Boundary Layer Phenomenon in the Limit Analysis of Reissner-Mindlin Plates

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Abstract. The Kirchhoff and Reissner-Mindlin plate theories are the most frequently used models for describing the behavior of linearly elastic plates. In what concern the limit analysis of plates based on the static theorem, the most significant difference between those theories lies in the fact that for a Reissner-Mindlin plate the transverse shear resistance is included in the yield criterion. It is numerically shown that the Kirchhoff plate theory increasingly overpredicts the collapse load of plates with decreasing side-to-thickness ratio. For this reason alone, the Reissner-Mindlin plate theory should be used in the limit analysis of relatively thick plates. On the other hand, transverse shear locking and boundary layer have been major sources of convergence delay of Reissner-Mindlin plate models, especially towards thin plate solutions. While the former is a well investigated finite element defect, the latter is a real physical phenomenon likely to be manifested in more refined theories, such as the Reissner-Mindlin plate theory, that nearly no attention has been paid in the framework of yield design. In this sense, it is used in the present work a recently developed shear locking-free finite element for the yield design of Reissner-Mindlin plates based on the static theorem to illustrate boundary layers along certain types of plate edges, which may not be of the same type as those in linear elastic solutions. Finally, a brief state-of-the-art review about limit analysis of Reissner-Mindlin plates is then provided.

Keywords: boundary layer, limit analysis, Reissner-Mindlin plates.

1 Introduction

The Kirchhoff and Reissner-Mindlin plate theories are the most frequently used models for describing the behavior of linearly elastic plates [1]. The latter theory was originally proposed to cope with first-order transverse shear effects. In what concern the limit analysis of Kirchhoff and Reissner-Mindlin plates based on the static theorem, the most significant difference between those theories lies in the fact that for a Reissner-Mindlin plate the transverse shear resistance is included in the yield criterion. As it will be seen in the numerical tests, the Kirchhoff plate theory increasingly overpredicts the collapse load of plates with decreasing side-to-thickness ratio. For this reason alone, the Reissner-Mindlin plate theory should be used in the limit analysis of relatively thick plates. On the other hand, transverse shear locking has been the fundamental obstacle to the design of effective plate elements based on the Reissner-Mindlin theory [2]. Depending on the numerical treatment of a Reissner-Mindlin plate model, such a locking defect occurs making the convergence slower as the plate thickness becomes smaller. Bleyer and de Buhan [3] show that no transverse shear locking, that is, no deterioration of the solution, with respect to a reducing thickness, occurs when using equilibrium finite elements in the framework of limit analysis. The pseudo-equilibrium element for the limit analysis of Reissner-Mindlin plates proposed by Cavalcante and Lucena Neto [4] is also not plagued by transverse shear locking. Another major source of convergence delay that appears in more refined theories, such as the Reissner-Mindlin plate theory, is the real physical phenomenon of boundary layer, which this paper focus on and that probably finds in [3, 4] the only references to mention it in the framework of limit analysis.

Most of the research in the limit analysis of plates has been devoted to Kirchhoff plates, just to mention a few, and, many of them have been carried out with computer aided methods [5-16]. The coupling of the static and kinematic theorems of limit analysis with finite element method gives rise to large-scale constrained optimization problems which can be solved by means of mathematical programming algorithms. The usefulness of this very powerful procedure was limited initially by the lack of robustness of the algorithms that were available for solving large-scale mathematical programming problems and by the low computational capability. The substantial advance in algorithms for nonlinear convex optimization [17-22] has played a key part in the recent progress of limit analysis applied to plate bending problems [3, 4, 23-33].

There are little works devoted to the limit analysis of Reissner-Mindlin plates [3, 4, 23, 24, 28, 30-32] and only the works of [3, 4] address the static approach to limit analysis. Both finite element formulations in [3, 4] do not lead to strict lower bounds. The three-node triangular element developed in [4] belongs to a class of mixed finite elements [9, 33-35], nowhere violates the adopted von Mises yield criterion, and naturally converges to the Kirchhoff plate theory as the thickness becomes smaller. The element also detects boundary layer along certain types of plate edges, which is numerically shown to be a source of convergence delay and that may not be of the same type as those in the linear elastic solutions [36-39].

