# Estimation of the Mass, Stiffness and Damping Matrices From Complex Frequency Response and Real Frequency Response Matrices


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**Abstract.** A frequency-domain method for estimating the mass, stiffness and damping matrices of the mass-spring-damper system is presented. The developed method is based on the extraction of real and imaginary parts of the Complex Frequency Response Matrix (Complex Transfer Matrix) and the Undamped Frequency Response Matrix (Normal Matrix or Real Frequency Response Matrix). A relationship among these matrices is used to obtain the Damping Matrix explicitly. The Mass and the Stiffness Matrices are calculated from the Undamped Frequency Response Matrix using the Least Squares Method. Three examples were employed in order to illustrate the applicability of the proposed method and the results were quite accurate.



**Keywords:** Frequency-Domain, Complex Frequency Response Matrix, Undamped Frequency Response Matrix


## 1 Introduction

A multi-degree-of-freedom (MDOF) structural system can be simulated in terms of mass, stiffness and damping matrices. The dynamic equilibrium equation in the time-domain, Clough and Penzien [1], Craig [2] and Rao [3], with N degrees of freedom is

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = \{f(t)\}. \tag{1}$$

with the initial conditions

$$\{x(0)\} = \{x_0\}, \qquad \{\dot{x}(0)\} = \{\dot{x}_0\}. \tag{2}$$

In eq. (1) , $[M](NxN)$, $[C](NxN)$ and $[K](NxN)$ are the mass, damping and stiffness matrices, $\{\ddot{x}(t)\}(Nx1)$, $\{\dot{x}(t)\}(Nx1)$ and $\{x(t)\}(Nx1)$ are the accelerations, velocities and displacements vectors while$\{f(t)\}(Nx1)$ is the external forces vector. In eq. (2), $\{x_0\}(Nx1)$ and $\{\dot{x}_0\}(Nx1)$ are the initials displacements and velocities vectors, respectively.

The mass matrix is assumed diagonal and the damping is considered as viscous. The elements of the matrices must be properly allocated in each matrix, according to the structural system disposition. If there are not the external forces $\{f(t)\}$, the the movement is natural and considered as free vibration. Then, eq. (1) becomes

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = 0. \tag{3}$$

For a single-degree-of-freedom system (SDOF), the undamped natural frequency $\omega_n$, the viscous damping factor $\zeta$ and the damped natural frequency $\omega_d$ are, respectively

$$\omega_n = \sqrt{k/m}. \tag{4}$$

$$\zeta = \frac{c}{c_{cr}} \qquad where \qquad c_{cr} = 2m\omega_n = \frac{2k}{\omega_n} = 2\sqrt{km}. \tag{5}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{6}$$

For the SDOF underdamped system ($\zeta < 1$), the analytical response of eq. (3) is

$$x(t) = \exp\left(-\zeta\omega_n t\right)\left[x_0\cos(\omega_d t) + \left(\frac{\dot{u}_0 + \zeta\omega_n u_0}{\omega_d}\right)\sin(\omega_d t)\right]. \tag{7}$$

For an undamped MDOF system, the governing equation of the free vibration is

$$[M]\{\ddot{x}(t)\} + [K]\{x(t)\} = 0. \tag{8}$$

## 2 Frequency Domain Approach

Applying the Fourier Transform into eq. (1), it is converted from time-domain to frequency-domain, Clough and Penzien [1], Craig [2] and Rao [3], and it becomes

$$\left(-\omega^2[M] + i\omega[C] + [K]\right)\{X(\omega)\} = \{F(\omega)\}. \tag{9}$$

where $\omega$ is the independent variable in the frequency-domain, $\{X(\omega)\}$ the Fourier Transform of $\{x(t)\}$ and $\{F(\omega)\}$ the Fourier Transform of $\{f(t)\}$.

Explaining $\{X(\omega)\}$ in eq. (9), it is obtained

$$\{X(\omega)\} = \left(-\omega^2[M] + i\omega[C] + [K]\right)^{-1}\{F(\omega)\}. \tag{10}$$

The expression $(-\omega^2[M] + i\omega[C] + [K]^{-1})$ from eq. (10) is called Complex Frequency Response Matrix $[H(\omega)]$, Transfer Matrix or Complex Frequency Response Function (CFRF) and has many applications in the frequency-domain.

$$[H(\omega)] = (-\omega^2[M] + i\omega[C] + [K])^{-1} \quad or \quad [H(\omega)]^{-1} = (-\omega^2[M] + i\omega[C] + [K]). \tag{11}$$

The Complex Frequency Response Matrix $[H]$ and the Undamped Frequency Matrix $[H_N]$ Matrices can be obtained by Modal Testing. The structure under analysis must be excited by a known kind of force, and the excitation force and resulting response vibrations, typically accelerations, are both measured properly by instruments and, after a software data processing, the Complex Frequency Response Function (CFRF) data can be obtained. There are two common methods of excitation: impact hammer and modal shaker, Allemang et. al. [4].

