Equivalent strain measures for micromorphic continuum damage models

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Abstract. Various applications of the micromorphic continuum theory have been proposed in the past mainly due to its regularization properties in strain localization problems as a consequence of its non-local character. The micromorphic theory is particularly suited for the analysis of quasi-brittle materials as the microstructural behavior is incorporated in its formulation through the consideration of additional degrees of freedom related to the material particle. In order to allow the application of the micromorphic theory associated to damage models, extending its regularization properties to different constitutive models, this work presents a generalization of classical scalar-isotropic damage models for the micromorphic theory implemented in a constitutive models framework for elastic degrading media. To guarantee conformity to a classical implementation, a compact tensorial formulation is used, allowing the application for the micromorphic theory of theoretical and numerical resources already defined for the classical theory. A homogenization strategy is also employed to obtain the micromorphic constitutive relations through the consideration of a Cauchy continuum in the micro-scale, what makes possible non-linear analysis of micromorphic media with only the definition of the material parameters of a classical continuum.

Keywords: Micromorphic continuum, Continuum damage models, Elastic-degradation, Equivalent strain measures.

1 Introduction

In the classical continuum mechanics the study of the material behavior is based on the hypothesis that every point in the material is occupied by a small element of the solid, i.e., a material particle. The dimensions of each particle are small compared to all characteristics lengths, but nevertheless large compared to atomic dimensions, leading to its idealization as mathematical points [1]. Hence, the medium kinematics is then described by the translational degrees of freedom of the material particles and the consequent measures of deformation.

When it comes to the modeling of inhomogeneous materials based on the classical continuum, the constitutive equations are developed assuming the concept of a material particle associated to the idea of a representative volume element (RVE). The RVE is structurally typical of the whole mixture on average, being statistically representative of the infinitesimal material neighborhood of a material point [2–4].

Therefore, the kinematics and statics descriptions of the medium consider only average macroscopic characteristics, disregarding the microstructure constituents behavior. For the modeling of usual structures in the engineering field these hypotheses are sufficient. However, in situations wherein the RVE concept does not fully represent the substructure influence in the material behavior or the structural dimensions are small comparatively to the microstructure, theories that incorporate information on the material substructure are required.

In order to accommodate the material microstructure into the analysis, generalized continuum mechanics were developed through the expansion of the basic working hypotheses of standard continuum mechanics of Cauchy [5, 6]. Two classes of generalized continua may be distinguished based on the considered generalizations: higher order continua, for the case of additional degrees of freedom, and higher grade continua, regarding higher order gradients of the displacement fields [7–10].

In this context, this work presents equivalent strain measures for continuum damage models applied to the micromorphic continuum theory, which can be placed under the category of higher order continua. In this theory, each material point is assumed to be a microcontinuum whose kinematics defines the additional degrees of freedom.

Due to its formulation, this generalized continuum theory is able to capture size-effects and it is particularly suited to account for materials possessing a significant microstructure, e.g., quasi-brittle media.

In addition, to overcome the two main drawbacks of the micromorphic theory, the definition of additional constitutive equations and the determination of the high number of constitutive parameters, a multiscale formulation proposed by Silva [11] is here employed to obtain the macroscopic micromorphic constitutive relations using parameters defined for the classical theory.

The implementations were held in the software INSANE (INteractive Structural ANalysis Environment) and were based on a tensorial format of a unified constitutive models formulation.

2 Continuum Damage Models

The modeling of damage and fracture has been an important study topic in the field of computational mechanics as, after a certain load, the structure of a given material may begin to deteriorate with the formation of cracks weakening the solid and reducing its load carrying capacity. By nature, these defects are discrete entities and an accurate analysis of their influence would require considering these disturbances of the material continuum.

Based on the same idea used for the formulation of constitutive equations, which describe the deformational process modeling the solid as a continuum, Kachanov [12] introduced the basis for the continuum damage theories. In Continuum Damage Models (CDM) the medium is modeled at the macro scale as a continuum body and the collective effect of damage is described by field variables denominated damage variables. This hypothesis of a continuum body is based on the definition of a representative volume element (RVE), allowing the transition from microscopic to macroscopic variables.