The resulting nonlinear convex optimization problem posed by the static theorem is treated as second-order cone programming and solved with a primal-dual interior-point algorithm [20] implemented in the MOSEK optimization package [40]. This paper makes use of this numerical procedure to show the presence of boundary layer in yield design (limit analysis) of Reissner-Mindlin plates.

2 Illustrative results

Predictions of collapse load are carried out for square plates assuming that $\sigma_0 = 250$ MPa and the computational procedure outlined in the last paragraph above. The computations are performed on a personal laptop computer (Intel Core i7-9750H CPU and 64 GB of RAM) running a 64-bit Windows 10.

Simply supported square plates of side length $L = 1$ m and thickness h are analyzed under uniformly distributed load q. As two kinds of simple and clamped supports must be distinguished in Reissner-Mindlin models [39], one kind of which named "hard" and the other "soft", the results to follow will take this distinction into account.

Due to symmetry consideration, only the bottom-left quarter of the plate is defined as the solution domain, as shown in Figure 1 modeled by the 4×4 mesh. The mechanical boundary conditions for the solution domain of a hard simply supported plate are

$$\left.\begin{array}{lll} M_n = 0 & & \text{hard} \\ M_n = 0 & M_{ns} = 0 & \text{soft} \end{array}\right\} \quad \text{on the left and bottom edges}$$

$$M_{ns} = 0 \qquad Q_n = 0 \qquad \text{on the right and top edges} \tag{1}$$

where M_n, M_{ns} and Q_n refer, respectively, to the bending moment, twisting moment and transverse shear force acting on the edges (n and s allude to the normal and tangential edge directions). According to the weak form (13) in Cavalcante and Lucena Neto [4], the weight function components ϕ_n, ϕ_s or w are made null on the edges for which M_n, M_{ns} or Q_n are unknown, respectively. For instance, on the right and top edges one must impose $\phi_n = 0$. For clamped plates, conditions (1) must be changed only on the left and bottom edges so that no mechanical boundary conditions are imposed on a hard edge while just $M_{ns} = 0$ takes place on a soft one.

The effect of the ratio L/h on the collapse load of hard or soft simply supported or clamped square plates is summarized in Figure 2 using the 128×128 mesh. No exact solution is available for any value of L/h. The

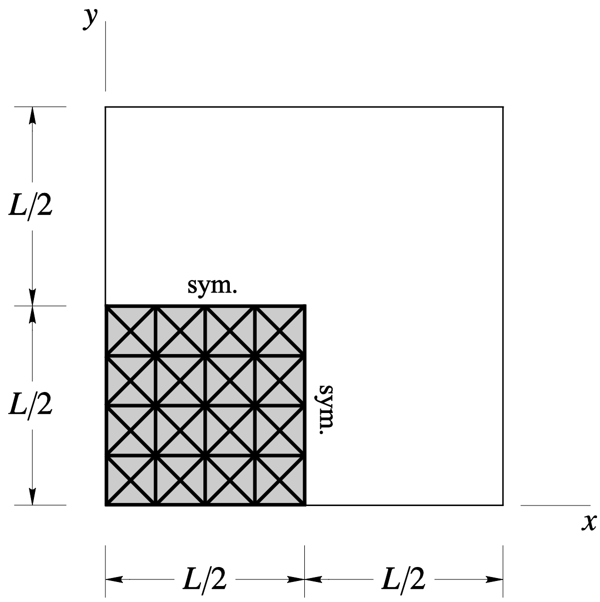

Figure 1. Square plate with solution domain modeled by the 4×4 mesh.

