Estimation of the Mass, Stiffness and Damping Matrices in the Frequency-Domain by Nonlinear Regression Techniques

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Abstract. A frequency-domain method for estimating the mass, stiffness and damping matrices of the mass-spring-dumper model is presented. The developed estimation is based on the Least Squares Method in the Nonlinear Regression Approach, where the input data are the elements of the Complex Frequency Response Inverse Matrix and frequencies, which were chosen randomly from the range of frequencies. The amount of frequencies for an accurate result is described in this work. It is shown that each element of the mass, stiffness and damping matrices can be estimated independently, using the corresponding element of the Complex Frequency Response Inverse Matrix, when the nonlinear regression is adopted properly. Although the method is developed for viscous damping, it can be generalized for other types of damping, material or hysteresis, for example. Three examples are employed in order to illustrate the applicability of the proposed method and the results are quite accurate.

Keywords: Frequency-Domain, Nonlinear Regression, Complex Frequency Response Matrix

1 Introduction

A multi-degree-of-freedom (MDOF) structural system is usually modeled in terms of mass, stiffness and damping matrices. The governing equation in the time-domain, Clough and Penzien [1] and Craig [2], considering N degrees-of-freedom is

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = \{f(t)\}. \tag{1}$$

with the initial conditions

$$\{x(0)\} = \{x_0\} \quad and \quad \{\dot{x}(0)\} = \{\dot{x}_0\}. \tag{2}$$

In eq. (1), $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices (N x N), $\{\ddot{x}(t)\}$, $\{\dot{x}(t)\}$ and $\{x(t)\}$ are the accelerations, velocities and displacements vectors (N x 1), while$\{f(t)\}$ is the external forces vector (N x 1). In eq. (2), $\{x_0\}$ and $\{\dot{x}_0\}$ are the initials displacements and velocities, respectively.

One can assume that the mass matrix is diagonal and the damping is considered viscous. The elements of the matrices must be properly allocated in each matrix, according to the structural system disposition. If the external forces $\{f(t)\}$ are equal to zero, the movement is natural and called free vibration. Then, eq. (1) becomes

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = 0. \tag{3}$$

For a single-degree-of-freedom (SDOF) system, the undamped natural frequency ω_n is

$$\omega_n = \sqrt{k/m}. \tag{4}$$

The viscous damping factor ζ is

$$\zeta = \frac{c}{c_{cr}} \qquad and \qquad c_{cr} = 2m\omega_n = \frac{2k}{\omega_n} = 2\sqrt{km}. \tag{5}$$

The damped natural frequency ω_d is

$$\omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{6}$$

For the underdamped SDOF system ($\zeta < 1$), the analytical response of eq. (3) is

$$x(t) = \exp\left(-\zeta\omega_n t\right)\left[x_0\cos(\omega_d t) + \left(\frac{\dot{u}_0 + \zeta\omega_n u_0}{\omega_d}\right)\sin(\omega_d t)\right]. \tag{7}$$

For an undamped MDOF system, the governing equation of the free vibration is

$$[M]\{\ddot{x}(t)\} + [K]\{x(t)\} = 0. \tag{8}$$

Considering the movement as harmonic, the displacement has the following form

$$\{x(t)\} = \{X\}\cos\left(\omega t - \alpha\right). \tag{9}$$

Substituting eq. (9) into eq. (8), the eigenvalue problem appears

$$\left([K] - \omega^2[M]\right)\{X\} = 0. \tag{10}$$

The eigenvalues of eq. (10) are the squared natural frequencies of the system and the eigenvectors are the modes of vibration, which are obtained substituting each eigenvalue into eq. (10), and the Mode Superposition Method can be used. If the damping components are added, for special types of damping, the system of equations can become uncoupled. These kinds of damping are called orthogonal, classical, modal or proportional, Craig [2]. One particular proportional damping is the Rayleigh Damping

$$[C] = \alpha[M] + \beta[K]. \tag{11}$$

where α and β are constants.

