Models for Representation of Cracks in One-dimensional Finite Element Method

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Abstract. Structures can develop discontinuities in their material in the form of cracks or generalized damage, changing their properties and resulting in malfunction or even collapse. Thus, to identify and monitor the existence of cracks, some numerical models have been proposed in recent decades, where different approaches regarding the representation of the crack are treated. Considering this scenario, the objective of this work is to study different ways of representing cracks in numerical models, based on the one-dimensional Finite Element Method (FEM). For crack representation in numerical models, 3 methods are highlighted in the literature, namely the reduced section method, the local flexibility method, and the continuous method. Thus, this work contemplates the numerical implementation of the mentioned methods, focusing on the analysis of free vibration of Euler-Bernoulli beams. The natural frequencies obtained by these 1D methods are compared to those obtained by other researchers using different models (Timoshenko, 2D, etc.). Different crack configurations and boundary conditions are tested numerically and compared to reference values presented in the literature, obtaining good results. In a comparative analysis, the models presented divergence in the answers for deeper cracks, evidencing a significant difference between these approaches.

Keywords: Cracked beam elements; Finite element method; Structural Health.

1 Introduction

The effect of cracks on the dynamic behavior of structures has been a subject of interest by many researchers. Numerous crack models for a cracked beam can be found in the literature, where different approaches are applied to the numerical treatment of cracks. To deal with this problem, three methods can be highlighted: an equivalent reduced section method, a local flexibility method and a continuous cracked beam method.

The equivalent reduced section method was first introduced by Kirmsher [1] and Thomson [2]. They simulated the effect of cracks using a local bending moment and a reduced section on crack region. It was the first and one of the simplest attempts to quantify the stiffness reduction in a crack boundary.

The local flexibility method considered a cracked beam as two beam segments with the same properties connected by a rotational spring at the cracked section. Spring stiffness is determined to adequately represent the local flexibility generated by the crack. One of the first papers with this approach was presented by Irwin [3]. The author realized that the phenomenon of local flexibilization in a structural element alters the dynamic responses of the system, changing its frequency and vibration modes. In its early stages, the study of crack models where mostly based on analytical formulations [4,5,6]. With the computational improvement and evolution of numerical techniques such as the Finite Element Method (FEM), crack models began to be implemented in discrete approaches. Eroglu and Tufekci [7] presented a new formulation for finite elements that includes rotational inertia and shear strain applied in the analysis of beams with cracks, implementing local flexibilization as a rotational spring similar to the one proposed by Rizos et al. [4], and experimentally validating results through free vibration tests. Bringing the application of the local flexibility method to the approach proposed by the one-dimensional Finite Element Method, Mehrjoo et al. [8] proposed a hybrid Euler-Bernoulli beam element where the rotational spring is not physically modeled in the coupling between elements but included in the element stiffness matrix. Subsequently, Mehrjoo et al. [9] extended the hybrid stiffness matrix formulation of their previous work for a Timoshenko beam. In both works, a comparative analysis was performed with analytical results from the literature and 2D FEM models.

The continuous model method is an approach where the crack is represented in terms of the stiffness variation that occurs in the vicinity of the crack, not restricting the flexibilization in just one section, but in a certain stretch close to the crack. Christides and Barr [10] noticed that the existence of cracks led to substantial changes in the stress distribution in and near of the cracked section. This distribution is not linear, being described by a function with maximum value at the tip of the crack and exponential decay with the distance of the crack section. This variation of the stress distribution can be considered numerically through the variation of the stiffness *EI* and implemented in different approximate methods. Lu and Law [11] applied the exponential stiffness decay formulation proposed by [10] in a model based on the Composite Element Method (CEM), obtaining good results compared to experimental data and other models present in the literature. With the same approach, Sinha et al. [12] implemented a FEM model where they propose a modified stiffness matrix with linear decay for cracked elements coupled to other intact elements.

