eXtended Isogeometric Analysis - a numerical investigation of simulation of two-dimensional elastic fracture.

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Abstract. When compared with the Finite Element Method (FEM), the Isogeometric Analysis IGA presents as advantages the exact representation of the problem geometry, the possibility of using the same basis functions to describe the geometry as well as the solution field and a straightforward and automatic scheme to refine the mesh. The eXtended Isogometric Analysis (XIGA) enlarges the approximate space of IGA incorporating customized functions, by the enrichment strategy of Generalized/eXtended Finite Element Method (G/XFEM). The present work evaluates the performance of the eXtended Isogeometric Analysis (XIGA) in the context of the Linear Elastic Fracture Mechanics. Several parameters related with the enrichment strategy, such as the region to be enriched, the number of nodes enriched, the type of enrichment functions are combined with the special features of IGA and the behavior of the solution is investigated. This work is a first step of the expansion of the INSANE (INteractive Structural ANalysis Environment) platform, originally developed with FEM functionalities and later expanded to G/XFEM simulations. It is an open source software developed at the Structural Engineering Department of the Federal University of Minas Gerais.

Keywords: eXtended Isogeometric Analysis Method; Computational Mechanics; Fracture Mechanics; Object Oriented Programming; JAVA.

1 Introduction

Isogeometric Analysis (IGA), introduced by Hughes et al. [1], seeks to combine the physical representation of problem with the approximate functions that reproduce the solution field. It uses base functions capable of performing this coupling, such as B-Splines and non-uniform rational B-splines (NURBS) functions. Thus, it is possible to describe the exact geometry of the problem being analyzed, in addition of reducing the computational effort arising from the mesh generation and processing task. Indeed, the mesh refinement process in IGA is automated, which enables the creation of simple meshes to describe the geometry of the physical model and more complex meshes to represent the solution field.

Another advantage of IGA is the smoother feature of the approximate functions, allowing them to reach higher levels of continuity of the derivatives that represent the problem solution, when compared to the FEM.

The aforementioned advantages can become drawbacks, to represent discontinuous information in a continuous medium, such as transition between materials or cracks. Within this context, eXtended Isogemetric Analysis (XIGA) proposed by Benson et al. [2] and later improved by Luycker et al. [3], combines the exact description of the geometry and the smoothness of the basic functions of IGA with the explicit enrichment strategy from the Generalized/eXtended Finite Element Method (G/XFEM) studied in Oden et al. [4], Strouboulis et al. [5] and Duarte et al. [6]. Similar to G/XFEM, an XIGA enriches the solution of the problem described by IGA with functions that notoriously represent well the discontinuous behavior to be incorporated. As a result, the smooth part of the solution is obtained by only IGA and the discontinuous part is described by the enrichment of the Partition of Unity base functions also given by IGA.

The present paper aims to discuss some results obtained in comparative analysis performed by XIGA and G/XFEM analyses in the context of linear elastic fracture mechanics. The convergence of the solution and the conditioning of the methods are evaluated under different types of polynomial degrees of the approximation and the combination of discontinuous and singular enrichment functions.

2 Isogeometric Analisys

The first works carried out in IGA used Non-Uniform Rational B-Splines as bases functions, because, according to Hughes et al. [1], the NURBS type functions are the most widely used in CAD systems. An isogeometric analysis, however, is not limited to the use of NURBS. Other functions can be used, as long as the geometry and solution of the analyzed model are in accordance with the description. We take, for example, functions that allow the local enrichment of the mesh, such as T-Splines, introduced by Sederberg et al. [7] and later applied in IGA by Bazilevs et al. [8].

IGA's field of application is very wide, especially when it comes to engineering problems. In Hughes et al. [1], IGA was primarily applied to linear structural problems and fluid flow problems. In Nguyen [9], IGA was discussed for classical solid mechanics problems. Other very common uses of IGA are in plate analysis (Veiga et al. [10]), shells (Uhm and Youn [11], Kiendl et al. [12], Benson et al. [13, 14, 15], Echter et al. [16]), composites (Ghafari and Rezaeepazhand [17]), hydraulic fracturing (Hageman and de Borst [18]) and problems involving the damage mechanics ([19]).

2.1 Overview

In isogeometric analysis there are two mesh concepts: the control mesh and the physical mesh. The control mesh is composed of points, called control points, which serve as a basis for the construction of NURBS that, in turn, describe the physical mesh. The degrees of freedom of the problem are linked to the control points (Rauen [20]).