Physically, the damage variable is “defined by the surface density of microcracks and intersections of microvoids lying on a plane cutting the RVE of cross section δS” [13]. For a scalar-isotropic damage model, this variable does not depend on the normal to this plane and the intrinsic variable is a scalar D.

The progressive material degradation may be represented by the deterioration of its elastic properties. In this case, for a uniaxial state, the original Young’s modulus is progressively degraded passing from an initial value E^0 to E^S that represents the modulus for the damaged material and evolves during the loading process. For a more general case, the process is represented by the degradation of the constitutive operator E^0_{ijkl},

$$E^S_{ijkl} = (1-D)E^0_{ijkl}. \tag{1}$$

3 Micromorphic continuum theory

The micromorphic continuum theory, as aforementioned, is a generalized continuum theory that incorporates additional degrees of freedom at each material point. As defined by Eringen [14], “a microcontinuum is a continuous collection of deformable point particles.” In order to represent the intrinsic deformation of a point, each deformable particle is replaced with a geometrical point P and some vectors attached to P that are related to the orientations and deformations of its material points. In addition, the vectors assigned to P also represent the additional degrees of freedom of each particle.

For such continuum, disregarding temperature variations, the free energy density ψ is approximated by

$$\begin{aligned}\psi \approx \psi^0 + \frac{1}{2}A_{klmn}\epsilon_{kl}\epsilon_{mn} + \frac{1}{2}B_{klmn}e_{kl}e_{mn} + \frac{1}{2}C_{klmnpq}\gamma_{klm}\gamma_{npq} + E_{klmn}\epsilon_{kl}e_{mn} + \\ + F_{klmnp}\epsilon_{kl}\gamma_{mnp} + G_{klmnp}e_{kl}\gamma_{mnp}\end{aligned} \tag{2}$$

where ψ^0 is the initial internal energy density; $U_0 = \psi - \psi_0$ is the strain energy density; A_{klmn}, B_{klmn}, C_{klmnpq}, E_{klmn}, F_{klmnp} and G_{klmnp} are the constitutive moduli; and ϵ_{kl}, e_{kl}, and γ_{klm} are the linear strain tensors.

From eq. 2 and applying symmetry regulations [14] the constitutive equations may be written as

$$t_{kl} = A_{klmn}\epsilon_{mn} + E_{klmn}e_{mn} + F_{klmnp}\gamma_{mnp} \tag{3}$$

$$s_{kl} = E_{mnkl}\epsilon_{mn} + B_{klmn}e_{mn} + G_{klmnp}\gamma_{mnp} \tag{4}$$

$$m_{klm} = F_{nplmk}\epsilon_{np} + G_{nplmk}e_{np} + C_{lmknpq}\gamma_{npq} \tag{5}$$

where t_{kl} is the stress tensor, s_{kl} is a symmetric stress tensor named micro-stress average [15], and m_{klm} is the stress moments tensor. The constitutive moduli may then be constructed by the product of the Kronecker delta δ_{kl}, i.e.,

$$\begin{aligned}
A_{klmn} &= \lambda\delta_{kl}\delta_{mn} + (\mu+\kappa)\delta_{km}\delta_{ln} + \mu\delta_{kn}\delta_{lm}, \\
E_{klmn} &= (\lambda+\nu)\delta_{kl}\delta_{mn} + (\mu+\sigma)(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) \\
F_{klmnp} &= 0, \\
B_{klmn} &= (\lambda+2\nu+\tau)\delta_{kl}\delta_{mn} + (\mu+2\sigma+\eta)(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}), \\
G_{klmnpq} &= 0, \\
C_{klmnpq} &= \tau_1(\delta_{kl}\delta_{mn}\delta_{pq} + \delta_{kq}\delta_{lm}\delta_{np}) + \tau_2(\delta_{kl}\delta_{mp}\delta_{nq} + \delta_{km}\delta_{lq}\delta_{np}) + \\
&\quad + \tau_3\delta_{kl}\delta_{mq}\delta_{np} + \tau_4\delta_{kn}\delta_{lm}\delta_{pq} + \tau_5(\delta_{km}\delta_{ln}\delta_{pq} + \delta_{kp}\delta_{lm}\delta_{nq}) + \\
&\quad + \tau_6\delta_{km}\delta_{lp}\delta_{nq} + \tau_7\delta_{kn}\delta_{lp}\delta_{mq} + \tau_8(\delta_{kp}\delta_{lq}\delta_{mn} + \delta_{kq}\delta_{ln}\delta_{mp}) + \\
&\quad + \tau_9\delta_{kn}\delta_{lq}\delta_{mp} + \tau_{10}\delta_{kp}\delta_{ln}\delta_{mq} + \tau_{11}\delta_{kq}\delta_{lp}\delta_{mn}
\end{aligned} \tag{6}$$

