CILAMCE PANACM RIO DE JANEIRO 2021

Comparison of Analytical and Numerical Solutions to the Stresses Problem in a Cylindrical Shell with a Circular Hole

Stanislava V Kashtanova[1], Alexey V Rzhonsnitskiy[2]

[1] *Institute for Problems of Mechanical Engineering of RAS*
Bolshoi prospect V.O., 61, 199178, Saint-Petersburg, Russia
kastasya@yandex.ru
[2] *Dept. of Mathematics, Saint-Petersburg State Institute of Technology*
Moskovsky prospect, 26, 190013, Saint-Petersburg, Russia
rzhonsnitskiy@yandex.ru

Abstract. The cylindrical shell with a circular hole under three types of boundary conditions is considered: axial tension, internal pressure and torsion. A new mathematical approach that allows reducing an infinite system and finding unknown coefficients for the deriving stress is offered. This approach lifts classical mathematical restrictions for curvature parameter. The comparison of analytical and numerical results by collocation method is described.

Keywords: Shell Theory, Cylindrical Shell, Circular Cutout

1 Introduction

In recent works, Kashtanova and Rzhonsnitskiy [1,2] reconsidered the classical approach to the solution of the problem of cylindrical shell a circular cutout under an axial tension. The protagonist of this problem is a certain geometric parameter β, that contains the ration between the radius of the hole, the radius of the cylindrical shell and the shell thickness. When this parameter is equal to zero, we have a plane problem (the Kirsch problem in case when the boundary condition is an axial tension). There were several reasons that forced us to look for a different approach: a very small range of applicability of the solution (close to Kirsch problem), a linear dependency in the system for finding coefficients for basis function, different results in the literature, a failure of the limit transition, no explicit formulas for the field of stresses and no opportunity to make an analytical analysis. The classical way was to decompose the solution and the coefficients into small parameter β, which immediately put restrictions on the range of applicability. The authors of the present paper offered to decompose the solution into Fourier series and to divide variables [1], then they found and exclude a linear-dependent equation and after some substitution of variables proofed the reducibility of the infinite system [2]. This method allows getting a model in range of β up to 4. In this paper this idea, in addition to axial tension, is applied to other boundary conditions: internal pressure and torsion. The comparison of analytical and numerical results by collocation method that were received by Van Dyke [4] is described.

2 The Problem

We consider a cylindrical shell of radius R and thickness h with a circular hole r_0 under various boundary conditions. The main parameter that is responsible for the ratio between geometric characteristics is

$$\beta^2 = r_0^2 \cdot \frac{\sqrt{3\,(1-\nu^2)}}{4Rh},$$

ν – Poisson's coefficient. Note that limit transition while $\beta \to 0$ leads us to the plane problem.

The government equation of the problem reduces by Lurie [3] to

$$\Delta\Delta\Phi + 8i\beta^2 \frac{\partial^2\Phi}{\partial x^2} = 0. \tag{1}$$

Here function $\Phi = \frac{Eh}{8\beta^2 R} w - iU$ is used, which contains the deflection w, the stress function U and E – Young's modulus. The stress function U is connected with stress tensor T by next correlation

$$\begin{pmatrix} T_x & T_{xy} \\ T_{xy} & T_y \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 U}{\partial y^2} & -\frac{\partial^2 U}{\partial y \partial x} \\ -\frac{\partial^2 U}{\partial y \partial x} & \frac{\partial^2 U}{\partial x^2} \end{pmatrix}, \tag{2}$$

median surface stress is $\sigma = T/h$.

We consider this problem with follow three types of boundary conditions:

I. axial tension p at infinity along x-coordinate

- at infinity $T_x = p, \quad T_{xy} = 0, \quad T_y = 0, \quad w = 0;$
- at the boundary of the hole in polar coordinates (r, ϑ) – free edge

$$T_{rr}|_{r=r_0} = 0, T_{r\vartheta}|_{r=r_0} = 0, M_r|_{r=r_0} = 0, Q_r|_{r=r_0} = 0; \tag{3}$$

II. uniform internal pressure q_0 ($q = \frac{q_0 r_0}{2}$):

- $T_x = q; T_y = 2q; T_{xy} = 0;$

or (in polar coordinates) $2T_r = q(3 - \cos 2\vartheta);\ 2T_{r\vartheta} = q \sin 2\vartheta;\ 2T_\vartheta = q(3 + \cos 2\vartheta)$