IGA shape functions, including NURBS, are built in parametric space from a "*knot vector*". The "*knot vector*" is a vector composed by a non-decreasing set of "*knots*" , which represent points in the parametric space and which constitute the necessary parameters for the creation of shape functions. Each direction of the model, whether one, two or three-dimensional, is composed of a parametric representation through the *knot vector*. In this paper two-dimensional analysis are presented.

For one-dimensional models, the concept of element is the domain between two "*knots*", of different values, within the "*knot vector*", the so-called "*knot span*". In two-dimensional problems, this concept is extended and the element is defined as the space between two "*knots*" in each direction considered (Nguyen [9]).

2.2 Formulation

Equilibrium Equation

Using the principle of virtual work, we have:

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) d\Omega = \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} d\Omega + \int_{\Gamma} \bar{\boldsymbol{t}} \cdot \boldsymbol{v} d\Gamma. \tag{1}$$

where the integral on the left side of the equality is the internal virtual work and the integrals on the right side represent the virtual work of the external forces. $\boldsymbol{v}$ is the field of virtual displacements, kinematically admissible, $\boldsymbol{b}$ are the body forces and $\bar{\boldsymbol{t}}$ are the surface forces, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ represents the stress and strain field respectively. $\boldsymbol{\Omega}$ is the problem domain and $\boldsymbol{\Gamma}$ is the boundary of the problem. In addition, it was considered linear elastic behavior so Hooke's law is assumed.

In this paper, it was used the NURBS basis functions in the IGA context.

Geometry and Displacements

The NURBS functions R_i, associated with the set of control points $P_i(x, y, z)$, $i = 1, 2, ..., n$ are used to describe the geometry of the problem by:

$$x = \sum_{i=1}^{n} R_i x_i, \qquad y = \sum_{i=1}^{n} R_i y_i. \tag{2}$$

Similar to the geometry description, displacements are approximated by the same NURBS functions:

$$u = \sum_{i=1}^{n} R_i u_i, \qquad v = \sum_{i=1}^{n} R_i v_i. \tag{3}$$

It is important to emphasize that the control points do not necessarily belong to the domain problem, therefore, the degrees of freedom associated with these control points have no physical meaning, serving only as a basis for calculating the displacements that actually occur inside the domain problem.

2.3 Bézier Extraction

The Bézier Extraction (Borden et al. [22]), is a technique that aims to facilitate the implementation of the IGA within a FEM computational system. It is a different and simpler way of writing NURBS and also tends to facilitate numerical integration as it reduces the need for mappings between different IGA meshes. This technique was used in the analysis shown in the present paper, since the implementation of XIGA was carried out on the computational platform INSANE where the FEM and G/XFEM are implemented.

The INSANE (INteractive Structural ANalysis Environment) is a computational system under development by the Department of Structural Engineering (DEES) of the Federal University of Minas Gerais (UFMG), implemented in JAVA programming language and that uses the Object-Oriented Programming paradigm (POO) (Fonseca and Pitangueira [21]). It was chosen because it is a dynamic system, so it allows several expansions and is open to new methods witch with small modifications can be added to its technical framework.

2.4 Refinement

The refinement used in IGA may be performed in a simple automated way, without the need of constant communication of the geometric description of the model. It is done in a way that preserves the description of the initial geometry and also preserves the range of parametric representation. Here, it is taken the advantage of the **k-Refinament**, a special IGA way to refine the mesh. The ideia of this refinament is to increase the degree of approximation, its continuity and the number of elements in the model. The formulation of such refinement can be found in Hughes et al. [1].

3 eXtended Isogeometric Analysis

In xEtended Isogeometric Analysis (XIGA) the space of approximation of the solution obtained by IGA is extended from the enrichment of the basis functions of the control points. It is interesting to carry out this operation in regions, with the presence of localized phenomena that require a more complex description. The enrichment strategy used in XIGA is derived from methods such as the Cloud Method (Duarte [23]), the Partition of Unit Finite Element Method (Babuška and Melenk [24]) and the G/XFEM (Duarte et al. [6]). The difference between these methods and XIGA is the construction of the partition of unity, PU, which in the latter is defined by the set of NURBS functions (Tran et al. [25]).

