MODELING BOUNDARY ELEMENT METHOD FOR THE ANALYSIS OF BEAMS COMPOSED OF DIFFERENT MATERIALS

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Abstract. In structural engineering more and more elements characterized as composites are being used. A previous understanding characterizes materials consisting of layers, in general, even varying the nature of the materials. The present paper presents an approach for analyzing the structural behavior of a beam formed of different materials from 2D modeling in numerical method. A computational implementation of the Boundary Element Method (BEM) was developed using the PYTHON programming language. The code "MEC 2D" was elaborated using Kelvin's fundamental solution with the use of continuous and discontinuous linear elements for the discretization of the boundary and interfaces of the subregions characterizing each material. The BEM formulation allows, therefore, the modeling of subregions for the assembly of the domain, in this case, the complete beam. The behavior is evaluated from the displacements obtained for the boundary and internal points, as well as the stresses, evaluating these fields conveniently in graph form. Applications were performed to test the implemented modeling.

Keywords: Beams of Different Materials, Boundary Element Method, 2D Analysis.

1 Introduction

Material properties can be determined by analytical, experimental or computational methods. However, analytical methods are limited by materials with simplified microstructure and low heterogeneity density (Milton [1]). Thus, in these problems, experimental or computational methods are used to obtain approximate solutions. Among the most popular numerical methods of analysis are the Finite Element Method (FEM), Finite Difference Method (FDM) and the Boundary Element Method (BEM), Fedelinski et al. [2].

The Boundary Element Method (BEM) was presented as an alternative to the domain discretization methods. Its formulation allows the problem to be solved by discretizing only the boundary of the region studied, Brebbia [3]. By using the BEM, the differential equations that govern the physical problem are transformed into boundary integral equations, therefore, taken to the surface of the problem. These integrals are solved at specific points of the elements discretized on the boundary. To this end, the BEM uses a weight function, known as the fundamental solution, which must meet the differential equation of the problem, but with very particular boundary conditions that make it possible to obtain it. The traditional formulation of the BEM allows the discretization of the domain in sub-regions that can be, therefore, composed of materials with different physical characteristics (Oliveira [4], Almeida [5]). Other techniques were studied in Eischen and Torquato [6], Liu [7] and Gurrum et al. [8].

In this context, the present work has the objective of developing a computational program in PYTHON language for the analysis of beams composed of different materials, by means of the Boundary Element Method, in its two-dimensional formulation for elastic-linear materials and with boundary discretization by discontinuous linear elements. The subdomain technique was implemented for the BEM-BEM coupling. The results obtained are compared with the results published in the specialized literature.

2 Boundary Element Method Formulation

2.1 The general problem

It is intended to investigate the behavior of a two-dimensional, homogeneous and isotropic elastic-linear solid whose domain is Ω and its respective boundary Γ. The equilibrium of the body characterized in the elastic problem can be analyzed from eq. (1), considering *j = 1,2*, the tensor of stresses σ_{ji} and body forces b_i.

$$\sigma_{ji,j} + b_i = 0. \tag{1}$$

Let us consider the boundary conditions given in eq. (2a, b), defined as a function of displacements (u) and surface tractions (p). Combined with eq. (1), one can analyze the elastic behavior of the body.

$$u = \overline{u}, \text{ em } \Gamma_1 \ , \qquad p = \sigma \cdot n = \overline{p}, \text{ em } \Gamma_2 \ . \tag{2a, b}$$

2.2 A numerical solution of the general problem

In this work, it was chosen to use the Boundary Element Method to analyze the elastic problem enunciated in the previous item. With some algebraic operations performed on eq. (1), we arrive at the Somigliana Identity, integral equation that governs the problem, eq. (3), which allows the analysis of the body using the variables distributed in nodes related to its boundary.

$$u_{lk}^i + \int_\Gamma p_{lk}^* \cdot u_k d\Gamma = \int_\Gamma u_{lk}^* \cdot p_k d\Gamma + \int_\Omega b_k \cdot u_{lk}^* d\Omega. \tag{3}$$