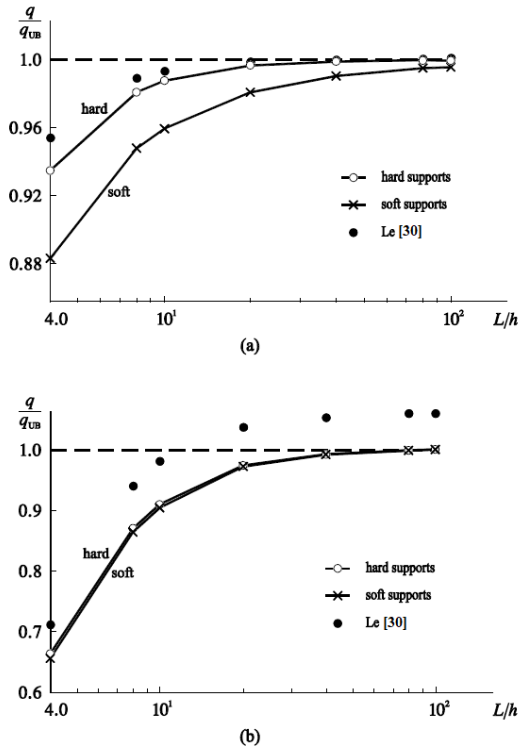

Figure 2. Variation of the collapse load predictions with respect to the ratio L/h for hard and soft square plates modeled by the 128×128 mesh: (a) simply supported ($q_{\mathrm{UB}} = 25.024\, M_0/L^2$); (b) clamped ($q_{\mathrm{UB}} = 44.196\, M_0/L^2$).

collapse load predictions with the von Mises yield criterion (see equation (8) in [4]) are normalized by the best upper bound q_{UB} found in the literature for each type of (hard) support and computed according to the Kirchhoff plate model [23, 29]. One can see that the results converge monotonically from below to q_{UB} with increasing L/h and, thus, the Kirchhoff plate model increasingly overpredicts the collapse load of plates with decreasing side-to-thickness ratio. The results for the hard simply supported condition exhibit a more pronounced convergence.

Figure 2 also depicts the upper bounds found in [30] for the most refined uniform mesh using a Reissner-Mindlin model. The predictions attained with the proposed element are systematically below those of [30], which

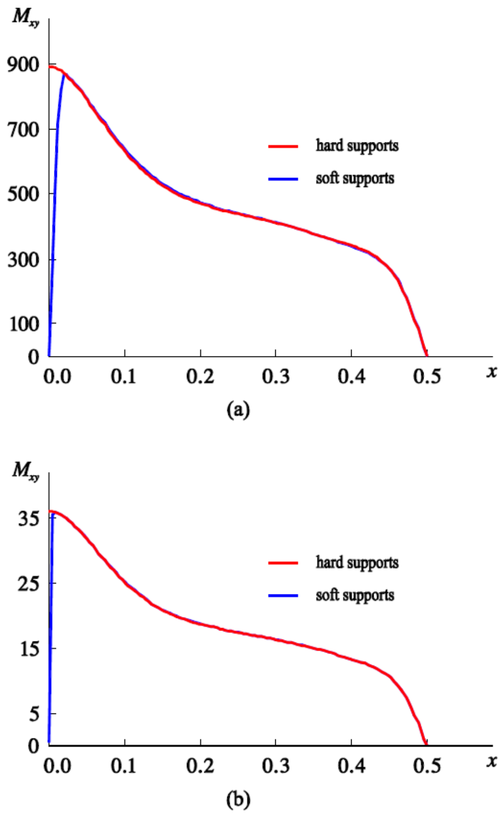

Figure 3. Distribution of M_{xy} along $y = 0.25$ at the collapse load of simply supported square plates modeled by the 128×128 mesh (SI units): (a) $L/h = 20$; (b) $L/h = 100$.

