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Consistency assessment of plate bending theories for the implementation of efficient hybrid finite elements in linear statics and dynamics

Ney Augusto Dumont[1], Renan Costa Sales[1]

[1]*Department of Civil and Enviromental Engineering, Pontifical Catholic University of Rio de Janeiro*
Rua Marquês de São Vicente 225, 22451-900, Rio de Janeiro, Brazil
dumont@puc-rio.br, renansales@aluno.puc-rio.br

Abstract. The present developments are part of a research work that aims to implement computationally efficient models of foldable and deployable structures, which are finding increasing application in the industry. In spite of the availability of powerful commercial codes, numerical models of large foldable and deployable systems would rather work with a minimum amount of degrees of freedom to allow for straightforward engineering conceptualizations while preserving the structure's basic static and dynamic properties. A thorough survey of the literature on moderately thick panels points out the existence of too many theoretical contradictions since the papers by Hencky, Timoshenko, Mindlin and Reissner, to mention just a few early researchers. This very critical literature review is part of the present contribution, which also presents a proposition that may claim some originality in terms of both mechanical consistency and ease of computational implementation.

Keywords: Variational methods, Hybrid finite elements, Moderately thick plates

1 Introduction

Our present motivation is the development of a solution for moderately thick plates that can be implemented in a robust code for the analysis of time dependent problems in the frame of the advanced modal analysis [1–3]. Our variational framework is the Hellinger-Reissner potential [4], which leads to a hybrid (Trefftz) finite element implementation [5]. The technical literature on the mechanics of moderately thick plates is overwhelming but we shall try to highlight some of the most interesting contributions of the distant and recent past. A more comprehensive overview is brought together by Sales [6, 7].

Different types of trial solutions, polynomials or non-polynomials, are reported in the literature [8–11] to approximate the internal stress field in the formulation of thick and thin finite plate elements. However, a proper evaluation not only of the trial solutions but also of the consistency of the transversal shear strains is required for arriving at consistent formulations.

It is not easy to assign a unique criterion to assess a plate theory's consistency [12–14]. Some criteria are based on the correct specification of the boundary conditions [15] and some may be related to the physical interpretation of the problem [16]. We present a consistent formulation for moderately thick plates in the time domain in the frame of the so-called first-order shear deformation theory [17, 18] ultimately setting the equations for a hybrid finite element implementation in the frequency domain with particular use in an advanced modal analysis [1–3], which is in progress[6, 7].

2 The plate equations

2.1 Kinematic assumptions

For a plate considered as a degenerated three-dimensional domain of constant thickness h much smaller than the dimensions in the (x, y) Cartesian directions and submitted only to transversal, distributed force $q(x, y)$, we make the kinematic assumptions for the in-plane displacements u, v and transversal displacement w

$$u(x, y, z) = \beta_y(x, y)z, \quad v(x, y, z) = -\beta_x(x, y)z, \quad w(x, y, z) = w(x, y), \tag{1}$$

where w and the rotations β_x, β_y are the problem's primary unknowns. Accordingly, the in-plane strains are

$$\varepsilon_{xx} = \beta_{y,x}z, \quad \varepsilon_{yy} = -\beta_{x,y}z, \quad \gamma_{xy} = (\beta_{y,y} - \beta_{x,x})\, z, \tag{2}$$

while stating that

$$\begin{aligned} &\varepsilon_{zz}, \sigma_{zz} \quad \text{are negligible, not considered in the equations} \\ &\gamma_{xz} = ?, \quad \gamma_{yz} = ? \quad \text{(no kinematic assumptions for the transverse direction).} \end{aligned} \tag{3}$$

The first assumption above is classical for the so-called first-order theory, which corresponds to Timoshenko's theory for a beam structure [19]. On the other hand, the explicit indication that no prior assumptions should be made for the transversal shear strains seems to have been first proposed by Reissner [20]. We actually proceed from here on in terms of equilibrium as well as of a virtual work statement, which seems more consistent.