For a system with one degree-of-freedom, with $m = 3kg$ , $\zeta = 0.05$ and $k = 2700N/m$ , the graphics of real and imaginary parts of $H$ versus frequency, magnitude of $H$ versus frequency and $H_N$ versus frequency are shown in Fig. 1, Fig. 2 and Fig. 3, respectively.

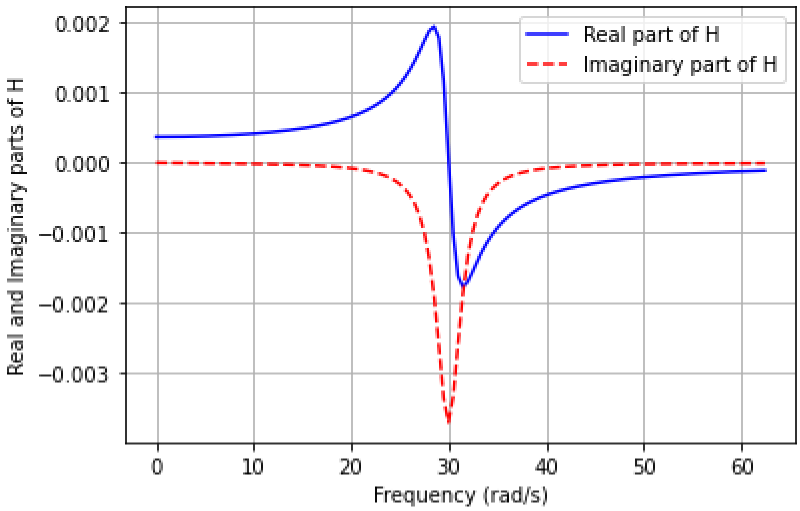


Figure 1. Real and Imaginary parts of H versus frequency

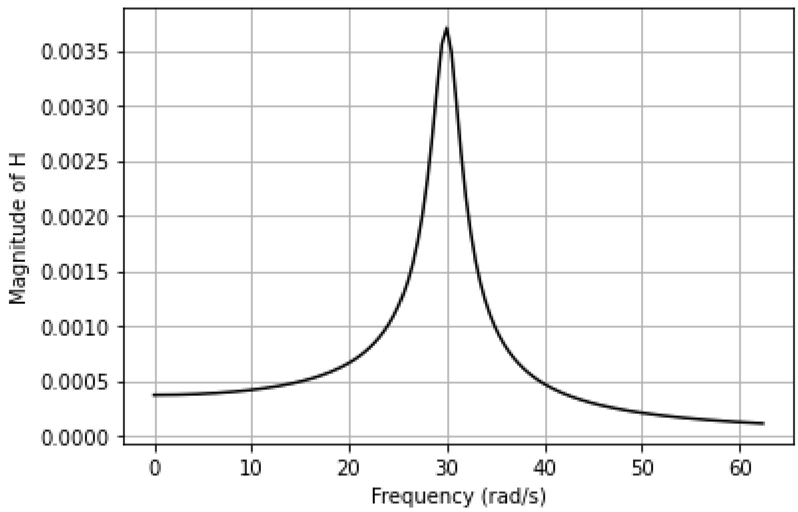


Figure 2. Magnitude of H versus frequency

## 3 Estimation of the Damping Matrix

For an undamped system, the matrix $[H]^{-1}$ from eq. (11) becomes

$$[H_N(\omega)]^{-1} = (-\omega^2[M] + [K]). \tag{12}$$

Subtracting eq. (12) from eq. (11), it results

$$[H]^{-1} - [H_N]^{-1} = i\omega[C]. \tag{13}$$

Post-multiplying eq. (13) by $[H]$, it turns

$$[I] - [H_N]^{-1}[H] = i\omega[C][H]. \tag{14}$$

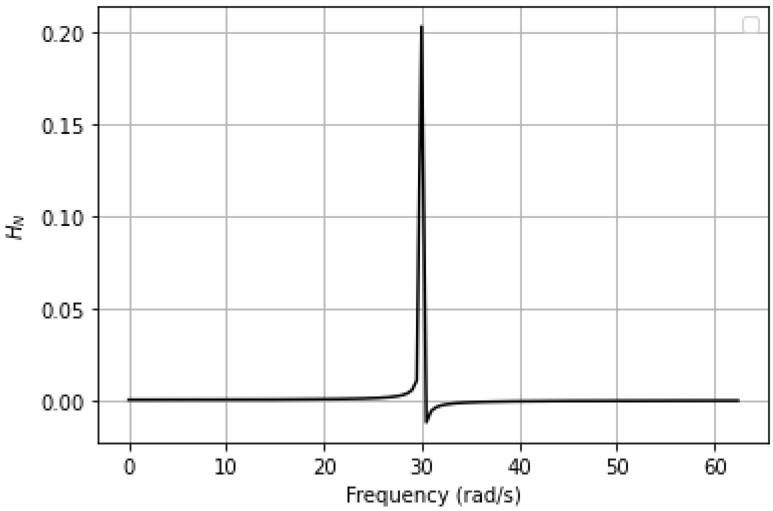


Figure 3. $H_N$ versus frequency

Pre-multiplying eq. (14) by $[H_N]$, it turns

$$[H_N] - [H] = i\omega[H_N][C][H]. \tag{15}$$

Moving $[H]$ from the left side of eq. (15) to the right side, it becomes

$$[H_N] = [H] + i\omega[H_N][C][H]. \tag{16}$$

Considering that $[H] = [H_R] + i[H_I]$ where $[H_R]$ and $[H_I]$ are the real and imaginary parts of $[H]$, respectively, and substituting into eq. (16), it turns

$$[H_N] = [H_R] + i[H_I] + i\omega[H_N][C]\left\{[H_R] + i[H_I]\right\}. \tag{17}$$

Rearranging eq. (17) it becomes

$$[H_N] = \left\{[H_R] - \omega[H_N][C][H_I]\right\} + i\left\{[H_I] + \omega[H_N][C][H_R]\right\}. \tag{18}$$

The matrix $[H_N]$ must be real. Then, the imaginary part of the right side of eq. (18) must be equal to $[0]$.

$$[H_I] + \omega[H_N][C][H_R] = [0] \qquad or \qquad [H_N][C][H_R] = -\frac{1}{\omega}[H_I]. \tag{19}$$