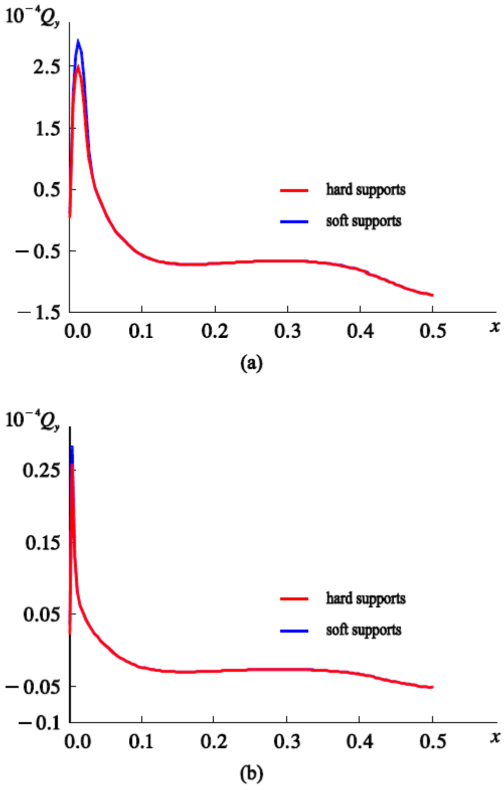

Figure 4. Distribution of Q_y along $y = 0.25$ at the collapse load of clamped square plates modeled by the 128×128 mesh (SI units): (a) $L/h = 20$; (b) $L/h = 100$.

Table 1. Limit loads computed with the von Mises infinite shear yield criterion.

Mesh	qL^2/M_0	
	Soft	Hard
4×4	24.8532 (−0.6587%)*	24.9109 (−0.4281%)*
8×8	24.9550 (−0.2518%)	24.9673 (−0.2027%)
16×16	25.0014 (−0.0664%)	25.0031 (−0.0596%)
32×32	25.0144 (−0.0144%)	25.0145 (−0.0140%)
64×64	25.0174 (−0.0024%)	25.0177 (−0.0012%)
128×128	25.0173 (−0.0028%)	25.0180 (0.0000%)
Pseudo Kirchhoff LB [3]	25.018	
Kirchhoff LB [26]	24.93	
Kirchhoff UB [3]	25.033†	

*Δ% to the pseudo Kirchhoff LB [3]

are based on a discrete shear gap formulation and combined with a stabilizing smoothing technique in order to avoid shear locking.

It is evident from Figure 2a that the collapse load of a simply supported square plate converges to that of the Kirchhoff plate, as the thickness becomes smaller, faster when hard condition is used. If the square plate is clamped, Figure 2b shows that the change from the hard to the soft condition has little effect on the convergence.

The distribution of the twisting moment M_{xy} in a simply supported square plate along $y = 0.25$ m is depicted in Figure 3a for $L/h = 20$ and Figure 3b for $L/h = 100$. One can see from the plot that the change from hard to soft condition introduces a noticeable difference in M_{xy} only in a narrow zone close to the edge, with the values of the soft plate varying rapidly. It is then said that M_{xy}, and also Q_x and Q_y (not shown here), has boundary layer (rapidly varying behavior) for the soft condition. While the narrow zone width decreases with the plate thickness, as in the linear elastic solution [36-39], the boundary layer intensity may or may not increase in the yield design. No boundary layer is detected for the bending moments M_x and M_y, whose values are insensitive to the change from the hard to the soft condition. The presence of boundary layers in the soft simply supported plate may explain its slower convergence toward the Kirchhoff plate shown in Figure 2a.

If the square plate is clamped, the change from the hard to the soft condition has little effect on the distribution of stress resultants. This may explain why the curves of Figure 2b are so close to each other. As illustrated in Figure 4, the shear force Q_y, and also Q_x (not shown here), has boundary layer with slightly higher intensity for the soft condition. Again, this may also explain the slow convergence shown in Figure 2b and why the curve of the soft plate is slightly below that of the hard plate. Unlike the yield design, the linear elastic solution does not detect any boundary layer close to straight soft clamped edges [38].