wherein λ, μ, κ, ν, τ, η and $\tau_1 \ldots \tau_{11}$ are 18 elastic parameters.

4 A unified formulation for elastic degradation in micromorphic continua

In this work, the modeling of the elastic degradation for micromorphic continua is based on the unified framework for constitutive modeling presented in Penna [16]. This framework is able to enclose a large amount of constitutive models (e.g., elasto-plastic, isotropic, orthotropic, and anisotropic elastic-degrading) based on multiple loading functions with the use of a tensorial format instead of vectorial-matricial one. This particularity increases the generality and the possibility of expansion of the code.

The theoretical basis for a unified formulation for constitutive models has been developed in the last years by a number of authors (see, e.g., [17–23]). The unified framework here presented [16] proposed an expansion based on the work of Carol et al. [18].

In a geometrically linear context, an elastic-degrading classical medium is characterized by total stress-strain relations

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \qquad \text{and} \qquad \varepsilon_{ij} = C_{ijkl}\sigma_{kl} \tag{7a,b}$$

where E_{ijkl} and C_{ijkl} are the components of the fourth-order stiffness and compliance tensors, inverse of each other (i.e., $C^{-1}_{ijkl} = E_{ijkl}$ and $E^{-1}_{ijkl} = C_{ijkl}$). The equations presented correspond to the assumption of an unloading-reloading process where the stiffness remains equal to the current secant one, i.e., a full unload leads to no permanent strains.

Similarly, a micromorphic elastic-degrading medium is characterized by three total stress-strain relations (eqs. 3 - 5) as, for the classical formulation, there is only a single total stress-strain relation (eq. 7). To approach this consistency problem, a compact tensorial formulation is proposed, in which the micromorphic total stress-strain expressions may be condensed in a single generalized secant relation, adopting the same formalism as Gori et al. [24]:

$$\Sigma_{\beta\nu} = \mathcal{E}^S_{\beta\nu\delta\psi}\Gamma_{\delta\psi}, \text{ for } \beta, \nu, \delta, \psi = 1, 2, ...9 \tag{8}$$

where the generalized stress operator $\Sigma_{\beta\nu}$ and the generalized strain operator $\Gamma_{\delta\psi}$ represent second-order tensors with dimension nine that group the stress and strain measures of the micromorphic continuum, respectively. The generalized secant operator $\mathcal{E}^S{}_{\beta\nu\delta\psi}$ gathers the four constitutive operators of the micromorphic theory for isotropic linear elastic solids, i.e., $A_{klmn}, B_{klmn}, C_{klmnpq}$, and E_{klmn}, in a fourth-order tensor with dimension nine.

Hence, the compatibility problem between both formulations is addressed enabling the extension of elastic-degrading models to the micromorphic theory within the same computational framework of classical models.

4.1 Scalar isotropic damage models

For scalar-isotropic damage models the degrading process is characterized by a single scalar damage. Extending this principle to micromorphic media and applying the generalized tensorial formulation presented, the resulting generalized constitutive operator can be expressed as

$$\mathcal{E}^{S}{}_{\beta\nu\delta\psi}(\mathcal{E}^{0}{}_{\beta\nu\delta\psi}, D) = (1 - D)\mathcal{E}^{0}{}_{\beta\nu\delta\psi} \tag{9}$$

where $\mathcal{E}^{0}{}_{\beta\nu\delta\psi}$ represents the initial elastic operator and D the damage variable, which varies from 0 for undamaged material to 1 for completely damaged material. The loading function can be written as

$$F(\Gamma_{eq}, D) = \Gamma_{eq}(\epsilon_{mn}, e_{mn}, \gamma_{npq}) - K(D) \tag{10}$$

with Γ_{eq} defined as the generalized equivalent strain and $K(D)$ is the history variable related to the equivalent strain written as a function of the damage.