2 Frequency Domain Approach

Applying the Fourier Transform into eq. (1), it is converted from time-domain to frequency-domain, Clough and Penzien [1] and Craig [2], and becomes

$$(-\omega^2[M] + i\omega[C] + [K])\,\{X(\omega)\} = \{F(\omega)\}\,. \tag{12}$$

where ω is the independent variable in the frequency-domain, $\{X(\omega)\}$ the Fourier Transform of $\{x(t)\}$ and $\{F(\omega)\}$ the Fourier Transform of $\{f(t)\}$. Explaining $\{X(\omega)\}$ in eq. (12), it is obtained

$$\{X(\omega)\} = (-\omega^2[M] + i\omega[C] + [K])^{-1}\,\{F(\omega)\}\,. \tag{13}$$

The expression $(-\omega^2[M] + i\omega[C] + [K]^{-1})$ from eq. (13) is called Complex Frequency Response Matrix $[H(\omega)]$, Transfer Matrix or Complex Frequency Response Function (CFRF) and has many applications in the frequency-domain.

$$[H(\omega)] = (-\omega^2[M] + i\omega[C] + [K])^{-1} \quad or \quad [H(\omega)]^{-1} = (-\omega^2[M] + i\omega[C] + [K]) \tag{14}$$

The Complex Frequency Response Matrix can be obtained by Modal Testing. The structure under analysis must be excited by a known type of force, and the applied excitation force and resulting response vibrations, typically accelerations, are both measured properly by instruments and, after a software data processing, the Complex Frequency Response Function (CFRF) data can be obtained. There are two common methods of excitation: impact hammer and modal shaker, Allemang et. al. [3].

For a system with one-degree-of-freedom, with $m = 3\,kg$, $\zeta = 0.05$ and $k = 2700\,N/m$.

3 Nonlinear Regression

Regression analysis uses statistical inferences equations, according to Seber and Wild [4], which take the form:

$$\{Y\} = f(\{X\}\,, \{\beta\}) + \{\epsilon\}\,. \tag{15}$$

where: $\{Y\}$ = a vector of response variables, $\{X\}$ = a vector of predictors or explanatory variables, $\{\beta\}$ = a vector of parameters, f = a known regression function, $\{\epsilon\}$ = an error term, with zero mean.

In eq. (15), if f is nonlinear in terms of $\{\beta\}$, the problem is called nonlinear regression analysis, which is the case of eq. (14), where the relation between the matrices $[H]^{-1}$ and $[M]$ is nonlinear with respect to frequency ω, in fact is of power 2. The most commons techniques to solve nonlinear regression are the Least-Squares Estimation and the Maximum-Likelihood Estimation. In this work The Least-Squares Estimation is adopted.

The least squares estimator of $\{\beta\}$ is called as $\left\{\hat{\beta}\right\}$ and is the point where $f(\left\{\hat{\beta}\right\})$ is closest to $\{Y\}$ in the sample space. The least squares estimator is calculated from minimizing the residual sum of equations

$$S(\beta)) = \sum_{i=1}^{n} \{y_i - f(x_i, \beta)\}^2 \tag{16}$$

where n is the size of the data set.

After calculating the differential of S with relation to each β, the system of equations is

$$\left.\frac{\partial\,\{S(\beta)\}}{\partial \beta_j}\right|_{\beta=\hat{\beta}} = 0, \qquad for \qquad j = 1, ..., m \tag{17}$$

where m is the number of parameters β.

Considering eq. (16) and eq. (17), the final system of equations turns

$$\sum_{i=1}^{n} \frac{\partial f(x_i,\beta)}{\partial \beta_j} \{y_i - f(x_i,\beta)\}\bigg|_{\beta=\hat{\beta}} = 0, \qquad for \qquad i = 1,...,n \qquad and \quad j = 1,...,m \tag{18}$$

or in Matrix Form

$$[V(\hat{\beta}]\,\{\hat{\epsilon}\} = 0 \tag{19}$$

where $[V(\hat{\beta})] = \frac{\partial f(x_i,\beta)}{\partial \beta_j}$

4 Examples of Application

Three examples of application are shown in order to illustrate the Methodology, which are in Ferro [5]. The first one is a single-degree-of-freedom system, the second one is a system with two degrees-of-freedom and the third a system with seven degrees-of-freedom. Python language was used in order to fit the data, where the input data are each element of the inverse of the matrix $[H]$ and four aleatory frequencies. At least four frequencies must be used for a good accuracy, because the relation between the matrices $[H]$ and $[M]$ and the frequency is of power 2. In fact, each element of $[H]^{-1}$, $[M]$, $[C]$ and $[K]$ are related separately and independently, according to the following relation

$$H^{-1}(\omega)(i,j) = -\omega^2 M(i,j) + i\omega C(i,j) + K(i,j) \tag{20}$$

The command polyfit.numpy of Python was used to calculate each $M(i,j)$, $C(i,j)$ and $K(i,j)$ from eq. (20) when $H^{-1}(i,j)$ is known and four frequencies are randomly chosen.