Additionally, according to Bleyer and de Buhan [3], the nondimensional collapse load qL^2/M_0 for a thin plate (Kirchhoff model) is independent of the plate thickness h and can be computed from a Reissner-Mindlin model by imposing the yield shear force Q_0 to be infinite (see equation (9) in [4]). For the simply supported condition, Table 1 condenses the soft and hard collapse predictions obtained with this von Mises infinite shear yield criterion ($Q_0 = \infty$) for all six meshes. It also details the relative error to the best solution computed in [3] with the same yield criterion and hard simple supports. As expected, the results are independent of L/h and this is verified by achieving the same values for ten different ratios ($L/h = 1, 2, 4, 8, 10, 20, 40, 80, 100, 200$).

The best limit load obtained for the hard simply supported square plate ($q\,L^2/M_0 = 25.0180$) coincides to the one computed by Bleyer and de Buhan [3]. The predictions for the soft and hard simply supported plates with all meshes are less than 1% of the reference hard value $q\,L^2/M_0 = 25.018$ and, except for the coarsest mesh, they are bracketed by the Kirchhoff lower and upper bounds found in Le *et al.* [26] and Bleyer and de Buhan [3], respectively. For the hard simply supported case, the results converge monotonically to $q\,L^2/M_0 = 25.018$. It is also evident from Table 1 that for each mesh the gap between the reference value $q\,L^2/M_0 = 25.033$ found in [3]

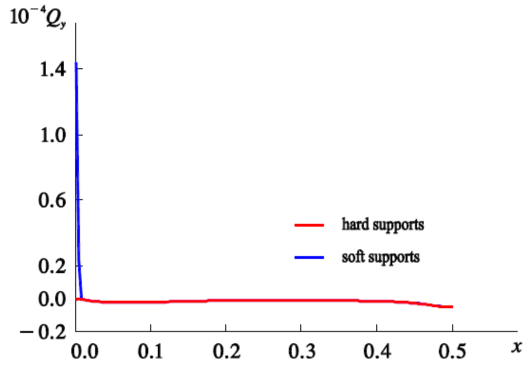

Figure 5. Distribution of Q_y along $y = 0.25$ at the collapse load of simply supported thin square plate modeled by the 128×128 mesh (SI units).

and the soft load predictions is always greater than the gap obtained with hard boundary conditions.

Again, the distribution of the shear force Q_y along $y = 0.25$ m for a simply supported thin plate is shown in Figure 5, which has the same distribution as Q_x along $x = 0.25$ m (not shown here). One can see from the plots that the change from the hard to the soft boundary condition introduces a noticeable difference in Q_y (and also Q_x) only in a narrow zone close to the edge, with the values of the soft plate varying rapidly. No boundary layer is detected for the bending moments M_x and M_y and the twisting moment M_{xy}, whose values are insensitive to the change from the hard to the soft condition. The presence of boundary layer in the soft simply supported thin plate may explain the larger gaps of the results in Table 1 to the Kirchhoff upper bound $q\,L^2/M_0 = 25.033$ found in [3]. The absence of boundary layer in any of the stress resultants for the hard simply supported boundary conditions, which describes more accurately the Kirchhoff model [3], agrees with linear elastic solutions [36-39]. Therefore, it can be argued that it is accurate the procedure of imposing hard boundary conditions and the yield shear force Q_0 to be infinite in a Reissner-Mindlin limit analysis model to simulate an equivalent Kirchhoff model, as suggested by Bleyer and de Buhan [3].

3 Conclusions

The pseudo-equilibrium finite element developed in [4] for limit analysis of Reissner-Mindlin plates detects boundary layer along edges that may not be of the same type as those in the linear elastic solutions. As a real physical phenomenon likely to be manifested in more refined theories, such as the Reissner-Mindlin plate theory, boundary layer should be taken into account in the yield design by means of some adaptive refinement strategy because it is a source of convergence delay.

