Simplified analytical-numerical study of the static behavior of the hemispherical dome of the Roman Pantheon

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Abstract. The interest in the recovery and maintenance of historical heritage requires mastery of the structural behavior of old buildings. The historical curiosity of contemporary scholars in the builders of the past, and in the construction methods adopted in ancient times, has currently sparked the development of knowledge, tools and innovative theoretical approaches, which allow the understanding of monumental works of architectural and historical interest, which has resisted time, as is the case with the Roman Pantheon. In order to understand how these monuments were built, it is necessary to understand their structural response (displacements, efforts, tensions, etc.), which is characterized by a complex problem. Current computational tools provide more realistic approaches, which in turn also provide the opportunity for robust testing of more consistent analytical theories, which allow in a simplified way to investigate the behavior of these structures, aiming to shed light on the understanding of this "art of construction". Thus, in this work, the static structural response of the hemispherical dome of the Roman Pantheon with simplified geometry is investigated. A progressive analytical methodology through the membrane and bending theory of spherical shells, as well as the finite element method (FEM) using the SAP2000 software are used and compared to model the problem. The results obtained have a good correlation with each other, which corroborates with the methodology used. In addition, with the analysis of the membrane and bending efforts of the hemispherical dome, it was possible to discuss and relate them to the structural and architectural conception adopted at the time.

Keywords: Roman Pantheon, domes, membrane theory, bending theory, finite element method.

1 Introduction

The name Pantheon (Fig. 1), in latin *Pantheum,* probably comes from the ancient Greek *Pantheion* (Πάνθειον), a term that conveyed different meanings, mainly the idea of "a temple of all the gods", according to Marder and Jones [1]. Although, for these authors, we cannot be absolutely sure that it was a pagan temple, we know that, as shown in MacDonald [2], about the year 609 the Byzantine emperor Phocas in Constantinople gave permission for the Pope Boniface IV to consecrate it as a Church, Sancta Maria ad Martyres. But before that, says the author, where the Pantheon stands, in the formerly grand *Campus Martius* district (a public area of about 2 km²), there had been another construction – a rectangular sanctuary. From Marder and Jones [1], this construction was completed in either 27 or 25 BC, build by Agrippa, a consul, general and statesman who served under Augustus, as we see from the inscription below the pediment that still stands in the Pantheon: "M · AGRIPPA · L · F · COS · TERTIVM · FECIT" (Marcus Agrippa, son of Lucius, thrice consul). This building was damaged by fire in AD 80 (restored by the emperor Domitian), burned again in AD 110 by a lightning and rebuild in its present form in around AD 125-8 during the reign of Hadrian. The beginning of this construction is somewhat dubious; it was previously thought to be in the time of Hadrian, but a new interpretation says it was in the reign of Trajan.

It is not by chance that the Pantheon was erected in one of the periods of greatest prosperity in ancient Rome, providing a visual reminder of the greatness that was its empire. It is almost certain, according to Addis [3], that Apollodorus of Damascus played an important part in its realization, as he was the author of great projects in Rome and one of the great builders of the time, first as a military engineer and later as chief engineer of public works (*praefectus fabrum*).

(a) (b)

Figure 1. (a) The Roman Pantheon outside (b) The dome and its *oculus* inside

The Pantheon is divided into 3 main parts: the entrance portico with its 16 Corinthian columns, entablature with the inscription of Agrippa, triangular pediment, intermediate block and bronze door; the cylindrical *rotunda*; and the semicircular dome with an *oculus*. The most important structural part of the Pantheon that makes its building so famous is its dome with 43.30 m in diameter, the largest dome in the world for at least 13 centuries and still today the largest dome in unreinforced concrete in the world.

According to Salvadori [4], a dome can be idealized as a perfect hemisphere of a small thickness in relation to its span. A dome must carry its own weight and the weight of the live load to be channeled to the ground, and a dome does this along its curved vertical lines or *meridians*, which become more and more compressed as they get closer to the dome's support. The continuity of the dome's surface introduces an action along its horizontal sections or *parallels* that prevents the meridians from opening up.