Pre and post-multiplying eq. (19) by $[H_N]^{-1}$ and $[H_R]^{-1}$, respectively, it becomes

$$[C] = -\frac{1}{\omega}[H_N]^{-1}[H_I][H_R]^{-1}. \tag{20}$$

In eq. (20) is shown the estimation of the damping matrix $[C]$ in terms of the undamped matrix $[H_N]$ and the real $[H_R]$ and imaginary $[H_I]$ parts of the transfer matrix $[H]$.

# 4 Estimation of the Mass and Stiffness Matrices

Considering only the real parts of eq. (18) it becomes

$$[H_N] = [H_R] - \omega[H_N][C][H_I]. \tag{21}$$

Substituting eq. (20) into eq. (21), it turns

$$[H_N] = [H_R] + [H_N][H_N]^{-1}[H_I][H_R]^{-1}[H_I]. \tag{22}$$

Then $[H_N]$ can also been calculated from the real and imaginary parts of $[H]$, or

$$[H_N] = [H_R] + [H_I][H_R]^{-1}[H_I]. \tag{23}$$

Taking eq. (12) and considering the uncoupling among the matrices $[H_N]^{-1}$, $[M]$, and $[K]$, each element of these matrices are related according to the following

$$[H_N(\omega)(i,j)]^{-1} = (-\omega^2[M(i,j)] + [K(i,j]). \tag{24}$$

# 5 Examples of Application

Three examples of application are shown in order to illustrate the Methodology, which are in Ferro [5]. The first one is a one-degree-of-freedom system, the second one is a two-degrees-of-freedom and the third a seven-degrees-of-freedom. A fitting method can be adopted to solve the nonlinear eq. (24). The Least Square Method was used in this work in order to fit the data, where the input data are each set of elements $H_N^{-1}(i,j)$ and four aleatory frequencies. The fitting results are the corresponding elements $M(i,j)$ and $K(i,j)$. At least four frequencies must be used for a good accuracy, because of the relation between the matrices $[H_N]^{-1}$ and frequencies is of power 2