2.2 Equilibrium of the cross section forces

The equilibrium equations for an infinitesimal plate element are, according to the scheme of Fig. 1,

$$M_{xx,x} + M_{yx,y} - Q_x - \frac{mh^2}{12}\ddot{\beta}_y = 0 \tag{4}$$

$$M_{yy,y} + M_{xy,x} - Q_y + \frac{mh^2}{12}\ddot{\beta}_x = 0 \tag{5}$$

$$Q_{x,x} + Q_{y,y} + q - m\ddot{w} = 0, \tag{6}$$

also considering the inertia effect, where $q \equiv q(x, y, t)$ and $m = \rho h$ are transversal force and mass density per unit area. A friction term for viscous damping might be considered with no substantial change in the developments – except that the frequency number to be obtained would be complex [2] – but this is disregarded presently. The expression $h^2/12$ in the first equation above comes from dI_y/dA_x, where $dI_y = dyh^3/12$ and $dA_x = dy\, h$ and accounts for Rayleigh's rotatory inertia. The term for the second equation has the same explanation (see, for instance, Graff [21], p. 180 and ff.). These equations are valid except for the higher order terms $-Q_{x,x}dx/2$, $-Q_{y,y}dy/2$, $(Q_{x,xy}dy + Q_{y,xy}dx)\,/2$, respectively, when special care must be taken if the shear forces experiment local jumps, which may happen at a plate's corner depending on the considered boundary conditions.

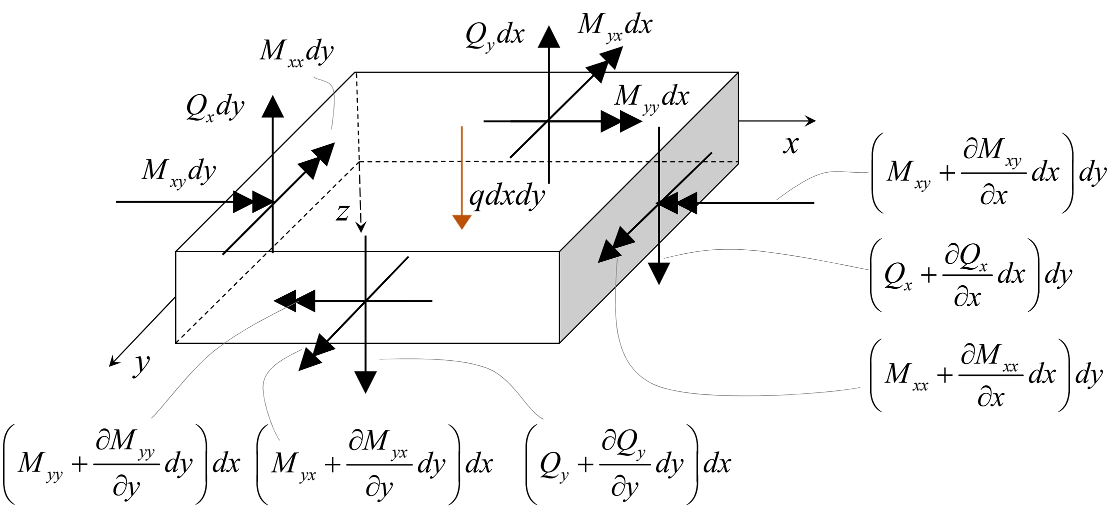

Figure 1. Forces that act on an infinitesimal element of the plate's domain

2.3 Stress expressions

The in-plane stress components become from eq. (2), according to Hooke's law, functions of the rotations β_x, β_y and from there we obtain by integrating through the thickness the bending moments, as well:

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) = \frac{Ez}{1-\nu^2}\left(\beta_{y,x} - \nu\beta_{x,y}\right) \quad \Rightarrow \quad M_{xx} = K\left(\beta_{y,x} - \nu\beta_{x,y}\right) \tag{7}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right) = \frac{Ez}{1-\nu^2}\left(-\beta_{x,y} + \nu\beta_{y,x}\right) \quad \Rightarrow \quad M_{yy} = K\left(-\beta_{x,y} + \nu\beta_{y,x}\right) \tag{8}$$

$$\tau_{xy} = G\gamma_{xy} = Gz\left(\beta_{y,y} - \beta_{x,x}\right) \quad \Rightarrow \quad M_{xy} = M_{yx} = K(1-\nu)/2\ \left(\beta_{y,y} - \beta_{x,x}\right), \tag{9}$$