Applying this general formulation, different damage models for the micromorphic continuum can be obtained when specific equivalent strain measures are defined. Considering the extension of the classical models proposed by Mazars and Lemaitre [25], Simo and Ju [26], Ju [27], and Marigo [28] the following equivalent strain measures were proposed:

$$\Gamma_{eq} = \begin{cases} \sqrt{\Gamma_{\delta\psi}\Gamma_{\delta\psi}} & \text{(Mazars-Lemaitre)} \\ \sqrt{2\psi^0} & \text{(Simo-Ju)} \\ \psi^0 & \text{(Ju)} \\ \sqrt{2\psi^0/E} & \text{(Marigo)} \end{cases} \tag{11}$$

with $\psi^0 = \frac{1}{2}\mathcal{E}^{0}{}_{\beta\nu\delta\psi}\Gamma_{\beta\nu}\Gamma_{\delta\psi}$ and E being the material Young's modulus.

5 Homogenization of a Classical continuum towards a micromorphic continuum

The analytical and discrete formulations of the micromorphic theory are well established in the literature, however the identification of the corresponding constitutive laws and the determination of the high number of constitutive parameters limit its practical application. As an alternative to circumvent these limitations, the micromorphic homogenization strategy proposed by Silva [11] and based on Hütter [29] is here employed, which consists in a multiscale formulation for the construction of macroscopic micromorphic constitutive relations in terms of homogenized microscopic quantities obtained from the solution of boundary value problems at the micro scale according to the classical continuum theory. This strategy begins with models of the classical continuum on the micro scale, without making any constitutive assumptions on the macroscale. Consequently, the necessary material parameters are those of the classical theory.

In this work, this formulation is applied so the initial elastic tensor $\mathcal{E}^{0}{}_{\beta\nu\delta\psi}$ is obtained only for the first step of the first iteration by subjecting the material particles to Cauchy stress states resulting from elementary states of strain, which consist of the successive application of component by component of macroscopic micromorphic strain with unit value, while the others components are kept as zero. From the Cauchy stress states, the components of macroscopic micromorphic stress are determined, which, as a result of elementary states of strain, consist of the terms of macroscopic micromorphic constitutive relations. For the subsequent iterations and steps, the initial constitutive relations are degraded through the investigation of the degraded state of the material based on the specified damage model. This strategy is illustrated in Figure 1.