Formulation

The local enrichment functions, $L_{ji}(\boldsymbol{x})$, specific to each problem to be analyzed, are q_j linearly independent functions, where q is the number of functions used to enrich the PU, defined for each base function or point of control, $\boldsymbol{x}_j$:

$$\{L_{ji}(\boldsymbol{x})\}_{i=1}^{q} = \{L_{j1}(\boldsymbol{x}), L_{j2}(\boldsymbol{x}), ..., L_{jq}(\boldsymbol{x})\}, \quad with \quad L_{j1}(\boldsymbol{x}) = 1. \tag{4}$$

The set of local enrichment functions multiplies the PU, which in the case of XIGA is formed by the NURBS functions, in order to create the set of enriched functions, $\phi_{ji}(\boldsymbol{x})$:

$$\{\phi_{ji}(\boldsymbol{x})\}_{i=1}^{q} = R_j^p(\boldsymbol{x}) \times \{L_{ji}(\boldsymbol{x})\}_{i=1}^{q} \qquad \textit{no sum in } j. \tag{5}$$

A generic approximation $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ is obtained through the linear combination of the shape functions:

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \sum_{j=1}^{n} R_j^p(\boldsymbol{x}) \left\{ \boldsymbol{u}_j + \sum_{i=2}^{q} L_{ji}(\boldsymbol{x}) \boldsymbol{b}_{ji} \right\}. \tag{6}$$

where $\boldsymbol{u}_j$ and $\boldsymbol{b}_{ji}$ are nodal parameters associated with each component $R_j^p(\boldsymbol{x})$ from IGA and $R_j^p(\boldsymbol{x}) \cdot L_{ji}(\boldsymbol{x})$ from XIGA, respectively.

Enrichment Functions

In the present work, under the approach of Fracture Mechanicals, three kind of enrichment funtions are adopted. They are the Heaviside functions to describe the discontinuity of the displacement field, polynomial functions to represent the smooth part of the solution and OD functions (Duarte et al. [6]) to introduce the singularity or the stress field around the crack tip. Its formulations can be found in Duarte et al. [6] and Moës et al. [26].

4 Numeric Example

The plate depicted in Figure 1(a) is considered here under plane stress conditions. The adopted material has the modulus of elasticity (E) equal to 1 c.u. (consistent units) and Poisson's coefficient (ν) equal to 0.3.

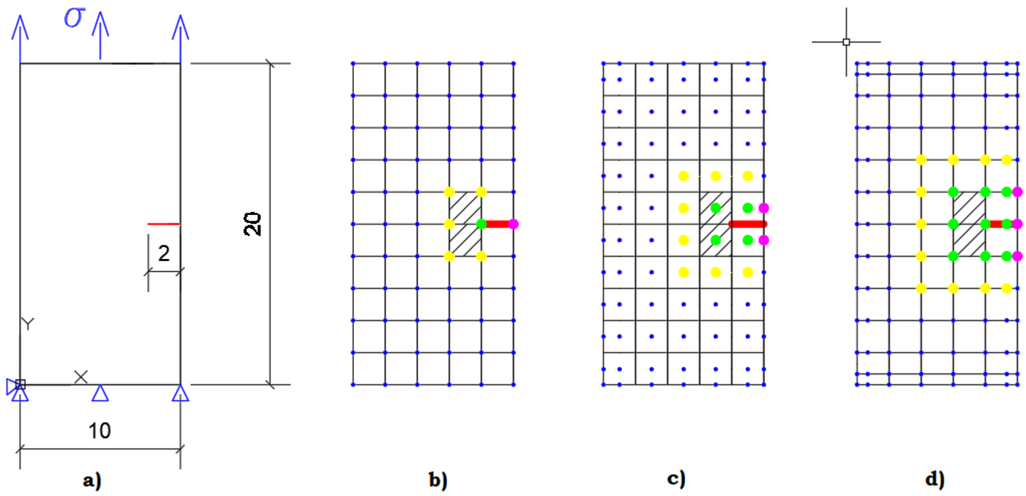

Figure 1. Plate Model Tensioned with Pre-Crack: a) The Model. b) Mesh for p=1 (XIGA and G/XFEM). c) Mesh for p=2 (XIGA). d) Mesh for p=3 (XIGA). The circles are the nodes/control points, the red line is the crack, the pink circles are the nodes/control points enriched with Heaviside function in the three types of analyses (*A*, *B* and *C*), the green circles are the nodes/control points enriched with the functions capable of describing the two crack opening modes in the *B* and *C* analyses and the yellow circles are the additional nodes/control points enriched with crack functions in the *C* analysis.