where $K = \dfrac{Eh^3}{12(1-\nu^2)}$. Moreover, we obtain from the stress equilibrium equations $\sigma_{ij,j} - \rho\ddot{u}_i = 0$ using eqs. (4)-(9) and after integrating in the z-direction with constants chosen such that $\tau_{xz}(z = \pm h/2) = \tau_{yz}(z = \pm h/2) = 0$:

$$\sigma_{xx,x} + \tau_{xy,y} + \tau_{xz,z} - \rho\ddot{u} = 0 \quad \Rightarrow \quad \tau_{xz} = \frac{12}{h^3}\left(\frac{h^2}{8} - \frac{z^2}{2}\right) Q_x \tag{10}$$

$$\tau_{xy,x} + \sigma_{yy,y} + \tau_{yz,z} - \rho\ddot{v} = 0 \quad \Rightarrow \quad \tau_{yz} = \frac{12}{h^3}\left(\frac{h^2}{8} - \frac{z^2}{2}\right) Q_y \tag{11}$$

$$\tau_{zx,x} + \tau_{zy,y} + \sigma_{zz,z} - \rho\ddot{w} = 0 \quad \text{negligible effect, not considered.} \tag{12}$$

2.4 Virtual-work equivalent mean transversal shear distortions

Trying to obtain expressions for γ_{xz} and γ_{yz} from eq. (1) would be completely inconsistent with the results that come from the latter equations by applying Hooke's law. In fact, the explicit expressions of γ_{xz} and γ_{yz} are inconsequential, as the plate kinematic assumptions of eq. (1) are an attempt to deal with a three-dimensional problem as a degenerated two-dimensional one and we are actually only interested in kinematic mean quantities obtained in terms of the cross-section equilibrium eqs. (4)-(11) in the frame of a virtual work statement.

We introduce the mean plate distortions γ_{0xz} and γ_{0yz} in such a way that shear-related vertical displacements $dh_{xz} = \gamma_{0xz}dx$ and $dh_{yz} = \gamma_{0yz}dy$ (in the direction of h but actually not an increment of it) between opposite faces of an infinitesimal plate element are obtained from eqs. (10) and (11) in terms of virtual work as

$$\delta Q_x dh_{xz} dy = dxdy \int_{-h/2}^{h/2} \delta\tau_{xz} \frac{\tau_{xz}}{G} dz = \delta Q_x dxdy \frac{144 Q_x}{Gh^6} \int_{-h/2}^{h/2} \left(\frac{h^2}{8} - \frac{z^2}{2}\right)^2 dz = \delta Q_x dxdy \frac{6}{5}\frac{Q_x}{Gh} \tag{13}$$

$$\delta Q_y dh_{yz} dx = dxdy \int_{-h/2}^{h/2} \delta\tau_{yz} \frac{\tau_{yz}}{G} dz = \delta Q_y dxdy \frac{144 Q_y}{Gh^6} \int_{-h/2}^{h/2} \left(\frac{h^2}{8} - \frac{z^2}{2}\right)^2 dz = \delta Q_y dxdy \frac{6}{5}\frac{Q_y}{Gh}, \tag{14}$$

according to the geometrical interpretation of Fig. 2. Note that $dh_{xz}dy$ and $dh_{yz}dx$ are the shaded areas corresponding to the parallelograms $[BB'C'''C'']$ and $[DC'C'''D']$. Moreover, $\overline{CC''} = \overline{C'C'''}$, $\overline{CC'} = \overline{C''C'''}$ and $\overline{CC'''} = dh_{xz} + dh_{yz}$ as the flat rectangle $[ABCD]$ becomes the distorted $[AB'C'''D']$. We obtain from above

$$\gamma_{0xz} = \frac{6}{5}\frac{Q_x}{Gh} \quad \text{and} \quad \gamma_{0yz} = \frac{6}{5}\frac{Q_y}{Gh}. \tag{15}$$

2.5 A geometric assessment

The mean plate distortions γ_{0xz} and γ_{0yz} exist only in terms of the virtual work statements of eqs. (13) and (14). However, should they have the geometric interpretation indicated on the left in Fig. 2, we obtain from the kinematic definitions in eq. (1) that they correspond to the differences between the derivatives of the transversal displacement w and the actual cross section rotations,

$$\gamma_{0xz} = \beta_y + w_{,x}, \quad \gamma_{0yz} = -\beta_x + w_{,y}, \tag{16}$$

and we check the consistency that no distortions occur for a thin plate ($h \to 0$), $\gamma_{0xz} = \gamma_{0yz} = 0$, as the normal to the reference plane remains normal when the plate deforms, according to Kirchhoff's theory.