The above methods provide different results for the same crack configuration. This paper proposes a comparative study where each of those numerical techniques are tested with different crack configurations. Each crack approach is implemented to one-dimensional Finite Element Method, focusing the analysis on free vibration of Euler-Bernoulli beams.

2 Theory

Figure 1 shows a simple beam with one transverse open crack. The crack is assumed to have uniform depth and do not change the mass of the beam. Before the incorporation into the Finite Element Method, the three crack models are introduced, here called model 1, model 2 and model 3.

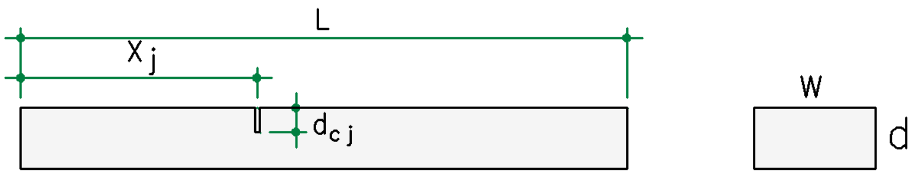


Figure 1. Beam with a single crack

2.1 Crack model 1

Fracture mechanics concepts show that the sections in the vicinity of cracks have alterations in the load stresses, contributing only partially to the local rigidity of the structure. This participation starts from a minimum stiffness in the exact position of the crack to a maximum value referring to a totally intact structure. The formulation of model 1 is based on the exponential stiffness decay (EI) proposed by Christides and Barr [10] and later adapted to a linear decay model proposed by Sinha et al. [12].

A local coordinate system is created to identify the function that represents the linear decay of stiffness as shown in Figure 2, where the variation of the product EI starts at a point located at a distance $l_c = d/\alpha$ on both sides of the crack position. In this work, the same value of α suggested by [10] was used, where α=0.667. It is important to emphasize that the Figure 2 does not represent an equivalent variation in the height of the beam, but rather a schematic variation of the second order moment of the beam cross section.

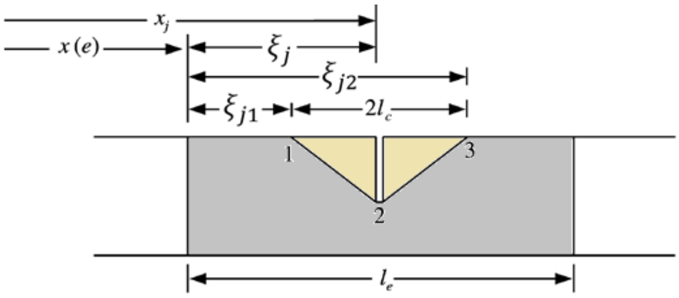


Figure 2. Beam element with linear rigidity variation

Considering the elementary local coordinate ξ, the function that governs the linear variation of stiffness can be given by:

$$EI_e(\xi) = \begin{cases} EI_0 - E\left(I_0 - I_{cj}\right)\frac{(\xi-\xi_{j1})}{(\xi_j-\xi_{j1})} \;\; se \;\; \xi_{j1} \le \xi \le \xi_j \\ EI_0 - E\left(I_0 - I_{cj}\right)\frac{(\xi_{j2}-\xi)}{(\xi_{j2}-\xi_j)} \;\; se \;\; \xi_j \le \xi \le \xi_{j2} \end{cases} \tag{1}$$

where ξ_j is the elementary local coordinate of the crack, and $\xi_{j1} = \xi_j - l_c$ and $\xi_{j2} = \xi_j + l_c$ represent the position where the stiffness variation starts and ends on both sides of the crack. Once the numerical treatment of the crack was determined, it was implemented in a FEM model based on a Euler-Bernoulli beam formulation. The cracked element stiffness matrix is given by:

$$\begin{aligned}\left[k_{e,ij}\right] = \int_0^{(\xi_j-l_c)} EI_0\left(\frac{\partial^2\psi_i^e}{\partial\xi^2}\right)\left(\frac{\partial^2\psi_j^e}{\partial\xi^2}\right)d\xi + \int_{(\xi_j-l_c)}^{\xi_j} EI_e(\xi)\left(\frac{\partial^2\psi_i^e}{\partial\xi^2}\right)\left(\frac{\partial^2\psi_j^e}{\partial\xi^2}\right)d\xi \, + \\ \int_{\xi_j}^{(\xi_j+l_c)} EI_e(\xi)\left(\frac{\partial^2\psi_i^e}{\partial\xi^2}\right)\left(\frac{\partial^2\psi_j^e}{\partial\xi^2}\right)d\xi + \int_{(\xi_j+l_c)}^{l_e} EI_0\left(\frac{\partial^2\psi_i^e}{\partial\xi^2}\right)\left(\frac{\partial^2\psi_j^e}{\partial\xi^2}\right)d\xi\end{aligned} \tag{2}$$

where ψ_i^e and ψ_j^e are the shape functions given by the Hermite polynomials as a function of the elementary local coordinate ξ with domain $\Omega = [0, l_e]$.

2.2 Crack model 2

The second model is based on the local flexibility method, where the flexibility generated by the crack is treated as a rotational spring, generating discontinuity between the elements to the right and left of the crack. In this work, the spring was implemented by decoupling the rotational degree of freedom of the elements adjacent to the crack. The equivalent stiffness K_t of the rotational spring was implemented according to the model used by [4], given by:

$$K_t = \frac{1}{\frac{5{,}346\,.d.f\left(\frac{d_{cj}}{d}\right)}{E.I_O}} \tag{3}$$

being d the height of the integral beam, E the modulus of elasticity, I_o the second order moment of the integral beam and $f(d_{cj}/d)$ a polynomial function of local conformity that varies according to the relationship between crack depth and beam height, described as:

$$f\left(\frac{d_{cj}}{d}\right) = 1.8624\left(\frac{d_{cj}}{d}\right)^2 - 3.95\left(\frac{d_{cj}}{d}\right)^3 + 16.375\left(\frac{d_{cj}}{d}\right)^4 - 37.226\left(\frac{d_{cj}}{d}\right)^5 + 76.81\left(\frac{d_{cj}}{d}\right)^6 - 126.9\left(\frac{d_{cj}}{d}\right)^7 + 172\left(\frac{d_{cj}}{d}\right)^8 - 143.97\left(\frac{d_{cj}}{d}\right)^9 + 66.56\left(\frac{d_{cj}}{d}\right)^{10} \tag{4}$$

2.3 Crack model 3

This model is based on the equivalent reduced section method. The local flexibilization generated by the crack is implemented with the reduction of the moment of inertia of an entire element. For this approach to be valid, it is necessary to work with a properly refined mesh in the crack vicinities (Figure 3), to generate a flexibility compatible with the one generated by the real crack.

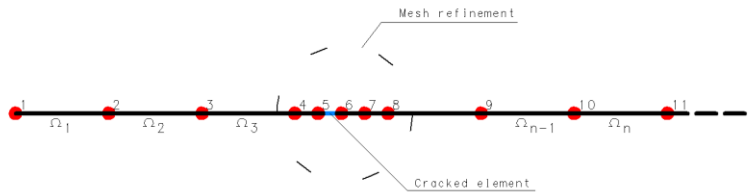


Figure 3. Mesh refinement for model 3

3 Applications

3.1 Comparison with other existing models

The presented models are compared with those presented by [7], [8] and [9]. The studied example consists of a beam with span L = 4 m, width w= 0,1 m and height d=0,2 m. The material considered is steel, with a modulus of elasticity of 200 GPa, with an 80mm deep crack positioned at 1,5 m from its left support. The cited authors calculated the natural frequencies for two support conditions, being clamped-free (cantilever beam) and simply supported. 4 elements are used in the FEM model. The results for the first three natural frequencies are presented in Table 1.