Salvadori [4] also says that the dome can be thought of as a series of identical arches set around a circular base and meeting at their top, where they have a common keystone. The continuity of the dome's surface is what allows its thickness to be relatively small, as it introduces, along the parallels, an action that prevents the meridians from opening up. The dome tends to come down at the top and to open at the bottom.

In the upper part of the dome, continues Salvadori [4], the parallels are compressed because they resist inward motions, while in the lower part they are tensed in resisting outwards motions. In domes, the deformations that occur are very small, as the alterations caused by compression or traction are minute when compared to those caused by bending. The tendency of the upper dome parallels is to shrink under load; the tendency of the lower ones is to elongate. But at some point, you need a parallel that neither shrinks nor elongates. In a dome under dead load this parallel makes an angle of about 52° with its vertical axis. All parallels above it are in compression, and those below are in tension.

Given the complexity of the spherical shell, several studies are applied in the area, in which the Group of Dynamics and Fluid-Structure Interaction (GDFE) of the University of Brasilia (UnB) has developed many studies on domes and other types of shells, e.g., we mention the works of Pedroso [5], Pedroso [6], Nunes [7], Lustosa [8] and Faria [9], among others.

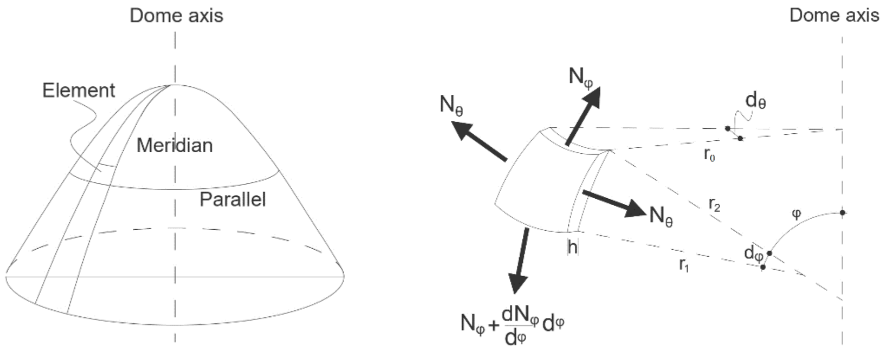
Therefore, in this work a simplified analysis of the dome of the Roman Pantheon is performed using analytical and numerical methods. The study was divided into three cases: domes with roller, hinge and fixed supports. With regard to analytical calculations, two theories were used – membrane theory and bending theory. The two theories interact to obtain the response of the dome structure through the force method. The numerical analysis of the dome was performed using the finite element method (FEM) using the SAP2000 software. The values of normal efforts in the meridians and parallels obtained by the analytical theory and numerical discretization were investigated and compared, in which they demonstrated a good agreement with each other, validating the analyses. Schematic sections were also made, favoring a better understanding of the normal efforts in the dome for different boundary conditions.

2 Theoretical foundation

2.1 Expressions for the primary (membrane theory) and secondary (bending theory) solution

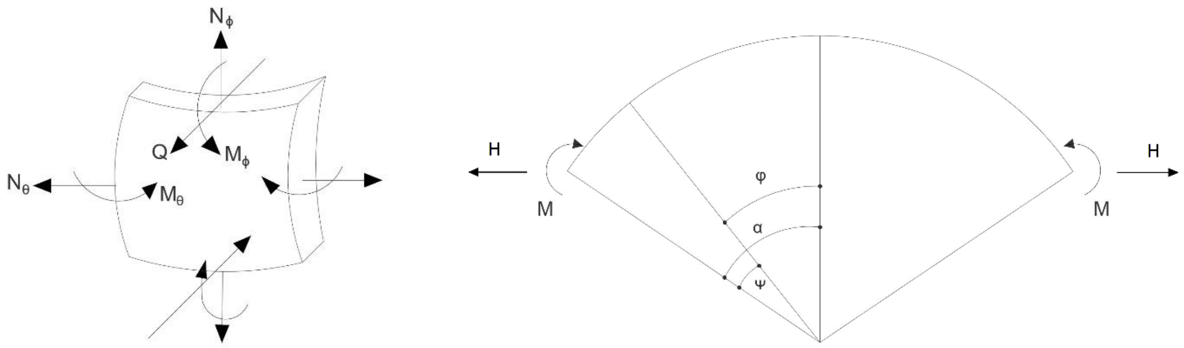
In this work, which is still under development until the submission date, the opening (*oculus*) at the top of the dome is not included, as well as other more realistic characteristics – due to difficulties of its geometry, constructive characteristics, materials used and boundary conditions – of the building, as a first step towards the complete treatment (through a progressive analysis) of the approach of analysis of this highly complex structure. The simplifying assumptions adopted in this work are: closed shells, thin (in relation to the average thickness), of constant thickness and axisymmetric, whose material is homogeneous, isotropic and linear elastic.