Using concepts of minimizing errors from weightings in eq. (3), adapting to the BEM that employs in the chosen weighting function the so-called fundamental solutions, of a fundamental problem, whose boundary tends to infinity. In this work, Kelvin's fundamental solution for two-dimensional elastic problems will be used, according to expressions (4a and b).

$$u_{lk}^* = \frac{1}{8\pi\mu(1-\nu)}\left((3-4\nu)\, ln\left(\frac{1}{r}\right)\delta_{lk} + r_{,l}r_{,k}\right), \qquad p_{lk}^* = -\frac{1}{4\pi(1-\nu)r}\left(\frac{\partial r}{\partial n}\left((1-2\nu)\delta_{lk} + 2r_{,l}r_{,k}\right) + (1-2\nu)\left(n_l r_{,k} - n_k r_{,l}\right)\right), \tag{4a, b}$$

where μ is the transverse modulus of elasticity and n is the vector normal to the boundary.

The study of equation (3) indicates that its application can be identified in the domain (containing the boundary) and outside it. Thus, in generalized form for any points, one arrives at eq. (5),

$$c_{lk}^i u_{lk}^i + \int_\Gamma p_{lk}^* \cdot u_k d\Gamma = \int_\Gamma u_{lk}^* \cdot p_k d\Gamma + \int_\Omega b_k \cdot u_{lk}^* d\Omega\,, \tag{5}$$

where c^i_{lk} is a constant that depends on the position of the source point i, and has value 1 for domain points, ½ for boundary points when smooth, and 0 for external points.

For the application of the numerical method, it is necessary to discretize the problem, determining the points (nodes) in which the equations of the solution found will be written. In the case of the boundary, being the problem two-dimensional, one-dimensional elements are used for its approximate conformation, appearing, therefore, the geometric end points that can be used as functional nodes. Therefore, equation (5) is discretized, assuming the non-existence of body forces (b_k), obtaining terms only at the boundary and whose variables are nodal surface displacements and forces. This new equation can be rewritten matrix-wise in the form of eq. (6), using the interpolating functions of the chosen elements and the nodal values of displacements and surface tractions.

$$[H]\{U\} = [G]\{P\}. \tag{6}$$

Apply the boundary conditions to eq. (6), with the appropriate column and row exchanges, and a system of linear equations is obtained, according to eq. (7), whose unknowns can be either displacements or surface forces. The order of the system reaches 4n x 4n (n representing the number of elements), if a discontinuous linear element is used, with two functional nodes for each element.

$$[A]\{X\} = \{B\} \tag{7}$$

2.3 Coupling BEM-BEM

In cases where the domain is composed of several subdomains of materials that present different physical characteristics, the use of subregions provides a higher precision to the analysis (Wutzow [9]). For this purpose, the boundary discretization of each homogeneous subdomain is performed and the equation of the connections between them is imposed by the equilibrium of forces and the compatibility of displacements at all interface points between the subregions, as presented in eq. (8) and eq. (9).

$$\{P\}^{1i} + \{P\}^{2i} = 0\,, \tag{8}$$

$$\{U\}^{1i} = \{U\}^{2i}. \tag{9}$$

The [G] and [H] matrices, representative of each domain independently, are generated, eq. (10).

$$\begin{bmatrix} [\mathrm{H}]_{11} & [\mathrm{H}]_{1i} \\ [\mathrm{H}]_{i1} & [\mathrm{H}]_{ii} \end{bmatrix} \begin{Bmatrix} \{u\}^1 \\ \{u\}^{1i} \end{Bmatrix} = \begin{bmatrix} [\mathrm{G}]_{11} & [\mathrm{G}]_{1i} \\ [\mathrm{G}]_{i1} & [\mathrm{G}]_{ii} \end{bmatrix} \begin{Bmatrix} \{P\}^1 \\ \{P\}^{1i} \end{Bmatrix}. \tag{10}$$