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References

[1] J.N. Reddy, *Theory and Analysis of Elastic Plates and Shells, 2nd ed*. CRC Press, 2007.
[2] S. Cen, Y. Shang, "Developments of Mindlin-Reissner plate elements". *Math. Probl. Eng.*, vol. 2015, pp. 1–12, 2015.
[3] J. Bleyer, P. de Buhan, "Lower bound static approach for the yield design of thick plates". *Int. J. Numer. Methods Eng.* vol. 100, n. 11, pp. 814–833, 2014.
[4] E.L.B. Cavalcante, E. Lucena Neto, "A pseudo-equilibrium finite element for limit analysis of Reissner-Mindlin plates". *Applied Math. Modelling* vol. 96, pp. 336–354, 2021.
[5] H.G. Hopkins, A.J. Wang, "Load-carrying capacities for circular plates of perfectly-plastic material with arbitrary yield condition". *J. Mech. Phys. Solids*, vol. 3, n. 2, pp. 117–129, 1954.
[6] H.G. Hopkins, "On the plastic theory of plates". *Proc. R. Soc. A*, vol. 241, n. 1225, pp. 153–179, 1957.
[7] E.H. Mansfield, "Studies in collapse analysis of rigid-plastic plates with a square yield diagram". *Proc. R. Soc. A*, vol. 241, n. 1226, pp. 311–338, 1957.