The membrane theory is based on the hypothesis that there are no bending and twisting moments in a shell, only normal forces in the main sections. The name *membrane* is due to the fact that the shells are very thin and do not have considerable stiffness for bending and torsion, so as to resist internal efforts with only normal forces. It is a theory that describes with a good approximation the structural behavior of real shells, as long as they satisfy certain geometric, support and load conditions. This means that, although there are bending and twisting moments, they are very small, and the actual torsional state resembles that predicted in the membrane theory. It is important to emphasize that this theory is valid for certain boundary conditions: In this case, only normal forces that are parallel to the direction of the meridians can be present at their edges. Figure 2 contains a diagram of the dome and membrane stresses and Fig. 3, bending stresses.



(a) (b)

Figure 2. (a) Dome (b) Infinitesimal surface element (both adapted from Rabello et al. [10])



(a) (b)

Figure 3. (a) Designations of differential element (adapted from Baker [11]) (b) Bending moment acting on the edge of the shell (adapted from Ramaswamy apud Pedroso [5])

In the effort analysis by the membrane theory, Rabello et al. [10] and Pedroso [6], after a series of algebraic manipulations, for a dome with roller support (primary solution) under dead load, N_Φ (normal force per unit of length in the direction of the meridian) is given by:

$$N_{\Phi} = -\frac{ap}{1 + \cos \Phi}. \tag{1}$$

Where Φ is the angle between the vertical line that passes through the center of the dome and the straight line that passes through its center and through some point of its meridian, $a = r_1 = r_2$ is the average radius of the dome, $p = \Upsilon h$ is the force per unit area, Υ is the specific weight of the dome material, h is the thickness of the dome (very small compared to r_1 e r_2) and $r_0 = r_2 \sin \Phi$ is the radius that is in a plane perpendicular to the axis of the dome. Furthermore, N_{θ} (normal force per unit of length in the direction of the parallel) is given by:

$$N_{\theta} = ap\left(\frac{1}{1 + \cos \Phi} - \cos \Phi\right). \tag{2}$$

Where θ is the angle between two points on the same parallel of the dome.

In the analysis with bending theory, Pedroso [5] presents that for a dome with roller support under horizontal forces and moment on the free edge (secondary solution), it is provided the following expression for N_{Φ}:

$$N_{\Phi} = -\sqrt{2}\cot(\alpha - \Psi)\sin \alpha\, e^{-\lambda \Psi} \sin\left(\lambda \Psi - \frac{\pi}{4}\right) H - \frac{2\lambda}{a}\cot(\alpha - \Psi)\, e^{-\lambda \Psi} \sin(\lambda \Psi)\, M. \tag{3}$$

Where $\Psi = \alpha - \Phi$ (in the case of a hemispherical dome we have $\alpha = 90°$), λ is a coefficient that attenuates the stresses of the dome (specially near the supports), H is a horizontal load on the edge of the dome and M is a moment on the edge of the dome. N_{θ} is given by:

$$N_{\theta} = -2\lambda \sin \alpha\, e^{-\lambda \Psi} \sin\left(\lambda \Psi - \frac{\pi}{2}\right) H - \frac{2\sqrt{2}\lambda^2}{a} e^{-\lambda \Psi} \sin\left(\lambda \Psi - \frac{\pi}{4}\right) M. \tag{4}$$

The two theories, as stated by Baker et al. [11], can be combined into an engineering method, called the force method (described below), which simplifies the problem and makes it possible to analyze complicated shells in a relatively short time.