4.1 Example 1 - SDOF System

In this example, the system has $m = 3\,kg$, $\zeta = 0.05$ and $k = 2700\,N/m$. Then, the expected values of m and k are 3 and 2700 and for c, the value is

$$c = 2 \times \zeta \times \sqrt{k \times m} = 2 \times 0.05 \times \sqrt{2700 \times 3} = 9\,N.s/m \tag{21}$$

The fitting equation is

$$H^{-1} = -\omega^2 m + i\omega c + k \tag{22}$$

The command numpy.polyfit of Python language was used to fit eq. (22) with 4 random frequency values, which are $\omega_1 = 0.0\,rad/s$, $\omega_2 = 0.6545\,rad/s$, $\omega_3 = 1.3090\,rad/s$ and $\omega_4 = 1.96350\,rad/s$.

The fitting results are $m = 3.0000 \quad c = 9.0000 \quad and \quad k = 2700.0000$, which indicate an excellent estimation.

4.2 Example 2 - 2 DOF System

In this example, from Craig [2], page 357, and shown in Fig. 1, where $m1 = m2 = 1\,kg$, $c1 = 0.6284\,N.s/m$, $c2 = 0.0628\,N.s/m$, $k1 = 987\,N/m$ and $k2 = 217\,N/m$, the mass $[M]$, damping $[C]$ and stiffness $[K]$ matrices are

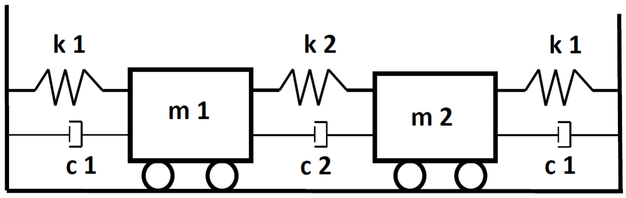

Figure 1. System of example 2

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [C] = \begin{bmatrix} 0.6912 & -0.0628 \\ -0.0628 & 0.6912 \end{bmatrix} \qquad [K] = \begin{bmatrix} 1204 & -217 \\ -217 & 1204 \end{bmatrix}$$

Again, the command numpy.polyfit was used for each element of $H^{-1}(\omega)$ and the four frequencies $\omega_1 = 0.0\,rad/s$, $\omega_2 = 20.9440\,rad/s$, $\omega_3 = 41.8879\,rad/s$ and $\omega_4 = 62.8319\,rad/s$. The fitting results are shown in Table 1.

Table 1. Results for Mass, Damping and Stiffness Matrices of Example 2

(i, j)	$M(i, j)$	$C(i, j)$	$K(i, j)$
(1,1)	1.0000	0.6912	1204.0000
(1,2)	-1.5468 e-17	-0.0628	-217.0000
(2,1)	-1.5468 e-17	-0.0628	-217.0000
(2,2)	1.0000	0.6912	1204.0000

4.3 Example 3 - 7 DOF System

In this example, from Chen [6], and shown in Fig. 2, the parameters are $m1 = m2 = m3 = m4 = m5 = m6 = m7 = 1,0\,kg$, $k1 = 10000\,N/m$, $k2 = 20000\,N/m$ and $c = 20\,N.s/m$. The mass $[M]$, damping $[C]$ and stiffness $[K]$ matrices of the system are

$$[M] = \begin{bmatrix} 1. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 1. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 1. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 1. \end{bmatrix} \qquad [C] = \begin{bmatrix} 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 20. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 20. & 0. & 0. & 0. & -20. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 20. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 20. & -20. \\ 0. & 0. & -20. & 0. & 0. & -20. & 40. \end{bmatrix}$$