[8] P.G. Hodge, T. Belytschko, "Numerical methods for the limit analysis of plates". *J. Appl. Mech.*, vol. 35, n. 4, pp. 796–802, 1968.
[9] E. Anderheggen, H. Knöpfel, "Finite element limit analysis using linear programming". *Int. J. Solids Struct.*, vol. 8, n. 12, pp. 1413–1431, 1972.
[10] E. Faccioli, E. Vitiello, "A finite element, linear programming method for the limit analysis of thin plates". *Int. J. Numer. Methods Eng.*, vol. 5, n. 3, pp. 311–325, 1973.
[11] E. Christiansen, S. Larsen, "Computations in limit analysis for plates". *Int. J. Numer. Methods Eng.*, vol. 19, n. 2, pp. 169–184, 1983.
[12] A. Sawczuk, J. Sokól-Supel, *Limit Analysis of Plates*, Polish Scientific Publisher, 1993.
[13] S. Krenk, L. Damkilde, O. Hoyer, "Limit analysis and optimal design of plates with equilibrium elements". *J. Eng. Mech.*, vol. 120, n. 6, pp. 1237–1254, 1994.
[14] M.A. Save, *Atlas of Limit Loads for Metal Plates, Shells and Disks*, Elsevier, 1995.
[15] D. Johnson, "Automated yield-line analysis of orthotropic slabs". *Int. J. Solids Struct.*, vol. 33, n. 1, pp. 1–10, 1996.
[16] M.A. Save, C.E. Massonet, G. de Saxce, *Plastic Limit Analysis of Plates, Shells and Disks, 2nd ed.* North Holland, 1997.
[17] Y.E. Nesterov, M.J. Todd, "Primal-dual interior-point methods for self-scaled cones" *SIAM J. Optimiz.*, vol. 8, n. 2, pp. 324–364, 1998.
[18] J.F. Sturm, "Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones". *Optim. Method Softw.*, vol. 11, n. 1-4), pp. 625–653, 1999.
[19] K.D. Andersen, E. Christiansen, A.R. Conn, M.L. Overton, "An efficient primal-dual interior-point method for minimizing a sum of Euclidean norms". *SIAM J. Sci. Comput.*, vol. 22, n. 1, pp. 243–262, 2000.
[20] E.D. Andersen, C. Roos, T. Terlaky, "On implementing a primal-dual interior-point method for conic quadratic optimization". *Math. Program. Ser. B*, vol. 95, n. 2, pp. 249–277, 2003.
[21] R.H. Tütüncü, K.C. Toh, M.J. Todd, "Solving semidefinite-quadratic-linear programs using SDPT3". *Math. Program. Ser. B*, vol. 95, n. 2, pp. 189–217, 2003.
[22] F. Pastor, E. Loute, "Solving limit analysis problems: an interior-point method". *Int. J. Numer. Meth. Biomed. Engng.*, vol. 21, n. 11, pp. 631-642, 2005.
[23] A. Capsoni, L. Corradi, "Limit analysis of plates - a finite element formulation". *Struct. Eng. Mech.*, vol. 8, n. 4, pp. 325–341, 1999.
[24] A. Cecchi, G. Milani, A. Tralli, "A Reissner-Mindlin limit analysis model for out-of-plane loaded running bond masonry walls". *Int. J. Solids Struct.*, vol. 44, n. 5, pp. 1438–1460, 2007.
[25] C.V. Le, M. Gilbert, H. Askes, "Limit analysis of plates using the EFG method and second-order cone programming". *Int. J. Numer. Methods Eng.*, vol. 78, n. 13, pp. 1532–1552, 2009.
[26] C.V. Le, H. Nguyen-Xuan, H. Nguyen-Dang, "Upper and lower bound limit analysis of plates using FEM and second-order cone programming". *Comput. Struct.*, vol. 88, n. 1-2, pp. 65–73, 2010.
[27] C.V. Le, H. Askes, M. Gilbert, "Adaptive element-free Galerkin method applied to the limit analysis of plates". *Comput. Methods Appl. Mech. Engrg.*, vol. 199, n. 37-40, pp. 2487–2496, 2010.
[28] A. Capsoni, M.V. Silva, "A finite element formulation of Mindlin plates for limit analysis". *Int. J. Numer. Meth. Biomed. Engng.*, vol. 27, n. 1, pp. 143–156, 2011.
[29] J. Bleyer, P. de Buhan, "On the performance of non-conforming finite elements for the upper bound limit analysis of plates". *Int. J. Numer. Methods Eng.*, vol. 94, n. 3, pp. 308–330, 2013.
[30] C.V. Le, "A stabilized discrete shear gap finite element for adaptive limit analysis of Mindlin-Reissner plates". *Int. J. Numer. Methods Eng.*, vol. 96, n. 4, pp. 231–246, 2013.
[31] T. Nguyen-Thoi, P. Phung-Van, M.H. Nguyen-Thoi, H. Dang-Trung, "An upper-bound limit analysis of Mindlin plates using CS-DSG3 method and second-order cone programming". *J. Comput. Appl. Math.*, vol. 281, pp. 32–48, 2015.
[32] J. Bleyer, C.V. Le, P. de Buhan, "Locking-free discontinuous finite elements for the upper bound yield design of thick plates". *Int. J. Numer. Methods Eng.*, vol. 103, n. 12, pp. 894–913, 2015.
[33] E. Christiansen, "Limit analysis of collapse states", in : P.G. Ciarlet, J.L. Lions (Eds.), *Handbook of Numerical Analysis*, Springer, 1996.
[34] H. Ciria, J. Peraire, J. Bonet, "Mesh adaptive computation of upper and lower bounds in limit analysis". *Int. J. Numer. Methods Eng.*, vol. 8, pp. 899–944, 2008.
[35] J. Bleyer, G. Hassen, "Automated formulation and resolution of limit analysis problems". *Comput. Struct.*, vol. 243, 106341, 2021.
[36] D.N. Arnold, R.S. Falk, "Edge effects in the Reissner-Mindlin plate theory, in: A.K. Noor, T. Belytschko, J.C. Simo (Eds.), *Proceedings of the Synposium on Analytical and Computational Models of Shells, 1989.*
[37] D.N. Arnold, R.S. Falk, "A uniformly accurate finite element method for the Reissner-Mindlin plate". *SIAM J. Numer. Anal.*, vol. 26, n. 6, pp. 1276–1290, 1989.
[38] B. Häggbald, K.J. Bathe, "Specifications of boundary conditions for Reissner/Mindlin plate bending finite elements". *Int. J. Numer. Methods Eng.*, vol. 30, n. 5, pp. 981–1011, 1990.
[39] A. Selman, E. Hinton, W. Atamaz-Sibai, "Edge effects in Mindlin-Reissner plates using adaptive mesh refinement". *Eng. Comput.*, vol. 7, n. 3, pp. 217–226, 1990.
[40] MOSEK ApS, *The Mosek Optimization Toolbox for MATLAB Manual, Version 8.1* (Available from http://docs.mosek.com/8.1/toolbox/index.html) [Accessed in May, 2019]