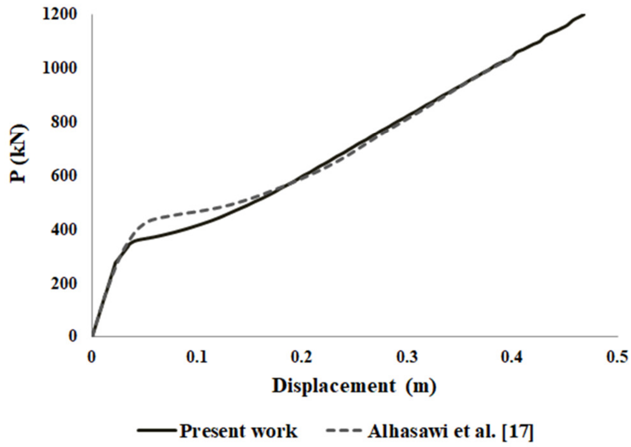

Figure 3: Clamped beam with an asymmetrical load

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