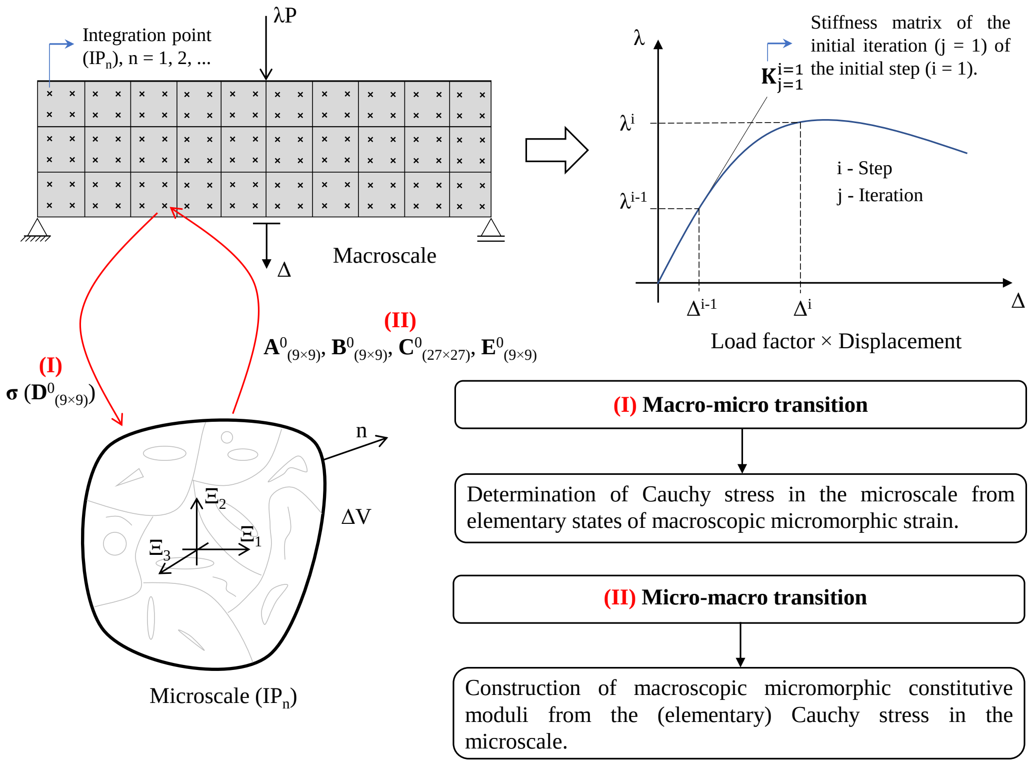
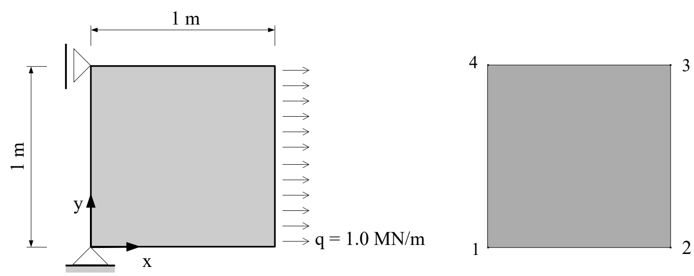


Figure 1. Micromorphic homogenization strategy

6 Numerical simulation

The implemented constitutive models presented are here illustrated considering the model in Figure 2: a square panel in a plane-stress state with unitary thickness composed of one plane element and loaded in the x direction.



(a) Geometry (b) Mesh

Figure 2. Uniaxial stress state

For obtaining the initial elastic tensor necessary for isotropic damage models, the homogenization strategy was applied with a square microcontinuum of dimension 0.05 m. The equivalent isotropic material is characterized by a Young's modulus of 20000 MPa and a Poisson's ratio of 0.2. The appropriated parameters for each constitutive model were adopted considering an exponential damage law [16].Hence: **Mazars-Lemaitre micromorphic model**: $\alpha = 0.999$, $\beta = 2000.0$, $\kappa_0 = 0.000104$; **Simo-Ju micromorphic model**: $\alpha = 0.999$, $\beta = 15.0$, $\kappa_0 = 0.0145$; **Ju micromorphic model**: $\alpha = 0.999$, $\beta = 15.0$, $\kappa_0 = 0.00011$; **Marigo micromorphic model**: $\alpha = 0.999$, $\beta = 2000.0$, $\kappa_0 = 0.000104$.

The parameters presented for each model vary as the conceptual differences between the models preclude the exact correspondence between the parameters even for the same damage law. The loading process is driven by the displacement control method assuming an increment of 5×10^{-6} m for the horizontal displacement of the loaded face in order to better describe the peak load behavior, and a tolerance for the convergence of 10^{-4} in load. The results for the analysis are presented in Figure 3 wherein the relation between the horizontal displacement for node

2 (Figure 2(b)) and the load factor is given. Consistent results for all the models are obtained, attesting that the implemented models for the micromorphic continuum are working properly.