By joining the two systems into one and applying the equilibrium conditions, compatibility and boundary conditions, a general system for the two analyzed subregions is obtained, as shown in eq. (11).

$$\begin{bmatrix} [\mathrm{H}]_{11}^1 & [\mathrm{H}]_{1\mathrm{i}}^1 & -[G]_{1j}^1 & [0] \\ [\mathrm{H}]_{\mathrm{i}1}^1 & [\mathrm{H}]_{\mathrm{ii}}^1 & -[G]_{ii}^1 & [0] \\ [0] & [H]_{ii}^2 & [\mathrm{G}]_{\mathrm{ii}}^2 & [\mathrm{H}]_{\mathrm{i}2}^2 \\ [0] & [H]_{2i}^2 & [\mathrm{G}]_{2\mathrm{i}}^2 & [\mathrm{H}]_{22}^2 \end{bmatrix} \begin{Bmatrix} \{u\}^1 \\ \{u\}^{1i} \\ \{P\}^{1i} \\ \{u\}^2 \end{Bmatrix} = \begin{bmatrix} [G]_{11}^1 & [0] \\ [G]_{i1}^1 & [0] \\ [0] & [G]_{i2}^2 \\ [0] & [G]_{22}^2 \end{bmatrix} \begin{Bmatrix} \{P\}^1 \\ \{P\}^2 \end{Bmatrix}. \tag{11}$$

The variables relating traction and displacement at the interfaces are assumed to be unknowns for eq. (11). Therefore, rewriting the system in a more simplified form, a linear system of equations is obtained, eq. (7), in which {X} represents the unknowns vector.

$$[A]\{X\} = \{B\} \qquad \text{(7) repeated}$$

The above procedure can be performed for every two subdomains, always considering as unknowns the variables of tractions and displacements at the interfaces, for as many subdomains as it is desired to discretize the body.

3 Application

Based on the formulation presented in the previous items, a computational program, written in free software language PYTHON, was developed to implement the discontinuous linear element, with the interactions of displacements and surface forces for the coupling of subdomains via the subregion's technique. The "MEC 2D" code, therefore, requires a simple computer, a microcomputer, and performs processing at a higher speed than platforms whose codes are interpreted.

To evaluate the use of the Boundary Element Method with the subdomain coupling technique (BEM-BEM) to analyze the behavior of beams made of different materials, the problem addressed in Muttashar et al. [10] was used. These are beams composed of overlapped (bonded) hollow profiles, here used the cases with two cells. The square profile in vinyl polyester reinforced with fiberglass, has external dimensions 125mm, with 6.5mm thickness and longitudinal elasticity modulus equal to 47.2GPa. The models were treated in three approaches: with only the vinyl profiles (hollow cross sections); filling of the compressed part with 15MPa and 32MPa concrete. Figure 1 (a) shows the beam design as taken from Muttashar et al. [10]. The concentrated forces (two), were applied in the central region of the beam, with a variation from zero to the failure value (values presented in Table 1).

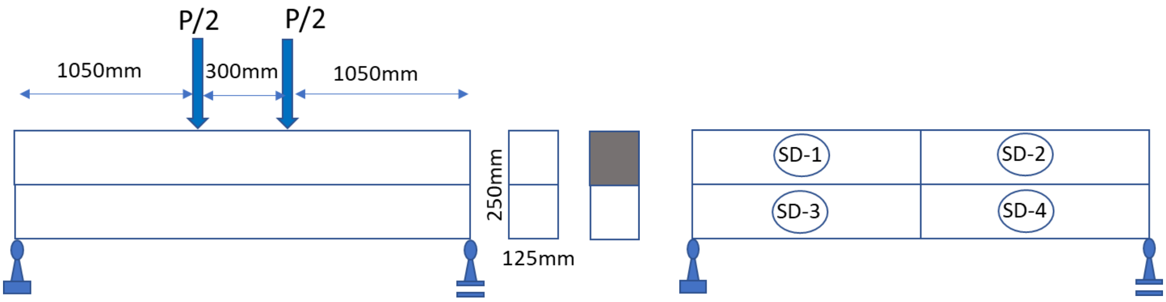

Figure 1. Beam analysis following Muttashar et al. [10]: a) definition of the beam in two bonded parts with its geometric coordinates and loads; b) definition of the subdomains to be coupled BEM-BEM

Figure 1 (b) shows the configuration in sub-regions (subdomains) for the changes in physical parameters required for each material analyzed. The discretization is done in each SD-i using linear elements, observing the interface nodes for proper coupling.