The drawing on the left in Fig. 2 shows a distortion γ_{0xz} that remains constant as we move from the edge $\overline{AB}$ to the edge $\overline{DC}$ at the infinitesimal distance dy, and the same occurs with the distortion γ_{0yz} when we advance an infinitesimal distance dx. As a result of such distortion variations the point C''' in the figure displaces an additional amount dh, as indicated in the right scheme, leading to, for geometric consistency,

$$dh = \gamma_{0xz,y} dy dx = \gamma_{0yz,x} dx dy \quad \Rightarrow \quad \gamma_{0xz,y} = \gamma_{0yz,x} \quad \Rightarrow \quad \Psi = 0. \tag{17}$$

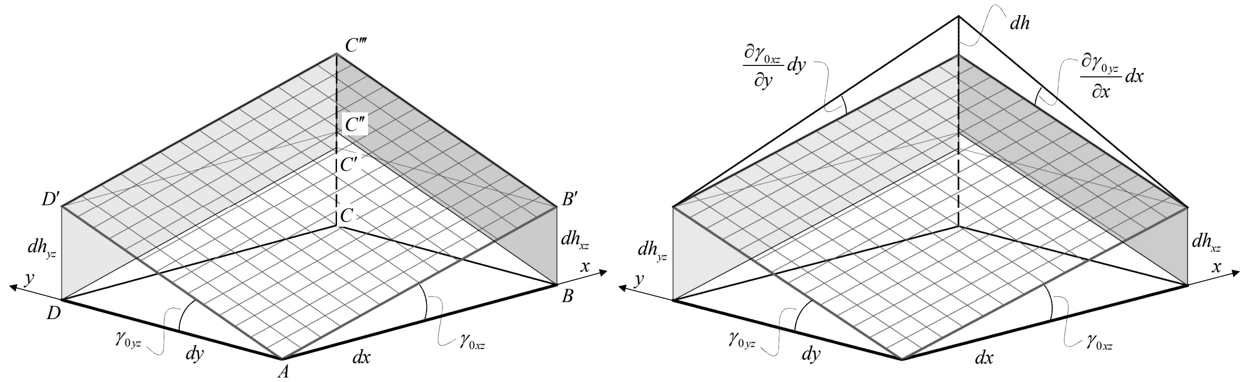

Figure 2. Infinitesimal plate reference layer element $[ABCD]$ subjected to plane distortions (given by dh_{xz} and dh_{yz}) due to shear forces (left), and scheme of differed distortions related to the additional displacement dh

3 Governing partial differential equations

For the development of hybrid finite plate elements we need solutions of w, β_x and β_y that satisfy the homogeneous part of the equilibrium eqs. (4)-(6) while keeping the virtual work consistency of Section 2.4. Such solutions are called "fundamental solutions" and should be as simple as possible while satisfying all the domain requirements outlined above, which should include the geometric consistency of Section 2.5. The boundary conditions are considered only in the frame of the proposed variational approach, which is not dealt with herein [7].

The complete differential equations of the problem are displayed in the following with aid of the auxiliary potential functions Φ and Ψ, also resorting to eq.(16):

$$\Phi = \beta_{y,x} - \beta_{x,y}, \quad \Psi = \beta_{y,y} + \beta_{x,x} = \gamma_{0xz,y} - \gamma_{0yz,x}. \tag{18}$$

Observe that, according to eq. (17), actually $\Psi = 0$. Nonetheless, we carry out all derivations considering Ψ explicitly and only afterwards propose a literature review on the subject, as this seems to very controversial. After a tedious manipulation of eqs. (4)-(9), (15), (16) we arrive at the problem's governing differential equations

$$K\nabla^4 w - \frac{mh^2}{60}\frac{17-5\nu}{1-\nu}\nabla^2\ddot{w} + m\ddot{w} + \frac{m^2h}{10G}\ddddot{w} + \left(\frac{h^2}{5(1-\nu)}\nabla^2 q - \frac{mh}{10}\frac{\ddot{q}}{G} - q\right) = 0 \tag{19}$$

$$\Phi = -\nabla^2 w + \frac{6m}{5Gh}\ddot{w} - \frac{6q}{5Gh} \tag{20}$$

$$\Psi = \frac{h^2}{10}\nabla^2\Psi - \frac{mh}{10G}\ddot{\Psi}. \tag{21}$$

As mentioned before, the homogeneous solution of these equations – non-singular fundamental or Trefftz solutions – are our ultimate goal for the development of finite elements in the particular framework of a frequency-domain hybrid variational formulation [1–3, 6, 7].