2.2 Force method

The force method, as shown in Baker [11], is an analytical tool in which the deflection relationships of the shell are expressed in terms of the redundant edge loads and/or moments (X_1 and X_2). The solution for the redundant leads to the solution for the shell. The final solution can be obtained by the superposition of the primary and secondary solutions, which are dependent on corrective edge loadings (Fig. 4).

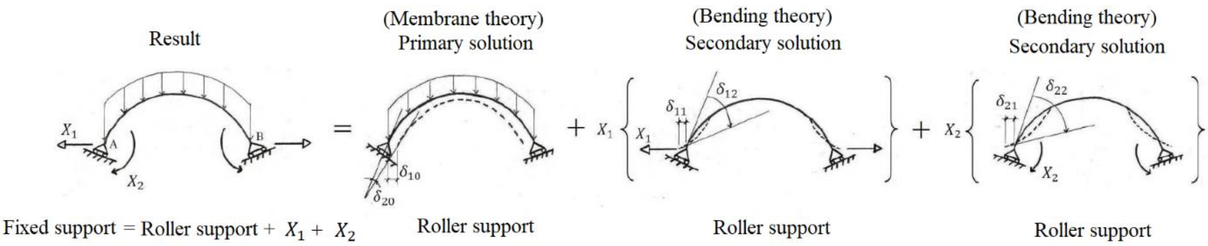
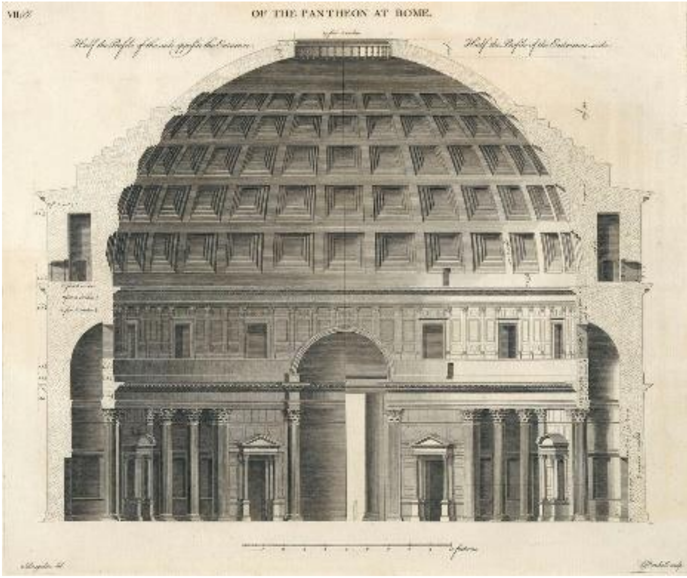


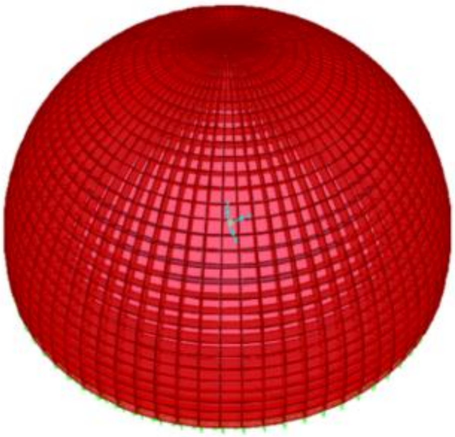
Figure 4. The force method solution (adapted from Pedroso [5])

3 Description of the models

Numerical analyzes for the simplified dome were broken down into 3 cases for a better understanding of the problem. The domes with roller, hinge and fixed supports are under dead load. The numerical analysis procedures are performed using the SAP2000 software, based on finite elements. In the static analysis of the shell, the *Shell Thin* element is used in the modeling, which consists of a finite element area, is a quadrilateral (four nodes), has six degrees of freedom and is suitable for evaluating membrane and bending efforts. Thus, the study of the convergence of the finite element mesh was carried out, adopting the final mesh with refinement presented in the numerical model of Fig. 5b of the Pantheon dome (Fig. 5a), with 80 subdivisions in the angular direction and 30 subdivisions in height. The study of mesh convergence can be found in Faria [9].