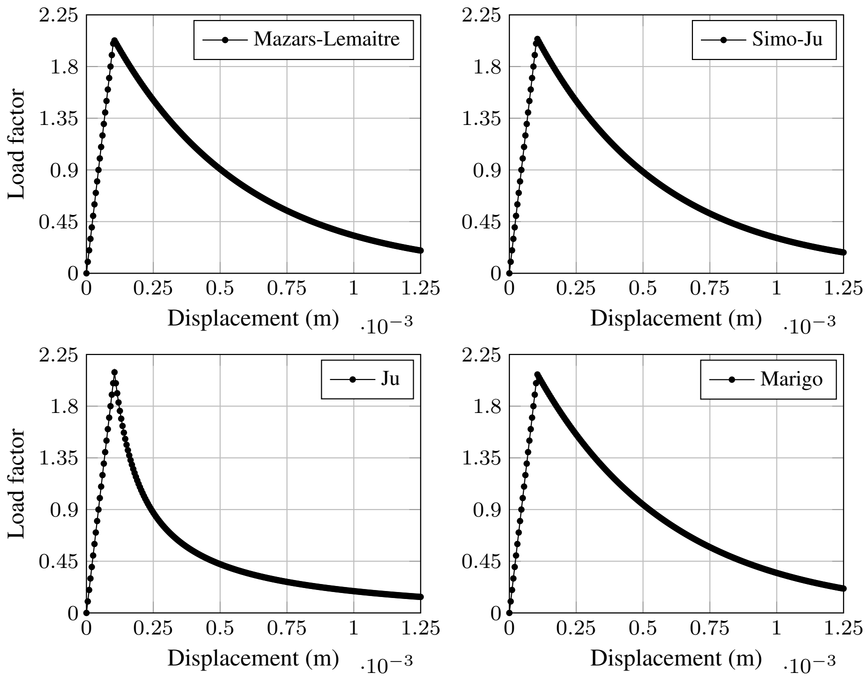


Figure 3. Uniaxial stress state: load factor versus horizontal displacement

7 Conclusions

The aim of the present work was to provide a basis for the modeling of damage by means of continuum damage model, more specifically scalar-isotropic model, with use of the micromorphic continuum theory in view of its ability to incorporate the microstructural behavior in the continuum formulation. Due to its formulation, this theory may address the strain localization phenomenon, characteristic of quasi-brittle media.

The computational implementation was held in the INSANE system within a unified constitutive framework first proposed for classical media. To solve the compatibility problem, a compact tensorial formulation was proposed allowing the inclusion of damage models for the micromorphic theory with minimum intervention in the code. Observing the results here presented, it is possible to assume that this formulation is viable as it yields consistent results. Furthermore, applying the homogenization technique to obtain the initial elastic tensor made possible the reduction of the number of elastic parameters necessary for the analysis (E and ν), solving one of the greatest disadvantages of this generalized continuum theory.

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References

[1] A. K. Mal and S. J. Singh. *Deformation of elastic solids*. Prentice Hall, Inc., Englewood Cliffs, NJ, USA, 1991.
[2] R. Hill. Elastics properties of reinforced solids: Some theoretical principles. *J. Mech. Phys. Solids*, vol. 11, pp. 357–372, 1963.