- at the boundary of the hole in polar coordinates (r, ϑ)

$$T_{rr}|_{r=r_0} = 0, T_{r\vartheta}|_{r=r_0} = 0, M_r|_{r=r_0} = 0, Q_r|_{r=r_0} = -\frac{q_0 r_0}{2}; \tag{4}$$

III. torsion

- at infinity $T_x = 0, \quad T_y = 0, \quad T_{xy} = \tau, \quad \tau = \frac{M}{2\pi R^2}$

or in polar coordinates $T_r = \tau \sin 2\vartheta, \qquad T_\vartheta = -\tau \sin 2\vartheta, \qquad T_{r\vartheta} = \tau \cos 2\vartheta$

- at the boundary of the hole in polar coordinates (r, ϑ)

$$T_{rr}|_{r=r_0} = 0, T_{r\vartheta}|_{r=r_0} = 0, M_r|_{r=r_0} = 0, Q_r|_{r=r_0} = 0. \tag{5}$$

The solution offered by authors can be written in the next form (for case I, II and III respectively):

$$\Phi_I = -i\frac{py^2}{2} + \sum_{n=0}^{\infty} (a_n + ib_n) \cdot f_n, \tag{6}$$

$$\Phi_{\text{II}} = -i\frac{qy^2}{2} - i\frac{2qx^2}{2} + \sum_{n=0}^{\infty} (a_n + ib_n) \cdot f_n, \tag{7}$$

$$\Phi_{\text{III}} = i\,\tau\,x\,y + \sum_{n=0}^{\infty} (a_n + ib_n) \cdot f_n, \tag{8}$$

where for I and II types of boundary conditions we have

$$f_n(r, \vartheta) = \frac{g(r, n, 0)}{2} + \sum_{l=1}^{\infty} g(r, n, l) \cdot \cos 2l\vartheta \tag{9}$$

and for III type

$$f_n(r, \vartheta) = \sum_{l=1}^{\infty} \tilde{g}(r, n, l) \cdot \sin 2l\vartheta. \tag{10}$$

The third type has a slightly different structure due to the antisymmetry of the problem, because of this $\sin 2l\vartheta$ appears, which is equal to zero for $l = 0$.

In its turn

$$g(r, n, l) = (-1)^{\left[\frac{n}{2}\right]+l} \cdot \frac{H_n^{(1)}\big((1+i)\beta r\big)}{H_n^{(1)}\big((1+i)\beta\big)} \cdot \Big(J_{n+2l}\big((1+i)\beta r\big) + J_{n-2l}\big((1+i)\beta r\big)\Big), \quad \begin{matrix} n = 0,1,\dots,\infty \\ l = 0,1,\dots,\infty \end{matrix} \tag{11}$$

$$\tilde{g}(r, n, l) = (-1)^{l+\left[\frac{n}{2}\right]} \frac{H_n^{(1)}\big((1+i)\beta r\big)}{H_n^{(1)}\big((1+i)\beta\big)} \Big(J_{n-2l}\big((1+i)\beta r\big) - J_{n+2l}\big((1+i)\beta r\big)\Big), \quad \begin{matrix} n = 1,\dots,\infty \\ l = 1,\dots,\infty \end{matrix}. \tag{12}$$

In formulas (6)-(8) we need to find unknown coefficients a_n and b_n. After getting the function Φ, using the correlation (2), we will find stresses for our problem with various boundary conditions.

3 Creating Systems

Finding unknown coefficients a_n and b_n from the system with an infinite number of equations that we get from boundary conditions of the stresses problem is the important part of the work. We proved that one equation is a linear combination of four others that helped us to solve the system [1]. In work [2] it was proved the reductubility of these infinite systems. For all types of boundary conditions, the matrixes of the system are received: by the susbstituing solution (6) into (3) for the first type, (7) into (4) for the second and, respectevely, (8) into (5) for the rhird case.

As you can see below, all systems have the same structure.