## 5.1 Example 1 - SDOF System

In this example, the system has $m = 3\,kg$ , $\zeta = 0.05$ and $k = 2700\,N/m$. Then, the expected values of m and k are 3 and 2700 , respectively, and for c, the value is

$$c = 2 \times \zeta \times \sqrt{k \times m} = 2 \times 0.05 \times \sqrt{2700 \times 3} = 9\,N.s/m. \tag{25}$$

For calculating the damping coefficient eq. (20) was used and for the mass and stiffness estimation the fitting equation is

$$H^{-1} = -\omega^2 m + i\omega c + k. \tag{26}$$

The command numpy.polyfit of Python language was used to fit eq. (26) with 4 random frequency values, which are $\omega_1 = 1.6000\,rad/s$ , $\omega_2 = 1.7212\,rad/s$ , $\omega_3 = 1.8423\,rad/s$ and $\omega_4 = 1.9635\,rad/s$.

The fitting results are $m = 3.0000 \quad c = 9.0000 \quad and \quad k = 2700.0000$, which indicate an excellent estimation.

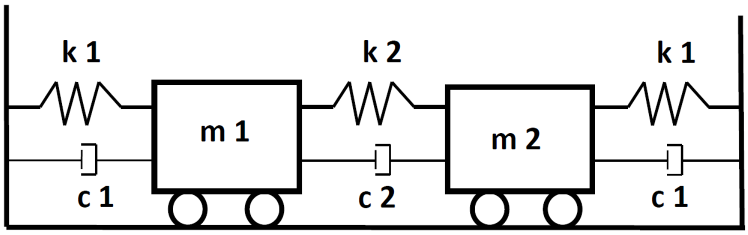


Figure 4. System of example 2

### 5.2 Example 2 - 2 DOF System

In this example, from Craig [2], page 357, and shown in Figure 4, where $m1 = m2 = 1\,kg$, $c1 = 0.6284\,N.s/m$, $c2 = 0.0628\,N.s/m$, $k1 = 987\,N/m$ and $k2 = 217\,N/m$, the Mass $[M]$, Damping $[C]$ and Stiffness $[K]$ matrices are

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [C] = \begin{bmatrix} 0.6912 & -0.0628 \\ -0.0628 & 0.6912 \end{bmatrix} \qquad [K] = \begin{bmatrix} 1204 & -217 \\ -217 & 1204 \end{bmatrix}$$

The damping matrix $[C]$ was calculated according to eq. (20) with $\omega = 3.1416\,rad/s$ and the command numpy.polyfit in Python language was used for each element of $H_N^{-1}(\omega)$ and four aleatory frequencies which are $\omega_1 = 1.00000\,rad/s$ , $\omega_2 = 1.7139\,rad/s$ , $\omega_3 = 2.42773\,rad/s$ , $\omega_4 = 3.1416\,rad/s$, for estimating the matrices $[M]$ and $[K]$. The results are shown in Table 1.

Table 1. Results for Mass, Damping and Stiffness Matrices of Example 2

| $(i,j)$ | $M(i,j)$ | $C(i,j)$ | $K(i,j)$ |
|---|---|---|---|
| (1,1) | 1.0000 | 0.6912 | 1204.0000 |
| (1,2) | 5.1908 e-14 | -0.0628 | -217.0000 |
| (2,1) | 5.1908 e-14 | -0.0628 | -217.0000 |
| (2,2) | 1.0000 | 0.6912 | 1204.0000 |

### 5.3 Example 3 - 7 DOF System

In this example, from Chen [6],and shown in Fig. 5, the parameters are $m1 = m2 = m3 = m4 = m5 = m6 = m7 = 1,0\,kg$, $k1 = 10000\,N/m$, $k2 = 20000\,N/m$ and $c = 20\,N.s/m$. The mass $[M]$, damping $[C]$ and stiffness $[K]$ matrices of the system are