The largest percentage differences occur for higher modes where the effect of shear deformation and rotational inertia are more relevant, as observed in the comparison with the 2D MEF model and the one proposed by Eroglu and Tufekci [7]. For the cantilever beam, the greatest difference occurred between Model 1 and MEF 2D model at the 3rd natural frequency, with an error of 6,04%, followed by the difference between Model 1 and Eroglu and Tufekci [7], also at the 3rd frequency, with an error of 5,35%. For the simply supported beam, the greatest difference occurred between Model 3 and MEF 2D model at the 3rd natural frequency, with an error of 6,63%, followed by the difference between Model 1 and 2D MEF, also at the 3rd frequency, with an error of 6,43%. For the first frequency, the results showed, in general, good convergence.

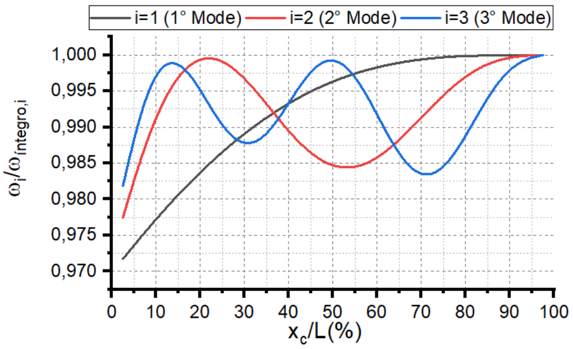
Table 1. Comparative analyses of natural frequencies (Hz)

Boundary Conditions	Natural Frequencies	Euler-Bernoulli model [8]	2D Element [8]	Timoshenko beam [9]	FEM [7]	Model 1	Model 2	Model 3
Cantilever beam	1st	9,78	9,71	9,77	9,79	9,85	9,88	9,77
	2nd	61,58	60,08	61,08	60,47	61,94	61,7	61,53
	3rd	174,97	165,67	172,07	166,75	175,67	172,83	171,17
Simply supported	1st	26,75	26,29	26,69	26,66	27,063	27,06	26,76
	2nd	111,64	108,11	110,65	108,89	112,06	111,26	110,17
	3rd	261,19	244,81	256,34	246,36	260,55	256,25	261,05

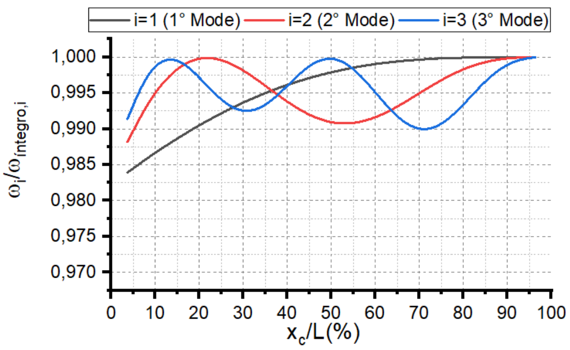
3.2 Parametric Analysis

To investigate the behavior of numerical models against different situations such as, crack depth and position, a parametric analysis was performed focused on identifying the first natural frequencies of the structure and the effect of each parameter on the dynamic response. To analyze the response of each cracked beam model, a cantilever beam was considered with 0,82 m length, 0,02 m width and 0,01 m height. The material is assumed to be aluminum with an elastic modulus of 70 GPa and mass density of 2700 kg/m^3. The investigation of the effect of crack position is carried out, keeping its depth fixed with a relation $d_{cj}/d = 0{,}3$,where x_j e d_{cj} represent the position and depth of the crack, respectively. Figure 4 shows the variation of the natural frequency for Models 1, 2 and 3. Depending on the analyzed mode, there is a different susceptibility to variation of the natural frequency for cracks in the same position.