SYSTEM FOR TYPE I AND II

n \ l	**0**		**1**		**2**		**3**			unknown	Free Type I	Free Type II
	Im	**Re**	**Im**	**Re**	**Im**	**Re**	**Im**	**Re**				
0	$t_3(0,0)$	$t_3(0,0)$	$t_3(1,0)$	$t_3(1,0)$	$t_3(2,0)$	$t_3(2,0)$	$t_3(3,0)$	$t_3(3,0)$		a_0	0	0
0	$t_4(0,0)$	$t_4(0,0)$	$t_4(1,0)$	$t_4(1,0)$	$t_4(2,0)$	$t_4(2,0)$	$t_4(3,0)$	$t_4(3,0)$		b_0	0	$16\beta^2 q$
1	$g(0,1)$	$g(0,1)$	$g(1,1)$	$g(1,1)$	$g(2,1)$	$g(2,1)$	$g(3,1)$	$g(3,1)$		a_1	$-\frac{p}{4}$	$\frac{q}{4}$
1	$g'(0,1)$	$g'(0,1)$	$g'(1,1)$	$g'(1,1)$	$g'(2,1)$	$g'(2,1)$	$g'(3,1)$	$g'(3,1)$	⋮	b_1	$-\frac{p}{2}$	$\frac{q}{2}$
1	$t_3(0,1)$	$t_3(0,1)$	$t_3(1,1)$	$t_3(1,1)$	$t_3(2,1)$	$t_3(2,1)$	$t_3(3,1)$	$t_3(3,1)$		a_2	0	0
1	$t_4(0,1)$	$t_4(0,1)$	$t_4(1,1)$	$t_4(1,1)$	$t_4(2,1)$	$t_4(2,1)$	$t_4(3,1)$	$t_4(3,1)$		b_2	0	0
2	$g(0,2)$	$g(0,2)$	$g(1,2)$	$g(1,2)$	$g(2,2)$	$g(2,2)$	$g(3,2)$	$g(3,2)$		a_3	0	0
2	$g'(0,2)$	$g'(0,2)$	$g'(1,2)$	$g'(1,2)$	$g'(2,2)$	$g'(2,2)$	$g'(3,2)$	$g'(3,2)$		b_3	0	0
...												

$$t_3(n,l) = i\left(-4l^2\, \nu\, g(n,l) + \nu\, g'(n,l) + g''(n,l)\right)$$

$$t_4(n,l) = i\left(12l^2 \cdot g(n,l) - \left(1+\nu+4l^2(2-\nu)\right) g'(n,l) + g'''(n,l)\right)$$

SYSTEM FOR TYPE III

n \ l	**1**		**2**		**3**		**4**			unknown	Free Type III
	Im	**Re**	**Im**	**Re**	**Im**	**Re**	**Im**	**Re**			
1	$\tilde{t}_3(1,1)$	$\tilde{t}_3(1,1)$	$\tilde{t}_3(2,1)$	$\tilde{t}_3(2,1)$	$\tilde{t}_3(3,1)$	$\tilde{t}_3(3,1)$	$\tilde{t}_3(4,1)$	$\tilde{t}_3(4,1)$		a_1	0
1	$\tilde{t}_4(1,1)$	$\tilde{t}_4(1,1)$	$\tilde{t}_4(2,1)$	$\tilde{t}_4(2,1)$	$\tilde{t}_4(3,1)$	$\tilde{t}_4(3,1)$	$\tilde{t}_4(4,1)$	$\tilde{t}_4(4,1)$		b_1	0
1	$\tilde{g}(1,1)$	$\tilde{g}(1,1)$	$\tilde{g}(2,1)$	$\tilde{g}(2,1)$	$\tilde{g}(3,1)$	$\tilde{g}(3,1)$	$\tilde{g}(4,1)$	$\tilde{g}(4,1)$		a_2	$-\frac{\tau}{2}$
1	$\tilde{g}'(1,1)$	$\tilde{g}'(1,1)$	$\tilde{g}'(2,1)$	$\tilde{g}'(2,1)$	$\tilde{g}'(3,1)$	$\tilde{g}'(3,1)$	$\tilde{g}'(4,1)$	$\tilde{g}'(4,1)$	⋮	b_2	$-\tau$
2	$\tilde{t}_3(1,2)$	$\tilde{t}_3(1,2)$	$\tilde{t}_3(2,2)$	$\tilde{t}_3(2,2)$	$\tilde{t}_3(3,2)$	$\tilde{t}_3(3,2)$	$\tilde{t}_3(4,2)$	$\tilde{t}_3(4,2)$		a_3	0
2	$\tilde{t}_4(1,2)$	$\tilde{t}_4(1,2)$	$\tilde{t}_4(2,2)$	$\tilde{t}_4(2,2)$	$\tilde{t}_4(3,2)$	$\tilde{t}_4(3,2)$	$\tilde{t}_4(4,2)$	$\tilde{t}_4(4,2)$		b_3	0
2	$\tilde{g}(1,2)$	$\tilde{g}(1,2)$	$\tilde{g}(2,2)$	$\tilde{g}(2,2)$	$\tilde{g}(3,2)$	$\tilde{g}(3,2)$	$\tilde{g}(4,2)$	$\tilde{g}(4,2)$		a_4	0
2	$\tilde{g}'(1,2)$	$\tilde{g}'(1,2)$	$\tilde{g}'(2,2)$	$\tilde{g}'(2,2)$	$\tilde{g}'(3,2)$	$\tilde{g}'(3,2)$	$\tilde{g}'(4,2)$	$\tilde{g}'(4,2)$		b_4	0
...											

