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Infinite Numerical Series Convergence Application to Slender Columns Mechanical Analysis

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Abstract. A numerical series represents the sum of terms in a numerical sequence. If the quantity of its terms is unknown, it constitutes an infinite series, and the sum of its terns must be obtained from the convergence analysis procedure. The series used to approach the values of given function, must be of the infinite mode because, in such cases, the quantity of terms that the series needs to attend such demand, is unpredictable. There are cases wherein the solution of engineering problems that may be aided, directly, by the modeling based on approach for infinite numerical series, as the slender columns analysis. The slender columns analysis cast by elastic and ductile material has been based on the concept of critical load. Such modeling version is not suitable to mechanical analysis of specimens cast by weak rupture pattern material, as the reinforced concrete, but even thus, some concepts involved in such a formulation are used as a classificatory reference. The Mechanics of the Materials uses the physical modeling capabilities in deriving the equations object of approach on the featured theme. The aim of this work is the slender columns mechanical performance modeling through infinite numerical series convergence concept.

Keywords: Slender, Columns, Analysis, Series.

1 Introduction

The infinite numerical series concept represents a relevant resource applied to tasks involving the approximation of values of functions of difficult algebraic manipulation for discrete points. It represents the sum of terms in a numerical sequence. If the quantity of its terms is unknown, it constitutes an infinite series, and the sum of its terns must be obtained from the convergence analysis procedure.

Often, the purpose to be reached is a good quality of approach numerical results, of a set of values of given function in a certain work domain and, for the accomplishment of such purpose, the series to be used must be of the infinite mode because, in such cases, the quantity of terms that the series need to contain to attend such demand, is unpredictable.

There are cases, including, wherein the solution of engineering problems may be aided, directly, by modeling based on infinite numerical series approach, as occur on regard the performance analysis of slender columns.

The analysis of slender columns constituted by isotropic, homogeneous, and ductile material, and that, whose mechanical work regimen obeys the Hooke's law, has been based on the critical load or Euler's load concept.

The modeling version mentioned in the preceding paragraph is not enough at all, for use in studies of slender columns made of heterogeneous material whose mechanical behavior is nonlinear, and that presents weak failure pattern, as the reinforced concrete, but even thus, there are concepts involved in such a formulation that can be used as a classificatory reference in its analysis methods.

The Mechanics of the Solid Materials, in its essence, uses its traditional physical-mathematical modeling resources for deriving the equations that are the object of approach of the featured theme.

The aim of this work is the mechanical performance modeling of slender columns through infinite numerical series convergence concept.

Such idealized modeling may consider, including, the geometric imperfections arising from construction faults that have occurred during the column cast, characterized by deviation from linearity, expressing the corresponding curvature in terms of a sin function branch.

2 Theoretical Basement

The simply supported column AC, Fig. 1 presents, initially, a rectilinear longitudinal axis. If such a structural member is submitted to the action of a load "**P**", applied in its top cross section, deviated from its gravity center by an eccentricity, namely "**e**", it will deform progressively and in a certain stage takes the ADC curved form so that the bending moment at an arbitrary point, stay given by:

$$M = P(e + h) \tag{1}$$

The "h = h (x)" function in Eq. 1 represents the horizontal displacement suffered by an arbitrary point, that is localized by a distance "x" from the column base cross section. The bending moment "M" due to the load "P", promotes the horizontal displacement increase that, consequently, induces the bending moment magnitude increase. In this way, an iteration involving displacements and bending moments is characterized. Such iteration may evolve into two distinct situations. At a first alternative, both the displacement and bending moment magnitudes, may hit the stationary condition at a certain value, so that, the column would be classified as a stable structural member. On the other hand, as a second behavioral pattern, both the displacement and the bending moment magnitudes would be increase, indefinitely, up to the level wherein the loosing of the column usefulness is culminated. If the second precede alternative is to be prevail the column is unstable, and the phenomenon is known as buckling.

The highlighted phenomenon involves flexure strains and displacements that, according to the mechanics of solid materials postulates, should be modeled upon the Deflected Curve Differential Equation concept, Timoshenko and GERE [1]. For the analyzed case in this work such an equation presents itself at the form:

$$\frac{d^2h}{dx^2} + \frac{P(e+h)}{EI} = 0 \tag{2}$$

Such that the "**E**" and the "**I**" parameters represent, respectively, the Young's Modulus of the column constituent material and its cross-section moment of inertia relating to the axis passing through its gravity center and parallel to the bending moment vector.