After processing using the computer program developed in PYTHON language for 2D analysis via BEM, result values were obtained very close to those referred in Muttashar et al. [10], both for displacements, Table 1, and moments, Table 2. Figure 2 shows the graphs of vertical displacements by forces, obtained for the process in the BEM and given by the reference.

Table 1. Maximum vertical displacement values for the tested/processed models: comparison between the reference (Muttashar et al. [10]) and the BEM

Model	P_p	Maximum Displac. (m)		$P_{máx}$	Maximum Displac. (m)	
	(kN)	Ref.	BEM	(kN)	Ref.	BEM
2H_C_0	100	-0.0022950	-0.021467	151.2	-0.0347	-0.032458
2H_C_15	100	-0.019677	-0.019455	217	-0.0427	-0.042217
2H_C_32	100	-0.019386	-0.018732	247.6	-0.048	-0.046380

Table 2. Maximum moment values for the tested/processed models: comparison between the reference (Muttashar et al. [10]) and the BEM

Model	P_p	Max. Bending Mom. (kN.m)		$P_{máx}$	Max. Bending Mom. (kN.m)	
	(kN)	Ref.	MEC	(kN)	Ref.	MEC
2H_C_0	100	52.51	53.33	151.2	79.4	80.63
2H_C_15	100	52.53	53.48	217	114	116.05
2H_C_32	100	52.50	53.69	247.6	130	132.94

Figure 2 shows the displacement values obtained from the failure loads indicated in the study of Muttashar et al. [10], by the use of and cited in that work. It is verified the adequacy of the BEM processing to obtain the responses obtained in Muttashar et al. [10], with small errors (below 4% on average), observing the values in Table 1.

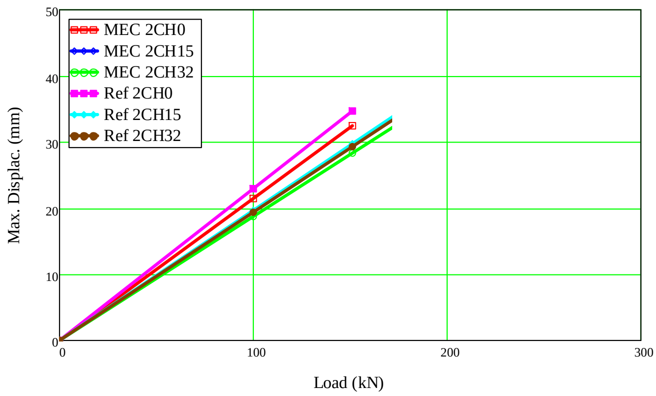

Figure 2. Graph of the vertical displacements by the load applied to the tested/processed models: comparison between the reference (Muttashar et al. [10]) and the BEM.

4 Conclusions

The original formulation of the Boundary Element Method leads to simple algorithms for adequate computational implementation. The programming language PYTHON, free software, proved to be adequate for the computational implementation, constituting a fast code for use on microcomputers. The BEM-BEM coupling technique for the subdomains with their respective physical parameters proved to be adequate to allow analysis of structures composed of parts of different materials.

The 2D BEM implementation was used to analyze beams made of two materials and presented results in strains and stresses with adequate accuracy, compared to results from published works.

The use of the BEM to more common engineering problems is adequate to expand the knowledge for the analysis of stresses and strains in structures for undergraduate engineering courses. It combines the stronger inclusion of numerical methods with the traditional study of the mechanics of bodies deformable.

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