(a)



(b)

Figure 5. (a) Drawing of the geometry of the Pantheon: back cross section (Source: Desgodetz [12]) (b) Mash of the numerical model

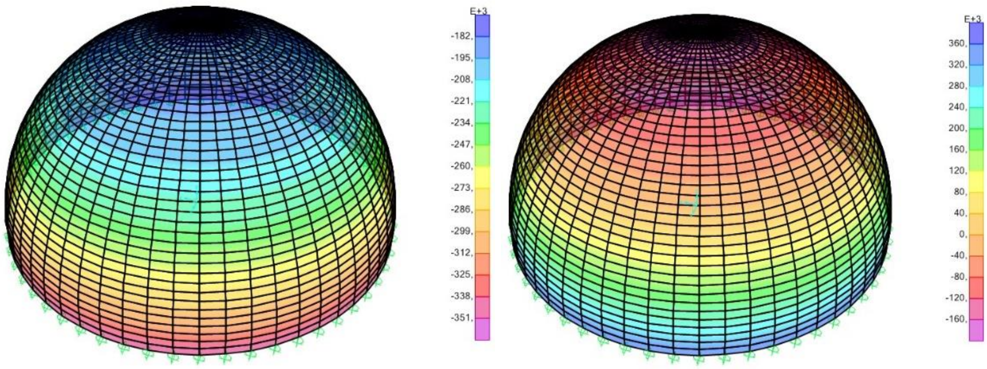
Table 1 presents the main parameters needed for the analysis of results, which were taken from Marder and Jones [1], MacDonald [2], Solheiro [13], Masi et al. [14] and Archeoroma [15]. Furthermore, for the simplified analysis some parameters were calculated (cases of h, p and λ) and can be found in detail in Faria [9].

Table 1. Summary of the main parameters of the Pantheon dome

Description	Nomenclature	Value
Average dome thickness	h	1.13 m
Radius of the dome	a	21.65 m
Weight of the dome	R	50 000 kN
Force per unit area in the dome	p	16.98 kN/m^2
Attenuation coefficient	λ	5.7
Specific mass of roman concrete	Υ	15 kN/m^3
Modulus of elasticity of roman concrete	E	2.9 Gpa
Poisson ratio of roman concrete	ν	0.2

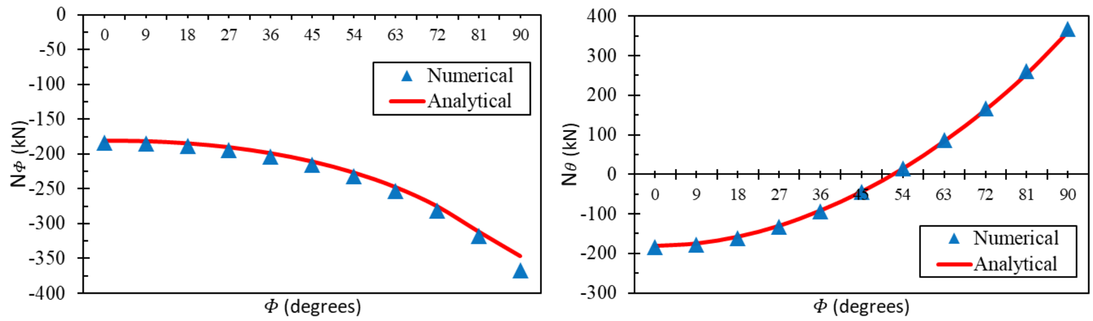
4 Results and discussions

With the help of SAP2000, the numerical analysis of the dome with roller support is illustrated in Fig. 6. Based on these figures, transverse and radial sections of the dome structure were extracted to evaluate the numerical results. In addition, the results were also obtained analytically, plotted through eqs. (1) and (2), and compared with the numerical results. Figure 7 presents the comparative graphs.