If it may be considered the algebraic artifice:

$$p^2 = \frac{P}{EI} \tag{3}$$

Some simple algebraic resources may transform Eq. 2 into:

$$\frac{d^2h}{dx^2} + p^2h = -p^2e \tag{4}$$

Whose solution may be described from the expression:

$$h(x) = A\cos(px) + Bsen(px) - e \tag{5}$$

The problem domain may be defined as:

$$D = \{x \in R \;/\; 0 < x < L\} \tag{6}$$

Where, the "L" parameter represents the column length. The boundary conditions applied to the case analyzed in this paper may be expressed as:

$$h(0) = h(L) = 0 \tag{7}$$

That, once considered in Eq. 5 results:

$$h(x) = e\left[\cos(px) + tang\left(\frac{pL}{2}\right)sen(px) - 1\right] \tag{8}$$

By using of the Eq. 8 it is possible to carry out the displacement trend analysis by a representative point of the slender column. If, for the fulfillment of such a subject, it is adopted the point placed at the column middle height, namely, to x = L/2, its horizontal displacement may be obtained from:

$$\delta = h(L/2) = e[\sec(pL/2) - 1] \tag{9}$$

According to the Eq. 9, the column deflection increases indefinitely as the **sec(PL/2)** term tends to the infinity, what happens if the pL/2 quantity tends to the **π/2** arch**.** At the limit it may considered:

$$\frac{pL}{2} = \frac{\pi}{2} \Rightarrow pL = \pi \; e \; p^2 = \frac{\pi^2}{L^2} \tag{10}$$

Whether Eq. 3 is compared with Eq. 10 it results:

$$p^2 = \frac{P}{EI} = \frac{\pi^2}{L^2} \Rightarrow P = \frac{\pi^2 EI}{L^2} \tag{11}$$

The Eq. 11 defines the Critical Load or the Euler's Load and represents the load magnitude "**P**" from which the column is induced to suffer the buckling phenomenon.

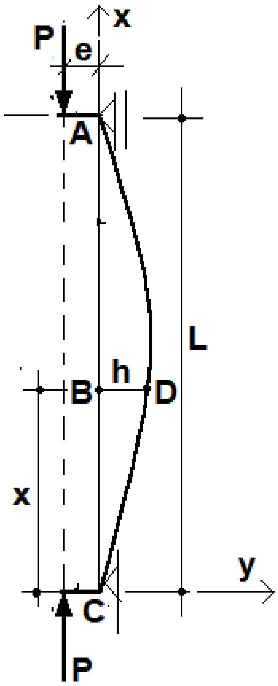

Figure 1. Column Model Number 1

The right condition that an infinite numerical series must attend is that, the sum of its terms tends for a certain value, even though by addition term after term indefinitely, Lang [2], expressing it, in other words, if:

$$\sum_{i=1}^{\infty} a_i \leq \alpha, \alpha \in IR \tag{12}$$

Where the "a_i" notation represents a generic term of the series.

Prior to realize whatever the procedure of infinite numerical series convergence study it is suitable to verify if the terms of the series present themselves in a decreasing way. For the formally purpose, it is enough to adopt a preliminary test and then to verify if:

$$\lim_{i \to \infty} a_i \rightarrow 0 \; (Zero) \tag{13}$$

The infinite series convergence analysis may be carried out by using of several methods one of them is the so called the Ratio Test, Lang [2]. According the referring test an infinite numerical series presenting terms "$\mathbf{a_i}$" converges, if:

$$0 < \lim_{n\to\infty} \frac{a_{(n+1)}}{a_n} < 1 \tag{14}$$

3 Modeling

As a suitable example of an alternative way to modeling the slender column buckling phenomenon, it may be considered the Member Structural Model presented on Fig. 2, that is constituted by homogeneous, elastic and ductile material and, in addition, a local geometric imperfection characterized by a deviation from its linearity condition, that's why, its longitudinal axis exhibit a curvature, even at the unloaded mode, described from the sin function branch:

$$y_o(x) = a.sin\left(\frac{\pi x}{L}\right), \quad \forall x \in R/\ 0 < x < L \tag{15}$$

Since the "$\boldsymbol{a}$" parameter is the deviation of the column longitudinal axis at the cross sectional placed at its middle height, as shown in Fig. 2.