$$\tilde{t_3}(n,l) = i\left(-4l^2\, \nu\, \tilde{g}(n,l) + \nu\, \tilde{g}'(n,l) + \tilde{g}''(n,l)\right)$$

$$\tilde{t_4}(n,l) = i\left(12l^2 \cdot \tilde{g}(n,l) - \left(1+\nu+4l^2(2-\nu)\right) \tilde{g}'(n,l) + \tilde{g}'''(n,l)\right)$$

4 Results

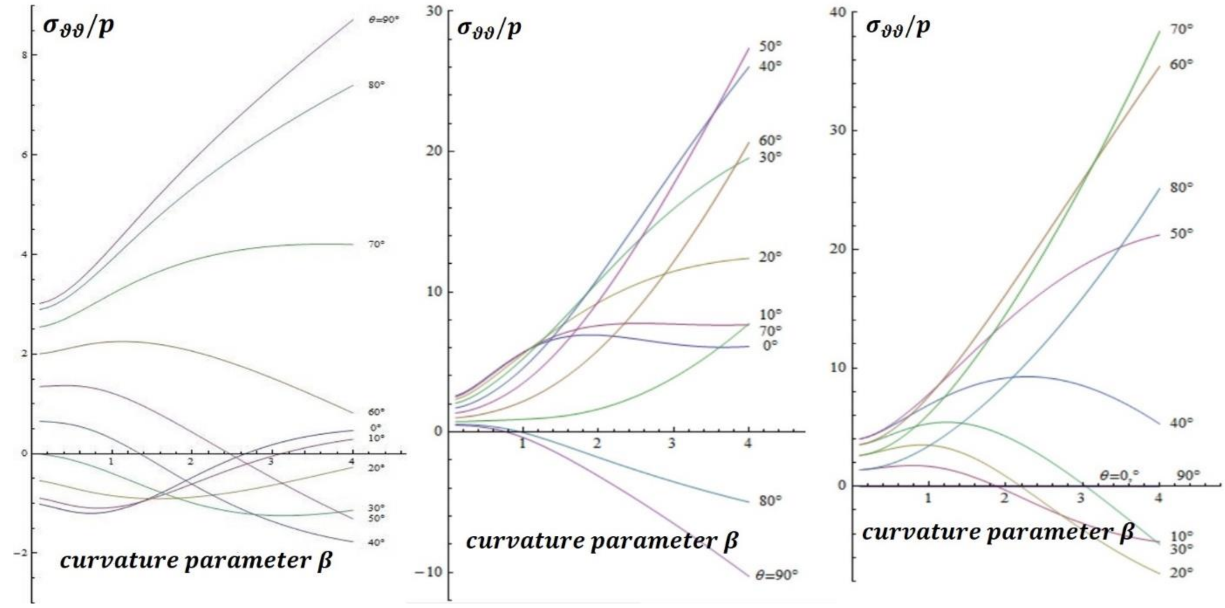

Figure 1. Stresses $\sigma_{\vartheta\vartheta}/p$ for types I, II and III respectively received in the present paper