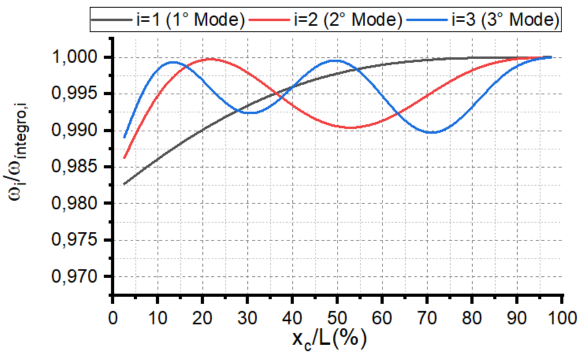
For the first mode, there is a greater frequency variation for cracks close to the clamped support, where the bending moment generated by the oscillatory process is higher. For modes 2 and 3, the same behavior is not observed due to the presence of modal nodes, points where the transverse displacement is null. For cracks positioned on modal nodes, no variation in the natural frequency is observed. Among the 3 flexibilization models, it is noted that Model 1 results in higher percentage variations in the natural frequency in relation to the crack position for the adopted depth.



(a)



(b)



(c)

Figure 4. Natural frequency variation for different crack positions – (a) Model 1; (b) Model 2; (c) Model 3

Regarding the crack depth (d_{cj}), analyzes were performed by fixing the crack position and varying the d_{cj}/d ratio in order to observe the influence on dynamic behavior. Figures 5, 6 and 7 show that the greater the crack depth, the smaller the natural frequency of its respective mode. However, the intensity of the frequency variation in relation to the integral beam depends on its position, as already observed in Figure 4. For Models 2 and 3, a greater frequency decay can be seen for higher d_{cj} /d ratios (figures 6 and 7), since in Model 1 there is a tendency to stabilize, evidencing a significant different response between the models for depth variation.

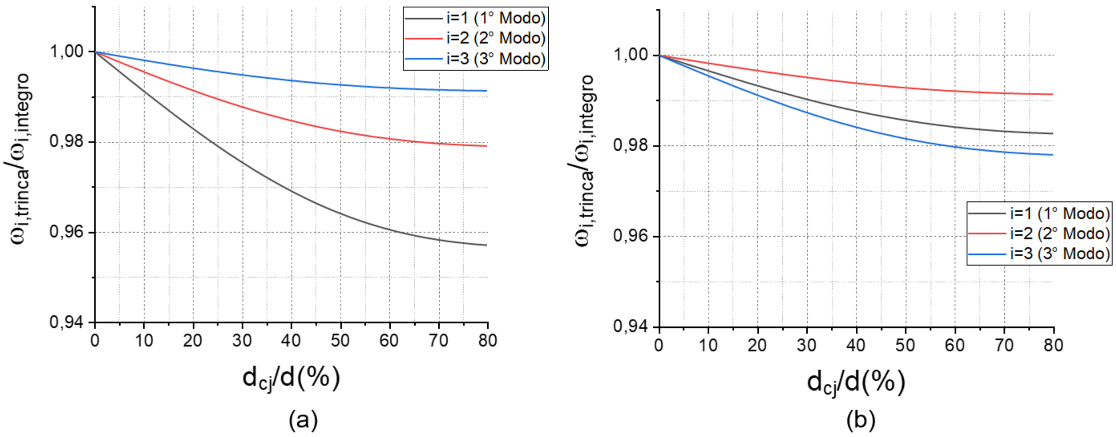


Figure 5. Natural frequency variation for different crack depth in Model 1 – (a) xj /L = 7,5%; (b) xj/L = 32,5%;