4 Proposed family of frequency-dependent non-singular fundamental solutions

This section presents the non-singular fundamental solution for the hybrid finite element formulation of moderately thick plates in the frequency domain, from which the particularization for the static case as well as for thin plates is straightforward. We are looking for the homogeneous solution of the governing eqs. (19)-(21), in which the latter one is actually void, as $\Psi = 0$ from the geometric consideration given in Section 2.5. The frequency-domain formulation of the homogeneous part of eq. (19) is obtained by separation of variables,

$$w(x, y, t) = w^*(x, y, \omega)\tau(t, \omega), \tag{22}$$

where ω is an in principle arbitrary separation constant, such that $\ddot{\tau} = -\omega^2\tau$, to be eventually related to the plate's natural frequencies. Then, after a tedious manipulation we obtain for the homogeneous part of eq. (19)

$$\nabla^4 w^* + k^2 \frac{h^2}{60}\frac{17-5\nu}{1-\nu}\nabla^2 w^* + k^2\left(\frac{k^2h^4}{60(1-\nu)} - 1\right) w^* = 0, \tag{23}$$

where $k = \omega\sqrt{\dfrac{m}{K}}$ is the frequency number for thin plates and $\dfrac{K}{G} = \dfrac{h^3}{6(1-\nu)}$. This may be rewritten as

$$\left(\nabla^2 + k_1^2\right)\left(\nabla^2 - k_2^2\right) w^* = 0, \tag{24}$$

where

$$k_1^2, k_2^2 = k\sqrt{1 + \left(\frac{5\nu+7}{1-\nu}\frac{h^2k}{120}\right)^2} \pm \frac{17-5\nu}{1-\nu}\frac{h^2k^2}{120} \tag{25}$$

and $\lim_{t\to 0} k_1^2, k_2^2 = k$. The general non-singular solution of eq. (24) is conveniently expressed as

$$\begin{aligned} w^* &= (C_{1n} sin(n\theta) + C_{2n}\cos(n\theta))\left(\frac{J_n(k_1 r)}{k_1^n} + \frac{I_n(k_2 r)}{k_2^n}\right) \\ &+ \frac{C_{3m} sin(m\theta) + C_{4m}\cos(m\theta)}{k_1^2 - k_2^2}\left(\frac{J_m(k_1 r)}{k_1^m} - \frac{I_m(k_2 r)}{k_2^m}\right), \quad m = max(n-2, 0), n = 0, \ldots, N \end{aligned} \tag{26}$$

in terms of polar coordinates (r, θ), where $J_n()$ and $I_n()$ are Bessel and modified Bessel functions of first kind and order n. The proposed solution is well suited for the expansion of w^* as a power series of the frequency ω in the frame of an advanced modal analysis [1, 2, 5, 6, 22], for $1, 2, 3, 4, 4, 4, \ldots, 4$ polynomial families in Cartesian coordinates (x, y) of degrees $n = 0, 1, 2, 3, 4, 5, \ldots, N$, respectively [7].

The proposed solution in terms of eqs. (24)- (26) assumes that $k_2^2 \geq 0$, that is, the plate's maximum thickness is in the present framework frequency dependent:

$$\frac{17-5\nu}{1-\nu}\frac{h^2k}{120} \leq 1. \tag{27}$$

Once w^* is adequately expressed, as proposed in eq. (26), we obtain the corresponding homogeneous frequency-domain eq. (20), for $\Phi(x, y, t) = \Phi^*(x, y, \omega)\tau(t, \omega)$, as

$$\Phi^* = -\left(\nabla^2 + \frac{h^2k^2}{5(1-\nu)}\right) w^* \tag{28}$$

as well as the frequency-dependent rotations introduced in eq. (1) in a similar way as in [15] for the static case:

$$\beta_x^* = \frac{12h^2}{60(1-\nu) - h^4k^2}\left(\frac{5(1-\nu)}{h^2} w^*_{,y} - \Phi^*_{,y} - \cancel{\frac{1-\nu}{2}\Psi^*_{,x}}\right) \tag{29}$$

$$\beta_y^* = \frac{-12h^2}{60(1-\nu) - h^4k^2}\left(\frac{5(1-\nu)}{h^2} w^*_{,x} - \Phi^*_{,x} - \cancel{\frac{1-\nu}{2}\Psi^*_{,y}}\right). \tag{30}$$