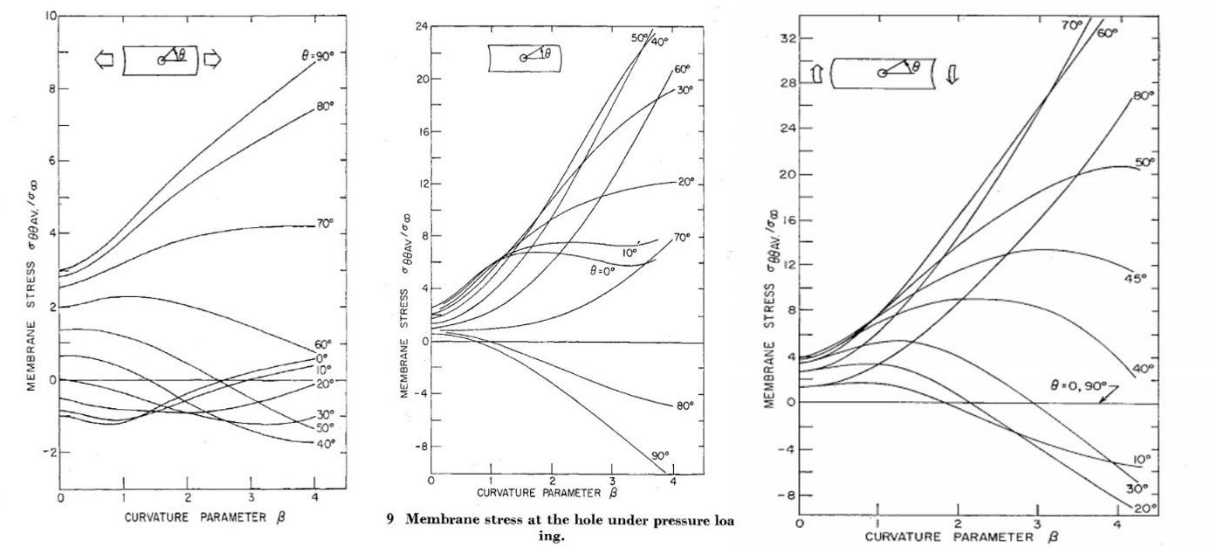

Figure 2. Stresses $\sigma_{\vartheta\vartheta}/p$ for types I, II and III respectively received in the work of Van Dyke [4] in 1965 by collocation method

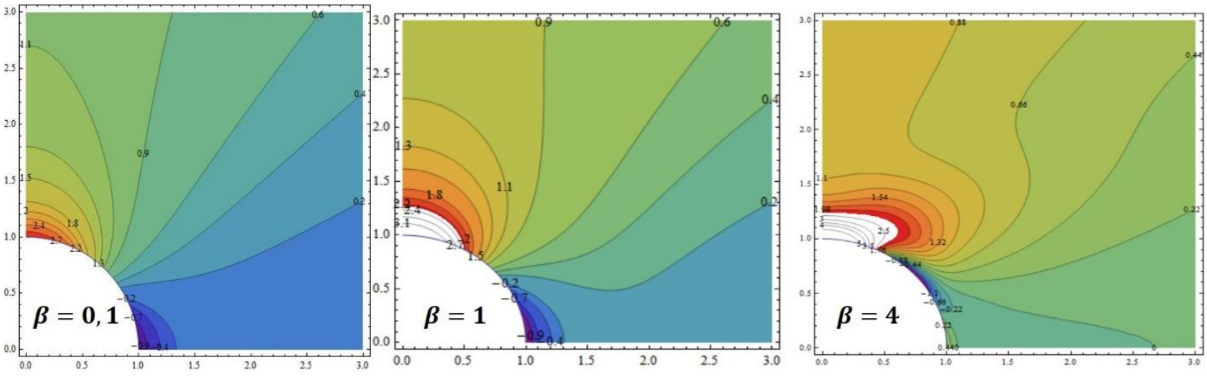

Figure 3a. Field of stresses $\sigma_{\vartheta\vartheta}/p$ for type I (axial tension) for different β