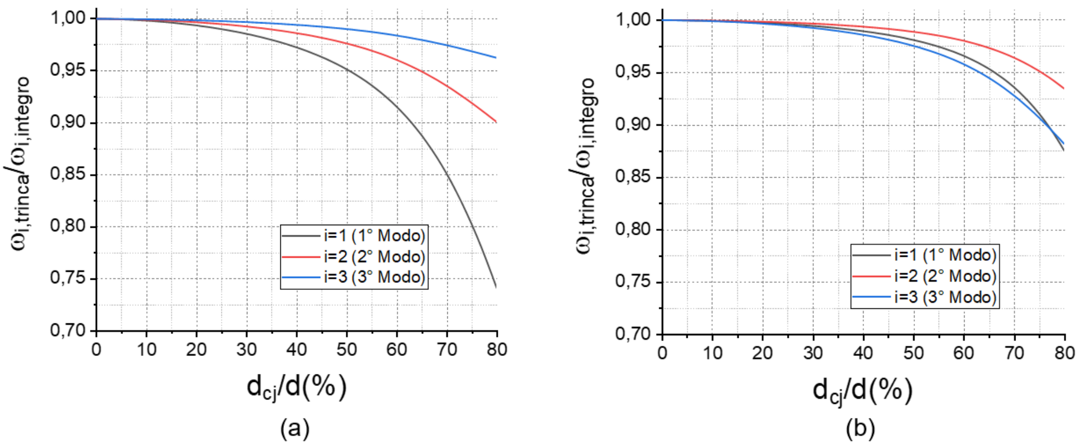


Figure 6. Natural frequency variation for different crack depth in Model 2 – (a) xj /L = 7,5%; (b) xj/L = 32,5%;

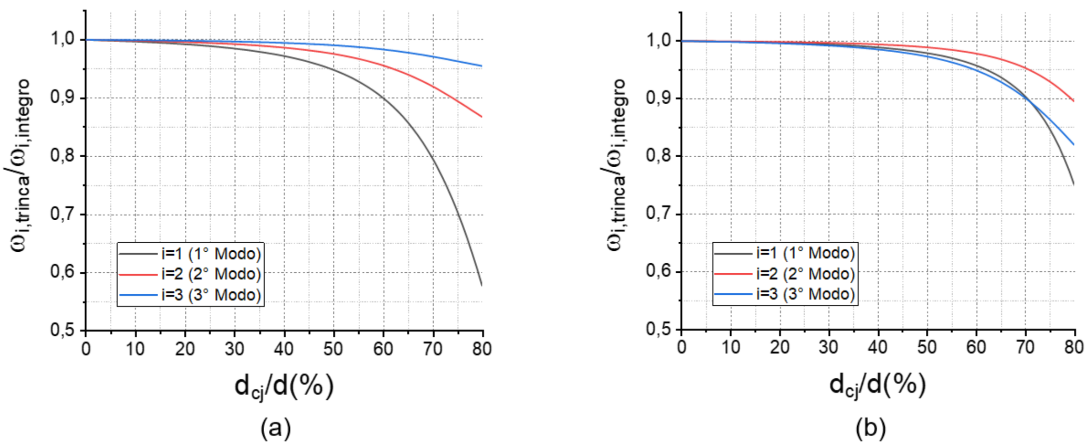


Figure 7. Natural frequency variation for different crack depth in Model 3 – (a) xj /L = 7,5%; (b) xj/L = 32,5%

4 Conclusions

Three crack models were computationally implemented according to assumptions presented in the literature. The influence of cracks in beams subjected to free vibration was numerically tested. The results obtained show a more accentuated decay of the first natural frequencies of a beam with increasing crack depth for Models 2 and 3, while Model 1 presents a stabilization trend. Regarding the crack position, in all models it was observed that changes in frequency depend on the presence of modal nodes, where cracks positioned in these nodes do not generate changes in the natural frequency of the beam relative to the mode of vibration with a modal node in that position. The numerical responses were compared to other approximate models of different approaches present in the literature, showing that a simplified one-dimensional formulation such as the Euler-Bernoulli presents good responses for the first frequencies of cracked beams.

A future work will compare the results of these three numerical models with experimental results.

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