(a) (b)

Figure 6. (a) Membrane stress N_Φ (b) Membrane stress N_θ



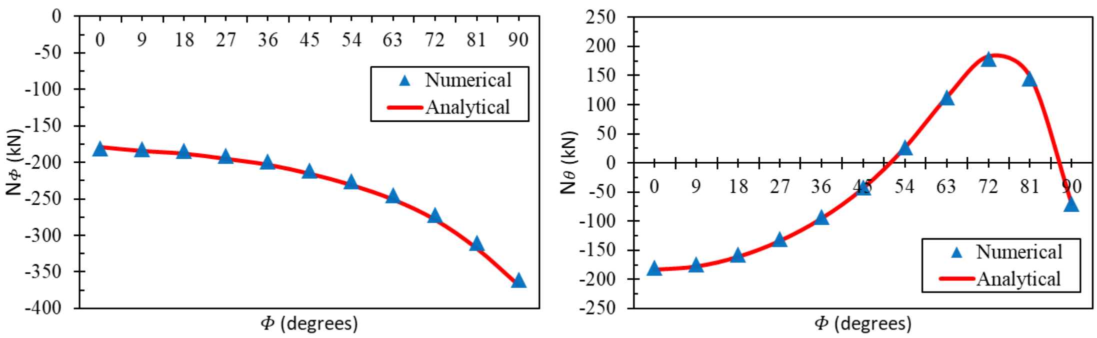
(a) (b)

Figure 7. Comparative graphics for the roller supported dome (a) N_Φ (b) N_θ

The results of the above graphs are consistent with Salvadori's words, shown in the introduction: A dome must carry its own weight and the weight of the live load to be channeled to the ground, and a dome does this along its curved vertical lines or *meridians*, which become more and more compressed as they get closer to the dome's support. This sentence reflects exactly what is mathematically exposed in the graph of N_Φ in Fig. 7a.

Again, repeating Salvadori's words: In a dome under dead load this parallel (that neither shrinks nor elongates) makes an angle of about 52° with its vertical axis. All parallels above it are in compression, and those below are in tension. And that is what is in the graph of N_θ in Fig. 7b.

Similar to what was done above, comparatives of analytical and numerical results are presented for the dome under dead load with hinge (Fig. 8) and fixed (Fig. 9) supports.



(a) (b)

Figure 8. Comparative graphics for the hinge supported dome (a) N_Φ (b) N_θ

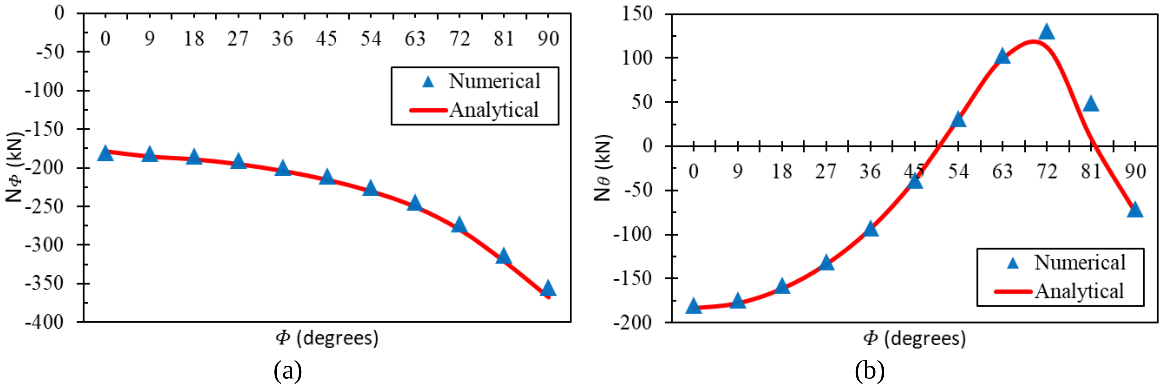


Figure 9. Comparative graphics for the fixed supported dome (a) N_{Φ} (b) N_{θ}

It can be seen that the analytical and numerical results are in good agreement. Furthermore, the efforts of N_{Φ} are practically coincident in all three types of support. However, the graphs of N_{θ} diverge a lot in the regions close to the edges, where efforts that appear in the domes with hinge and fixed supports did not exist in the dome with the roller support; it is also noticed that the efforts are quite similar for $0 < \Phi < 60°$. However, for $\Phi > 60°$, the bonds at the edges influence the efforts and impose a certain difference.

5 Conclusions

This work showed, in a simplified way, the result of the main efforts of a dome – inspired by the geometry of the Roman Pantheon – under dead load. The simplifications, together with analytical calculations and numerical modeling, proved to be coherent, revealing good results.

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