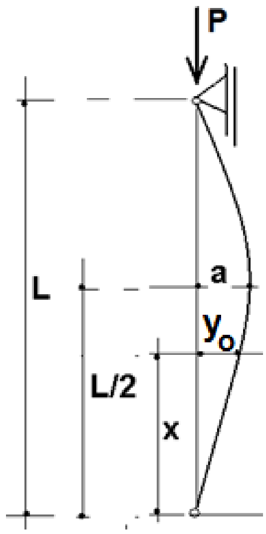

Figure 2. Column Model Number 2

If the column is to be submitted to a load P, a whatever cross section placed along its longitudinal axis, which presented, at first, a deviation measured as $\mathbf{y = y_o}$, will be submitted, too, by a bending moment, whose magnitude must be expressed in terms of the Equation:

$$M_o(x) = Py_o(x) = Pa.sin\left(\frac{\pi x}{L}\right) \tag{16}$$

The transverse displacement due the referring bending moment, that an arbitrary point placed on the column axis will suffer, may be modeled from the Differential Equation of the Deflection Curve:

$$\frac{d^2y_1}{dx^2} + \frac{M_O(x)}{EI} = 0 \ \Rightarrow\ \frac{d^2y_1}{dx^2} + \frac{Pa}{EI} sin\left(\frac{\pi x}{L}\right) = 0 \tag{17}$$

The parameter "$\mathbf{y_1(x)}$", represents thus such transverse displacement, Fig. 3.

If it may be let:

$$k = \frac{P}{EI} \tag{18}$$

so, the Eq.17 becomes itself:

$$\frac{d^2y_1}{dx^2} + kasin\left(\frac{\pi x}{L}\right) = 0 \quad (19)$$

Whose solution is:

$$y_1(x) = ka\frac{L^2}{\pi^2}sin\left(\frac{\pi x}{L}\right) + Cx + D \quad (20)$$

The problem domain may be characterized in a similar way as expressed by Eq. 6. The boundary conditions referring to the column subject of analysis, Fig. 3, are:

$$y_1(0) = 0 \; e \; y_1(L) = 0 \quad (21)$$

That, if are applied to the Eq. 20, induce:

$$y_1(x) = ka\frac{L^2}{\pi^2}sin\left(\frac{\pi x}{L}\right) \quad (22)$$

Once the column has suffered the displacement $y_1(x)$, Fig. 3, bending moment increase on the considered cross section occurs, and it is expressed by the form:

$$M_1(x) = Py_1(x) = Pka\frac{L^2}{\pi^2}sin\left(\frac{\pi x}{L}\right) \quad (23)$$

Such moment parcel promotes an additional displacement "$y_2(x)$" of the column axis, Fig. 3, at the analyzed section, that must be obtained from the resolution of Differential Equation:

$$\frac{d^2y_2}{dx^2} + k^2a\,\frac{L^2}{\pi^2}sin\left(\frac{\pi x}{L}\right) = 0 \quad (24)$$

If the same procedure applied to obtain the Eq. 22 is used to solve Eq. 24, results

$$y_2(x) = k^2a\frac{L^4}{\pi^4}sin\left(\frac{\pi x}{L}\right) \quad (25)$$

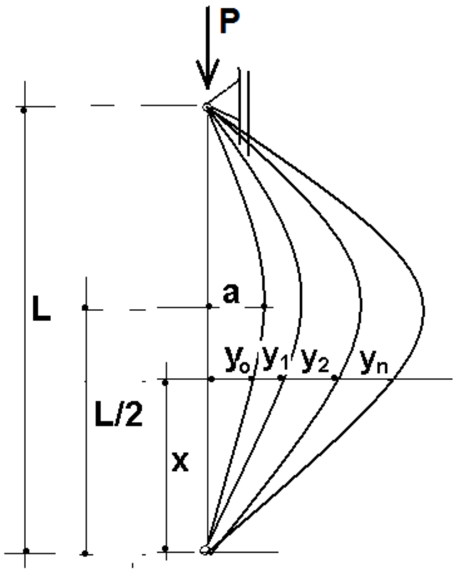

Figure 3. Column Model axis deformed shape

If a calculus philosophy identical to that one employed do obtain the Eq. 22 and 25 is adopted, it may be derive the equations to determine the successive displacements increases "$y_3(x)$", "$y_4(x)$", . . ., "$y_{n-1}(x)$" e "$y_n(x)$", resulting the forms of Eq. 26.