According to Alves [27], the reference solution of this problem was obtained using the software *ANSYS*® with 12087 mesh of quadrilateral p-elements, considering the symmetry of the problem and from the extrapolation to polynomial approximations of degree p=1, 2 and 3. The strain energy result found for the whole domain was 10.98326746 c.u. and the stress component σ_{yy} at the point x=7.99944 and y=10.00 is 66.769 c.u..

For the analysis performed by G/XFEM a mesh of 50 bilinear quadrilateral finite elements (4 nodes) was used. For the XIGA analysis the physical mesh has the same 50 quadrilateral elements, but there are 3 different control meshes in XIGA: the first one is the same as G/XFEM mesh where the nodes coincides with control points only, representing a linear analysis (p=1); the second one is associated with the quadratic analysis (p=2), and the last represents the cubic analysis (p=3). To reproduce the quadratic and cubic analysis in G/XFEM, polynomial enrichment functions were used to enrich the PU, so the mesh keeps unchanged. The crack was described in the mesh by a combination of Heaviside functions (Moës et al. [26]), to describe the discontinuity of the displacement field and the OD enrichment functions, used in Duarte et al. [6], to introduce the singularity or the stress field around the crack tip.

Figure 1 shows the physical mesh used in both analysis and the control meshes used in XIGA analysis. For numerical integration, 12x12 points are used in the hatched elements in Figure 1, aiming to accurately represent the proposed approximate solutions. In the other elements only 4x4 integration points were used.

Three types of analyses were adopted, *A*, *B* and *C*. In analysis *A*, only the discontinuity is inserted in the model, through the Heaviside function enriching the pink points showed in Figure 1(b), 1(c) and 1(d). In analysis *B*, the OD functions, capable of describing the Mode I and Mode II of crack opening, are inserted as PU enrichment only for the nodes/control points around the crack tip, green points in Figure 1(b), 1(c) and 1(d). In the *C* analysis the OD functions are used again but one more layer of nodes/control points is enriched with them, the yellow ones in Figure 1(b), 1(c) and 1(d).

The graphs of Figures 2, 3 and 4 show the results of the analyses in terms of error in strain energy, error in stress component σ_{yy} near to the crack tip and condition number of the stiffness matrix provided by the investigated methods, as a function of the number of degrees of freedom, respectively.

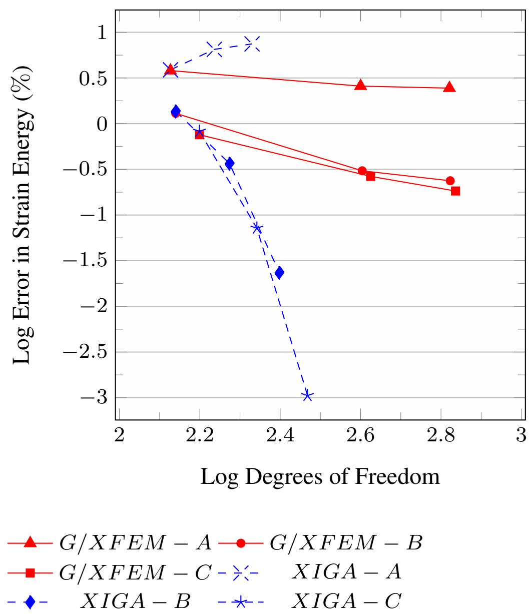

Figure 2. Log Error in Strain Energy for the various models presented.

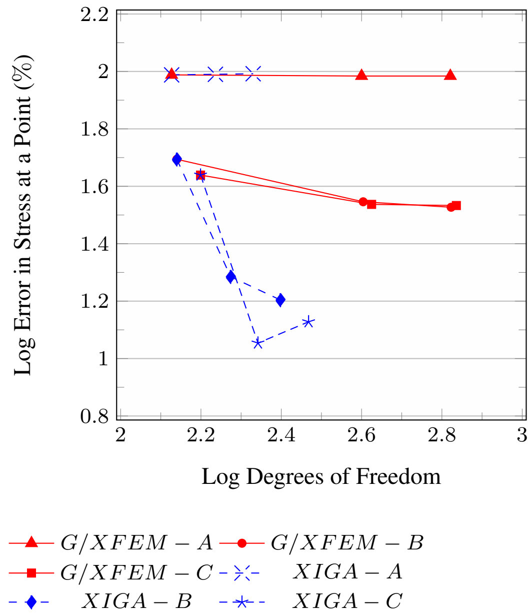

Figure 3. Error in Stress at a Point for the various models presented.