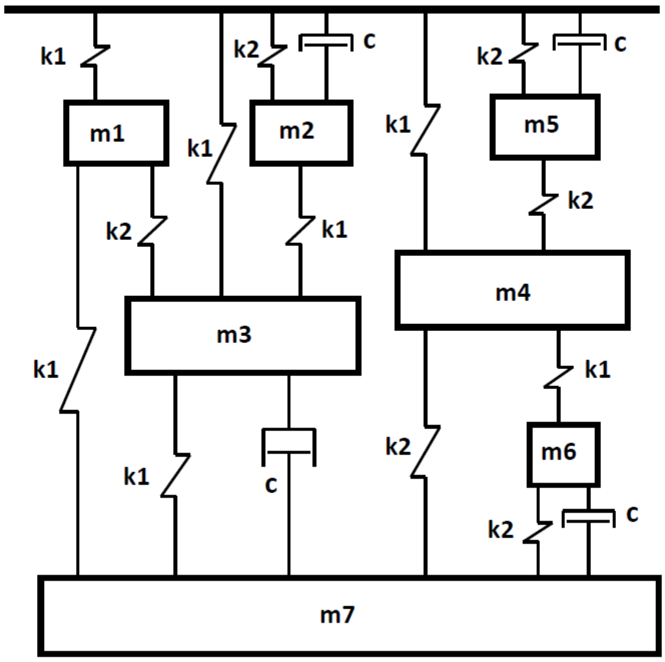


Figure 5. System of example 3

$$[M] = \begin{bmatrix} 1. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 1. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 1. \end{bmatrix} \qquad [C] = \begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 20. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 20. & 0. & 0. & 0. & -20. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 20. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 20. & -20. \\ 0. & 0. & -20. & 0. & 0. & -20. & 40. \end{bmatrix}$$

$$[K] = \begin{bmatrix} 4. & 0. & -2. & 0. & 0. & 0. & -1. \\ 0. & 3. & -1. & 0. & 0. & 0. & 0. \\ -2. & -1. & 5. & 0. & 0. & 0. & -1. \\ 0. & 0. & 0. & 6. & -2. & -1. & -2. \\ 0. & 0. & 0. & -2. & 4. & 0. & 0. \\ 0. & 0. & 0. & -1. & 0. & 3. & -2. \\ -1. & 0. & -1. & -2. & 0. & -2. & 6. \end{bmatrix} \times 10^4$$

Again, the damping matrix was calculated according to eq. (20) with $\omega = 3.1416\,rad/s$ and the command numpy.polyfit in Python language was used in each element of $[H_N]^{-1}$ , $[M]$ and $[K]$.The results are shown in Table 2

Comparing the results on tables 1 and 2 with the matrices $[M]$ , $[C]$ and $[K]$ of examples 2 and 3, respectively, it can be concluded that they are quite good.

Table 2. Results for Mass, Damping and Stiffness Matrices of Example 3

| $(i,j)$ | $M(i,j)$ | $C(i,j)$ | $K(i,j)$ |
|---|---|---|---|
| (1,1) | 1.0000 | -1.3553 e-16 | 4.000 e+04 |
| (1,3) | -6.2040e-12 | -2.5404 e-16 | -2.0000 e+04 |
| (1,7) | -3.1020 e-12 | 6.0987 e-16 | -1.0000 e+04 |
| (2,2) | 1.0000 | 20.0000 | 3.0000 e+04 |
| (2,3) | -3.1020 e-12 | -2.7756 e-16 | -1.000 e+04 |
| (3,3) | 1.0000 | 20.0000 | 5.0000 e+04 |
| (3,7) | -3.1020 e-12 | -20.0000 | -1.000 e+04 |
| (4,4) | 1.0000 | 8.1294 e-15 | 6.0000 e+04 |
| (4,5) | -6.2040 e-12 | -1.0850 e-15 | -2.0000 e+04 |
| (4,6) | -3.1020 e-12 | -1.0853 e-15 | -1.0000 e+04 |
| (4,7) | -6.2040 e-12 | 1.3448 e-16 | -2.000 e+04 |
| (5,5) | 1.0000 | 20.0000 | 4.0000 e+04 |
| (6,6) | 1.0000 | 20.0000 | 3.0000 e+04 |
| (6,7) | -6.2040 e-12 | -20.0000 | -2.0000 e+04 |
| (7,7) | 1.0000 | 40.0000 | 6.0000 e+04 |

## 6 Conclusions

A formulation based on the Complex Frequency Response Matrix and the Undamped Response Matrix was used in order to estimate the Mass, Damping and Stiffness Matrices of spring-mass-damper systems in the frequency-domain. The Damping Matrix can be calculated explicitly and the Mass and Stiffness matrices are estimated by the Least Square Method. It was shown that the equations for the fitting problem are uncoupled and the Mass and Stiffness elements of the corresponding column and row can be calculated separately . Three examples were used for illustrate the methodology and the results were quite good in all of them. This approach can be adopted for different kinds of damping and continuous structural systems, waves propagation, for example, see Mansur [7] .

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