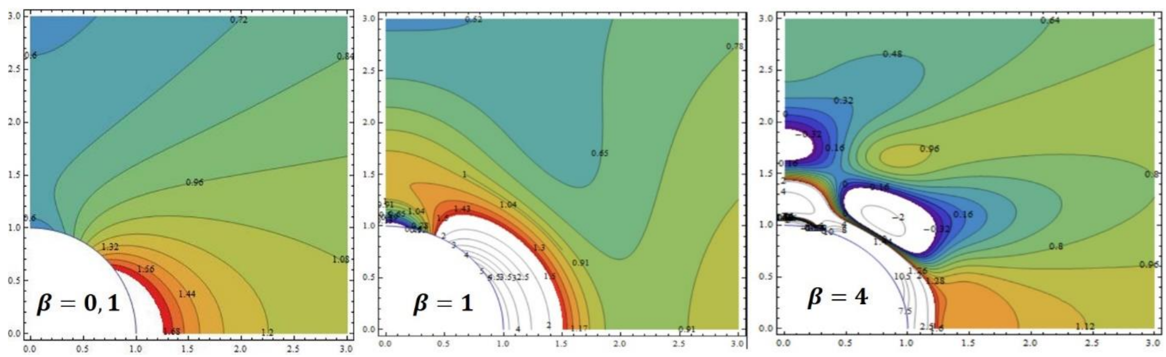

Figure 3b. Field of stresses $\sigma_{\vartheta\vartheta}/p$ for type II (internal pressure) for different β

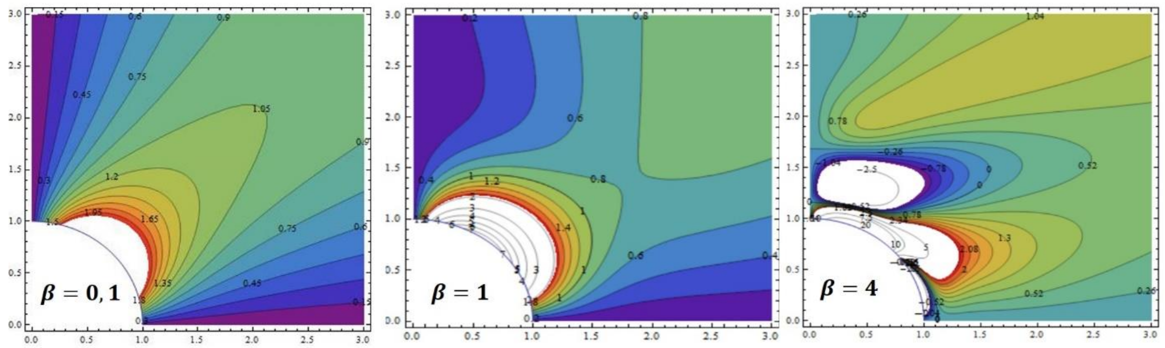

Figure 3c. Field of stresses $\sigma_{\vartheta\vartheta}/p$ for type III (torsion) for different β

5 Conclusions

The results that were received by new analytical approach are absolutely coincide with results that were obtained by collocation method by Van Dyke [4] in 1965 for all three cases of boundary conditions. Our model has no mathematical restrictions, as it was before, and from the point of view of mechanics it works up to 4.

Acknowledgements. The reported study was funded by RFBR, project number 19-31-60008.

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] Kashtanova S.V., Rzhonsnitskiy A.V. *Analytical Approach to the Derivation of the Stress Field of a Cylindrical Shell with a Circular Hole under Axial Tension* // Recent Approaches in the Theory of Plates and Plate-like Structures, Chapter 10, Springer, 2021 (*in print*)
[2] Kashtanova S.V., Rzhonsnitskiy A.V. Investigation of Systems of the Stress Field Problem of a Cylindrical Shell with a Circular Cutout under Various Boundary Conditions // *Acta Mechanica (to be published)*.
[3] Lurie, A.I.: Concentration of stresses in the Vicinity of an Aperture in the Surface of a Circular Cylinder (in Russian), *Prikladnaya Matematica I Mekhanica, Vol.10,p.397*, 1946 (English transl. by N. Brunswick, New York University, Inst. of Math. Sci., 1960)
[4] Van Dyke, P: Stresses about a Circular Hole in a Cylindrical Shell, *AIAA Journal, Vol.3, № 9, pp.1733-1742*, 1965