Analyzing the results of strain energy and stress (σ_{yy}) next to the crack tip in Figures 2 and 3, it is possible to note that there is a coincidence of results from the G/XFEM and XIGA when the polynomial approximation degree is p=1. This result was already expected, as the NURBS shape functions coincide with the Lagrangian shape function in this case. When higher degrees are used (p=2, p=3), there is an improvement in the results, as the number of nodes/control points increases, in both methods. The advantage of XIGA in this regard, is the achievement of a satisfactory solution, as it is enriched with functions capable of representing the crack opening modes, increasing few degrees of freedom of the model, when compared to G/XFEM. Surely, a small number of DOFs reduces the computational cost of the problem. Nonetheless, other factors such as the complexity of the algorithm of creating the XIGA approximation should also be investigated. Additionally, in the higher order XIGA analysis, values of strain energy and stress near the crack tip were found closer to the reference. The results found in the analyses *A* can be explained by the poor description of the approximate solution around the crack. Indeed, only Heaviside functions are used, which doesn't contribute to simulate the expected singularity of the stress field in the crack tip. Particularly, for XIGA, this poor description is combined with the increase of the number of control points enriched with the Heaviside functions, which seems to deteriorate the accuracy of the solution. In

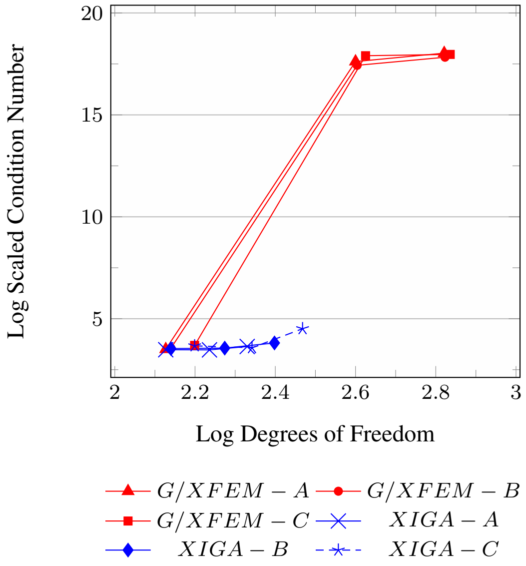

Figure 4. Scaled Condition Number for the various models presented.

terms of the stress component σ_{yy}, a similar behavior is observed in Figure 3. The only difference is the result for XIGA-C with p=3, which presents a slightly higher error when compared with the previous point of the same curve. Nevertheless the XIGA-C still provides more accurate values than the other simulations. Numerical perturbations from the computation of stress component σ_{yy} on the element edge and from the numerical integration need to be investigated. This investigation and the evaluation of the SIF will be part of the next steps of this work.

Regarding the condition number presented in Figure 4, it is possible to see that as the degree of approximation increases in the G/XFEM models, the condition number is less stable than in the XIGA models. Thus, matrices solved in G/XFEM are much more likely to add numerical errors to the model since they deal with linear dependence of their variables and small perturbations in the stiffness matrix can lead to large changes in the vector solution making the results numerically less reliable.

5 Conclusions

XIGA is a methodology that seeks to combine the advantages of accurate geometry description and the smoother feature of its shape functions coming from IGA with the ease way of inserting discontinuities in the model by PU enrichment coming from G/XFEM. In the present work, a cracked painel is used to compare different configurations of discretizations by G/XFEM and XIGA.

In the analyses carried out, it was possible to observe the equivalence between G/XFEM and XIGA for linear approximations. For higher order approximations, in XIGA, it was possible to achieve better results in terms of strain energy and stress near the crack tip without substantially increasing the number of degrees of freedom. In addition, the non polynomial character of the PU built with NURBS functions provides smaller and better behaved condition numbers when compared with the ones obtained with the polynomial enriched G/XFEM.

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