$$y_3(x) = k^3 a \frac{L^6}{\pi^6} sin\left(\frac{\pi x}{L}\right);\ y_4(x) = k^4 a \frac{L^8}{\pi^8} sin\left(\frac{\pi x}{L}\right);\ \cdots;$$

$$y_{(n-1)}(x) = k^{(n-1)} a \frac{L^{2(n-1)}}{\pi^{2(n-1)}} sin\left(\frac{\pi x}{L}\right);\ y_n(x) = k^n a \frac{L^{2n}}{\pi^{2n}} sin\left(\frac{\pi x}{L}\right) \quad (26)$$

Considering the point placed at the middle height cross sectional column as the representative one, therefore, wherein **x = L/2,** every successive displacement increase will be described as:

$$\delta_1 = ka\frac{L^2}{\pi^2};\ \delta_2 = k^2 a \frac{L^4}{\pi^4}; \delta_3 = k^3 a \frac{L^6}{\pi^6};\cdots;$$

$$\delta_{(n-1)} = k^{(n-1)} a \frac{L^{2(n-1)}}{\pi^{2(n-1)}};\ \delta_n = k^n a \frac{L^{2n}}{\pi^{2n}} \quad (27)$$

The total transverse displacement may be understood as the algebraic sum:

$$\delta = \delta_1 + \delta_2 + \delta_3 + \cdots + \delta_{(n-1)} + \delta_n \quad (28)$$

And, therefore:

$$\delta = ka\frac{L^2}{\pi^2} + k^2 a \frac{L^4}{\pi^4} + k^3 a \frac{L^6}{\pi^6} + \cdots + k^{(n-1)} a \frac{L^{2(n-1)}}{\pi^{2(n-1)}} + k^n a \frac{L^{2n}}{\pi^{2n}} \quad (29)$$

It may be noted that Eq. 29 represents an infinite numerical series whose terms are drafted in the form:

$$\delta_n = k^n a \frac{L^{2n}}{\pi^{2n}} \quad (30)$$

If the series expressed from Eq. 29 converges so the displacement "**δ**" tends to certain finite values, characterizing, in this way, the column stability. In this case, the ratio test is to be able to define the convergence condition of that series. The forward term of the series may be expressed by:

$$\delta_{(n+1)} = k^{(n+1)} a \frac{L^{2(n+1)}}{\pi^{2(n+1)}} \quad (31)$$

The ratio involving those terms defined from Eq. 31 and Eq. 30 may be expressed as:

$$r = \frac{\delta_{(n+1)}}{\delta_n} = \frac{k^{(n+1)} a \frac{L^{2(n+1)}}{\pi^{2(n+1)}}}{k^n a \frac{L^{2n}}{\pi^{2n}}} = k\frac{L^2}{\pi^2} \quad (32)$$

And, if:

$$C = \lim_{n\to\infty} r = \lim_{n\to\infty} k\frac{L^2}{\pi^2} = k\frac{L^2}{\pi^2} \quad (33)$$

The series presented in Eq. 30 converges if 0 < C < 1, or, in other words, if:

$$0 < k\frac{L^2}{\pi^2} < 1 \quad (34)$$

In the other hand, once:

$$k\frac{L^2}{\pi^2} < 1 \Rightarrow k < \frac{\pi^2}{L^2} \quad (36)$$

And considering the Eq. 18, it gets:

$$\frac{P}{EI} < \frac{\pi^2}{L^2} \Rightarrow P < \frac{\pi^2}{L^2} EI \tag{37}$$

Thus, the slender column stability is guaranteed if the maximum load magnitude is:

$$P = \frac{\pi^2}{L^2} EI \tag{38}$$

It may observe that the form of Eq. 11 is similar to the one of Eq. 38.

4 Conclusions

This paper refers itself to a slender column mechanical performance modeling through infinite numerical series convergence concept.

Such an analysis is compared with that one performed from the Euler's Load concept that is part of the Mechanics of the Solids Material postulates.

From the theoretical basement presented in the section 2 and the mathematical analysis developed in the section 3, it may be concluded that the Eq. 38 is identical to the Eq. 11, consequently, the efficacy of the alternative way of modeling that is proposed in this work may be considered validated.

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References

[1] S. P. Timoshenko and J. E. Gere. Mecânica dos Sólidos. Vol. 1. Livros Técnicos e Científicos, 1998.
[2] S. Lang. Cálculo. Vol. 1. Ao Livro Técnico, 1973.