We have left in these equations the derivatives of Ψ^* – from $\Psi(x, y, t) = \Psi^*(x, y, \omega)\tau(t, \omega)$ –, while indicating that they should actually be void, according to eq. (17). As a matter of fact, the frequency dependent, homogeneous expression of Ψ^* above would come from eq. (21) as the Helmholtz equation

$$\cancel{\nabla^2\Psi^* + \left(\frac{k^2h^2}{6(1-\nu)} - \frac{10}{h^2}\right)\Psi^* = 0}, \tag{31}$$

which is disconnected from w^* and Φ^*, and should be simply disregarded not least because we do not get a smooth transition to a thin plate, $h \to 0$, as obtained for all other expressions [6].

The frequency dependent constitutive equations for a cross section of the general plate problem are obtained from eqs. (7)-(9), (15), (16), (26), (29) and (30) as

$$\begin{Bmatrix} M^*_{xx} \\ M^*_{yy} \\ M^*_{xy} \end{Bmatrix} = K \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \beta^*_{y,x} \\ -\beta^*_{x,y} \\ \beta^*_{y,y} - \beta^*_{x,x} \end{Bmatrix}; \quad \begin{Bmatrix} Q^*_x \\ Q^*_y \end{Bmatrix} = \frac{5Gh}{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \beta^*_y + w^*_{,x} \\ -\beta^*_x + w^*_{,y} \end{Bmatrix}. \tag{32}$$

5 A very brief literature review

Many authors such as Marguerre and Woernle [15] propose transversal shear distortions similar to eq. (16) as if they are actual distortions and not equivalent mean quantities in terms of virtual work. They introduce a thickness $h_s = h/1.2$ and call it the "energy correction" in a way that is not very clear (although ultimately correct) and generally related to the "Mindlin-Reissner" plate theory [8, 23–28], although other authors base their works exclusively on Mindlin's theory [9]. There are several implementations that correctly consider that Ψ is void or not depending on the plate's boundary conditions[13, 15, 29, 30]. Several publications are dedicated to classical and higher order plate theories [13, 19] although missing the virtual work considerations of the present paper in favor of overly stated kinematic assumptions. For the sake of arriving at different types of hybrid (Trefftz) finite plate elements, several authors [8, 11, 24, 25] correctly consider, as presently conceptualized, that $\Psi = 0$, although without a clear justification.

Huang et al [27] assess the effect of Ψ on plate edge phenomena briefly and conclude that it is in most cases negligible. Other authors [23, 26] propose solutions with $\Psi \neq 0$ as modified Bessel or exponential functions that are rather involved in terms of code implementation, specially for the sake of numerical integration, suggesting their use only for finite elements along the plate boundaries, where the edge effect may be pronounced. However, we should never overlook that our simple proposition has a sound mechanical basis and applies to domain equations only – whereas the boundary conditions are taken into account by means of a variational principle. A thorough numerical comparison of literature results is being prepared by Sales [7].

6 Conclusions

We propose a hybrid formulation for moderately thick plates to be modeled in the frequency domain and more precisely in the frame of a generalized modal analysis that resorts to frequency-dependent stiffness and mass matrices for the more accurate solution of the associated eigenvalue problem. Our plate formulation is hybrid in the sense that in-plane strains are obtained from kinematic assumptions whereas the transversal shear distortions are obtained as virtual-work equivalent mean quantities from the plate's cross section equilibrium equations. We present a simple geometric assessment of the mean transversal shear distortions to conclude that a Helmholtz equation related to the distortion function Ψ should be just void. The originality of our contribution lies in the fully consistent developments as well as in the mentioned geometric assessment. These developments build the theoretical basis of the second author's PhD work that aims to propose new families of fully consistent plate and shell elements for the solution of general time-dependent problems.

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Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material of the present paper is the property (and authorship) of the authors.

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