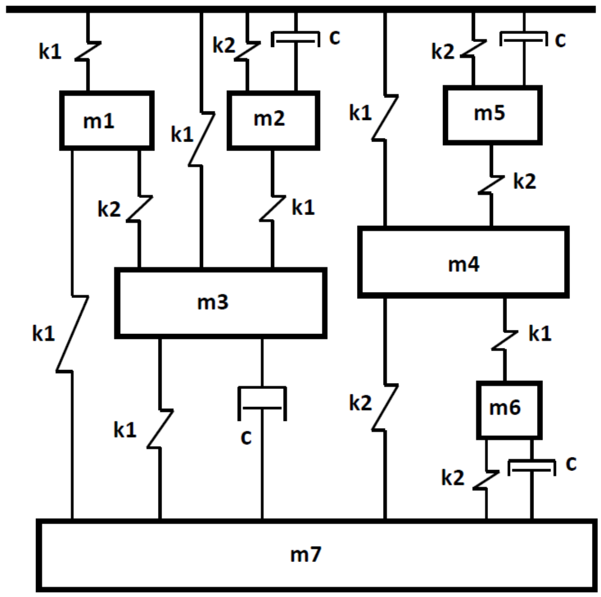

Figure 2. System of example 3

$$[K] = \begin{bmatrix} 4. & 0. & -2. & 0. & 0. & 0. & -1. \\ 0. & 3. & -1. & 0. & 0. & 0. & 0. \\ -2. & -1. & 5. & 0. & 0. & 0. & -1. \\ 0. & 0. & 0. & 6. & -2. & -1. & -2. \\ 0. & 0. & 0. & -2. & 4. & 0. & 0. \\ 0. & 0. & 0. & -1. & 0. & 3. & -2. \\ -1. & 0. & -1. & -2. & 0. & -2. & 6. \end{bmatrix} \times 10^4$$

After using the command numpy.polyfit from Python language in each element of $[H]^{-1}$ and the four frequencies $\omega_1 = 0.0\,rad/s$, $\omega_2 = 20.9440\,rad/s$, $\omega_3 = 41.8879\,rad/s$ and $\omega_4 = 62.8319\,rad/s$, the fitting results are shown in Table 2.

Comparing the results on tables 1 and 2 with the correspondent matrices $[M]$, $[C]$ and $[K]$, it can be concluded that they are quite good.

5 Conclusions

A Nonlinear Regression technique was used in order to estimate the Mass, Damping and Stiffness Matrices of spring-mass-damper systems in the frequency-domain. It was shown that the equations for the fitting problem is uncoupled. Then, each element of the matrices can be calculated independently and separately. Three examples were used in order to illustrate the methodology and the results were quite good in all of them. This approach can be adopted for different kind of damping and continuous structural systems, waves propagation, for example, see Mansur [7] .

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Table 2. Results for Mass, Damping and Stiffness Matrices of Example 3

(i,j)	$M(i,j)$	$C(i,j)$	$K(i,j)$
(1,1)	1.0000	0.0000	4.0000 e+04
(1,3)	-1.5510 e-14	0.0000	-2.0000 e+04
(1,7)	-7.7550 e-15	0.0000	-1.0000 e+04
(2,2)	1.0000	20.0000	3.0000 e+04
(2,3)	-7.7550 e-15	0.0000	-1.0000 e+04
(3,3)	1.0000	20.0000	5.0000 e+04
(3,7)	-7.7550 e-15	-20.0000	-1.0000 e+04
(4,4)	1.0000	0.0000	6.0000 e+04
(4,5)	-1.5510 e-14	0.0000	-2.0000 e+04
(4,6)	-7.7550 e-15	0.0000	-1.0000 e+04
(4,7)	-1.5510 e-14	0.0000	-2.0000 e+04
(5,5)	1.0000	20.0000	4.0000 e+04
(6,6)	1.0000	20.0000	3.0000 e+04
(6,7)	-1.5510 e-14	-20.0000	-2.0000 e+04
(7,7)	1.0000	40.0000	6.0000 e+04

6 References

[1] R. W. Clough and J. Penzien. *Dynamics of Structures, 3rd ed.. Computers and Structures, Inc, 1995*.
[2] R. R. Craig. *Structural Dynamics - An Introduction to Computer Methods. John Wiley and Sons, Inc, 1981*.
[3] R. J Allemang, D.L. Brown, R.W. Rost. *Multiple Input Estimation of Frequency Response Functions. Proceedings of the 2nd International Modal Analysis Conference, 1984*.
[4] G.A.F. Seber, C.J. Wild. *Nonlinear Regression, John Wiley and Sons, Inc, 2003*.
[5] Ferro, M. A. C. *Estimation of Mass, Damping and Stiffness Matrices from Frequency Response Function. Master in Science Dissertation. Civil Engineering Program, Federal University of Rio de Janeiro, Alberto Luiz Coimbra Institute for Graduate Studies and Research in Engineering. 1997*.
[6] S. Y. Chen, M. S. Ju, Y. G. Tsuei. *Estimation of Mass, Stiffness and Damping Matrices from Frequency Response Functions. Transactions of the ASME, v. 118, pp. 78-82. July 1996*.
[7] Mansur, W. J. *A Time Stepping Technique to Solve Wave Propagation Problems Using the Boundary Element Method. Ph. D. Thesis. Faculty of Engineering and Applied Science, Civil Engineering, University of Southampton. 1983*.