[3] R. Hill. The essential structure of constitutive laws for metal composites and polycrystals. *J. Mech. Phys. Solids*, vol. 15, pp. 79–95, 1967.
[4] S. Nemat-Nasser and M. Hori. *Micromechanics: Overall Properties of Heterogeneous Materials*. Elsevier Science Publishers, 1993.
[5] W. Voigt. Theoretische studien über die elasticitätsverhältnisse der krystalle. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften*, vol. 34, pp. 3–52, 1887.
[6] E. Cosserat and F. Cosserat. *Théorie des corps déformables*. Librairie Scientifique A. Hermann et Fils, Paris, 1909.
[7] P. Germain. La méthode des puissances virtuelles en mécanique des millieux continus premiere partie: théorie du second gradient. *Journal de Mécanique*, vol. 12, n. 2, pp. 235–274, 1973a.
[8] P. Germain. The method of virtual power in continuum mechanics. part 2: Microstructure. *SIAM Journal of Applied Mathematics*, vol. 25, n. 3, pp. 556–575, 1973b.
[9] S. Forest. Mechanics of generalized continua: construction by homogenization. *Journal de Physique IV*, vol. 8, pp. Pr4–39, Pr4–48, 1998.
[10] C. B. Hirschberger. *A Treatise on Micromorphic Continua. Theory, Homogenization, Computation*. PhD thesis, Technische Universität Kaiserslautern, Kaiserslautern, Germany, 2008.
[11] L. L. Silva. *Estratégia Multiescala para Descrição Micromórfica do Contínuo a partir do Contínuo Clássico*. PhD thesis, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brasil, 2019.
[12] L. Kachanov. On time to rupture in creep conditions. *Izvestia Akademii Nauk SSSR, Otdelenie Tekhnicheskikh Nauk*, , n. 8, pp. 26–31. in russian, 1958.
[13] J. Lemaitre and R. Desmorat. *Engineering Damage Mechanics: Ductile, Creep, Fatigue and Brittle Failures*. Springer, 2005.
[14] A. C. Eringen. *Microcontinuum Field Theories: I. Foundations and solids*. Springer, New York, 1999.
[15] A. C. Eringen and E. S. Şuhubi. Nonlinear theory of simple micro-elastic solids – i. *International Journal of Engineering Science*, vol. 2, n. 2, pp. 189–203, 1964.
[16] S. S. Penna. *Formulação Multipotencial para Modelos de Degradação Elástica - Unificação Teórica, Proposta de Novo Modelo, Implementação Computacional e Modelagem de Estruturas de Concreto*. PhD thesis, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brasil, 2011.
[17] de R. Borst. Smeared cracking, plasticity, creep, and thermal loading–a unified approach. *Computer Methods in Applied Mechanics and Engineering*, vol. 62, pp. 89–110, 1987.
[18] I. Carol, E. Rizzi, and K. Willam. A unified theory of elastic degradation and damage based on a loading surface. *International Journal of Solids and Structures*, vol. 31, n. 20, pp. 2835–2865, 1994.
[19] de R. Borst and M. A. Gutiérrez. A unified framework for concrete damage and fracture models including size effects. *International Journal of Fracture*, vol. 95, pp. 261–277, 1999.
[20] F. Armero and S. Oller. A general framework for continuum damage models. I. infinitesimal plastic damage models in stress space. *International Journal of Solids and Structures*, vol. 37, n. 48-50, pp. 7409–7436, 2000a.
[21] F. Armero and S. Oller. A general framework for continuum damage models. II. integration algorithms, with applications to numerical simulation of porous metals. *International Journal of Solids and Structures*, vol. 37, n. 48-50, pp. 7437–7464, 2000b.
[22] I. Carol, E. Rizzi, and K. Willam. On the formulation of anisotropic elastic degradation. I. theory based on a pseudo-logarithmic damage tensor rate. *International Journal of Solids and Structures*, vol. 38, n. 4, pp. 491–518, 2001a.
[23] I. Carol, E. Rizzi, and K. Willam. On the formulation of anisotropic elastic degradation. II. generalized pseudo-rankine model for tensile damage. *International Journal of Solids and Structures*, vol. 38, n. 4, pp. 519–546, 2001b.
[24] L. Gori, S. S. Penna, and R. L. Pitangueira. An enhanced tensorial formulation for elastic degradation in micropolar continua. *Applied Mathematical Modelling*, vol. 41, pp. 299–315, 2017.
[25] J. Mazars and J. Lemaitre. *Application of Continuous Damage Mechanics to Strain and Fracture Behavior of Concrete*, pp. 507–520. Springer Netherlands, Dordrecht, 1985.
[26] J. Simo and J. Ju. Strain- and stress-based continuum damage models–i. formulation. *International Journal of Solids and Structures*, vol. 23, n. 7, pp. 821 – 840, 1987.
[27] J. Ju. On energy-based coupled elastoplastic damage theories: Constitutive modeling and computational aspects. *International Journal of Solids and Structures*, vol. 25, n. 7, pp. 803–833, 1989.
[28] J. Marigo. Modelling of brittle and fatigue damage for elastic material by growth of microvoids. *Engineering Fracture Mechanics*, vol. 21, n. 4, pp. 861 – 874, 1985.
[29] G. Hütter. Homogenization of a Cauchy continuum towards a micromorphic continuum. *Journal of the Mechanics and Physics of Solids*, vol. 99, pp. 394–408, 2017.