A Finite Difference Energy Method to Arbitrary Grids Applied the Plate Bending Problems

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Abstract. The finite difference energy method (FDEM), although already used in several structural problems so far, including dynamic and nonlinear analysis, has always been applied to regular grids, making it limited to applications with geometries formed by (or mapped to) rectangles. In order to make it competitive with other methods, such as finite elements and meshless, it is necessary to generalize the FDEM to arbitrary domains and boundaries, endowing it with ability to manipulate arbitrary grids. Thus, this paper presents a FDEM formulation for regular and irregular grids applied to the thin plate bending problem. The coefficients were obtained at each point of the arbitrary grid by expanding the Taylor series. The results show that using the proper choice of points for the stencils we can keep the order of approximation of the case regular, but now allowing the generalization of the use of the method that was only applied to rectangular grids.

Keywords: finite difference, energetic method, arbitrary meshes, plate bending

1 Introduction

Until the first half of the 20th century, the finite difference method (FDM) was the main numerical method used to find approximate solution of differential equations using expansion of the Taylor series on uniform grid of points of the discretized domain. However, with the emergence of the finite element method (FEM) at the end of the 50s, provided by the development of computer, the FDM lost its prominence in problem solving, as the FEM proved to be of more general use, regardless of the domain, boundary conditions, behavior, loading etc. Unlike FEM, the classical FDM has limitations in its development and application, mainly related to the domain and discretization of the problem, being limited to a grid of structured points and domains, in general, rectangular or mapped in rectangles.

Among the methods that can be considered variants of the classic FDM, the finite energetic differences method (FDEM) can be highlighted, which is based on the same ideas as the classic method in relation to the discretization strategy, but uses the weak form or variational of the problem, in the same way as the FEM, thus reducing the order of the derivatives that appear in the equilibrium equations of the problem. This method has great potential for application, however, like FDM, it has as a gap the difficulty of being applied to problems of complex geometry and curved contours, such as non-rectangular plates.

The generalized finite differences method (GFDM) is an attempt, developed from studies by several authors (*e.g.* Liszka and Orkisz [1], Gavete et al. [2]), to apply FDM to non-rectangular domains and grids of arbitrary points. These studies have gradually allowed the application of this method in the solution and analysis of problems with complex geometries. In parallel, variants of the FEM have been proposed to adequately account for specific situations of cracks and large plastic deformations, among others, giving rise to a set of methods based on point grids such as GFDM, as opposed to those based on elements, called mesh free or meshless methods, which can have general applications such as FEM.

FDEM has already been used in several applications in structural modeling, such as in nonlinear static and dynamic analysis of thin and thick plates bending made by Graça [3], nonlinear static and dynamic bending of bars by Lima [4] and analysis of cylindrical shells by Mittelbach [5]. Although the FDM has reached its generalized version through the advent of the GFDM, an generalized version of the FDEM requires additional techniques and care, such as the need for integration in the subdomains, with the consequent definition of regions and integration techniques. In the same way as FDM in its generalized form, FDEM is expected to be able to handle any domains and arbitrary grids of points, bringing with it the advantage of using lower order derivatives and the simpler consideration of boundary conditions natural, what makes the method more advantageous.

2 Finite Difference Energy Method to Arbitrary Grids

2.1 Approximation of Derivatives

Given a function $f : \Omega \rightarrow \mathcal{R}$, with the domain $\Omega \subset \mathcal{R}^2$ defined by grid of scattered points, using the technique of the GFDM, it is possible to approximate second-order derivatives in a point x_o using the collocation method in 5 neighborhood points. However, the use of more points brings as advantage a lesser difficulty in having uniqueness in the generation of the coefficients via matrix inversibility, requiring, however, the use of least squares method. But for circumvent such a situation we can increase equally the terms of the Taylor series, improving the convergence rate . Thus, around a central balance point $P_o(x_o, y_o)$ (*star*), it is possible, for example, to 8 selected points $\mathbf{x}_i = (x_i, y_j)$, $i = 1...8$ from the neighborhood. Thus, according to Perrone and Kao [6] and Liszka and Orkisz [1], the derivates at each point can be obtained through the from *Taylor* series expansion:

$$\begin{aligned} f_i \quad = \quad & f_o + h\frac{\partial f_o}{\partial x} + k\frac{\partial f_o}{\partial y} + \frac{h^2}{2}\frac{\partial^2 f_o}{\partial x^2} + \frac{k^2}{2}\frac{\partial^2 f_o}{\partial y^2} + hk\frac{\partial^2 f_o}{\partial x \partial y} + \\ & \frac{hk^2}{2}\frac{\partial^3 f_o}{\partial x \partial y^2} + \frac{h^2 k}{2}\frac{\partial^3 f_o}{\partial x^2 \partial y} + \frac{h^2 k^2}{4}\frac{\partial^4 f_o}{\partial x^2 \partial y^2} + 0(\Delta^3) \end{aligned} \tag{1}$$

being:

$$f_i = f(x_i, y_i); \quad f_o = f(x_o, y_o); \quad h = x - x_o; \quad k = y - y_o; \quad \Delta = \sqrt{(h^2 + k^2)};$$

generating the linear system of algebraic equations represented in a matrix by

$$\begin{bmatrix} h_1 & k_1 & \frac{h_1^2}{2} & \frac{k_1^2}{2} & h_1 k_1 & \frac{h_1 k_1^2}{2} & \frac{h_1^2 k_1}{2} & \frac{h_1^2 k_1^2}{4} \\ h_2 & k_2 & \frac{h_2^2}{2} & \frac{k_2^2}{2} & h_2 k_2 & \frac{h_2 k_2^2}{2} & \frac{h_2^2 k_2}{2} & \frac{h_2^2 k_2^2}{4} \\ h_3 & k_3 & \frac{h_3^2}{2} & \frac{k_3^2}{2} & h_3 k_3 & \frac{h_3 k_3^2}{2} & \frac{h_3^2 k_3}{2} & \frac{h_3^2 k_3^2}{4} \\ h_4 & k_4 & \frac{h_4^2}{2} & \frac{k_4^2}{2} & h_4 k_4 & \frac{h_4 k_4^2}{2} & \frac{h_4^2 k_4}{2} & \frac{h_4^2 k_4^2}{4} \\ h_5 & k_5 & \frac{h_5^2}{2} & \frac{k_5^2}{2} & h_5 k_5 & \frac{h_5 k_5^2}{2} & \frac{h_5^2 k_5}{2} & \frac{h_5^2 k_5^2}{4} \\ h_6 & k_6 & \frac{h_6^2}{2} & \frac{k_6^2}{2} & h_6 k_6 & \frac{h_6 k_6^2}{2} & \frac{h_6^2 k_6}{2} & \frac{h_6^2 k_6^2}{4} \\ h_7 & k_7 & \frac{h_7^2}{2} & \frac{k_7^2}{2} & h_7 k_7 & \frac{h_7 k_7^2}{2} & \frac{h_7^2 k_7}{2} & \frac{h_7^2 k_7^2}{4} \\ h_8 & k_8 & \frac{h_8^2}{2} & \frac{k_8^2}{2} & h_8 k_8 & \frac{h_8 k_8^2}{2} & \frac{h_8^2 k_8}{2} & \frac{h_8^2 k_8^2}{4} \end{bmatrix} \begin{Bmatrix} \frac{\partial f_o}{\partial x} \\ \frac{\partial f_o}{\partial y} \\ \frac{\partial^2 f_o}{\partial x^2} \\ \frac{\partial^2 f_o}{\partial y^2} \\ \frac{\partial^2 f_o}{\partial x \partial y} \\ \frac{\partial^3 f_o}{\partial x \partial y^2} \\ \frac{\partial^3 f_o}{\partial x^2 \partial y} \\ \frac{\partial^4 f_o}{\partial x^2 \partial y^2} \end{Bmatrix} = \begin{Bmatrix} f_1 - f_o \\ f_2 - f_o \\ f_3 - f_o \\ f_4 - f_o \\ f_5 - f_o \\ f_6 - f_o \\ f_7 - f_o \\ f_8 - f_o \end{Bmatrix}; \tag{2}$$

or symbolically by

$$[A]\{Df\} = \{f\}. \tag{3}$$

Thus, through the inversion of the matrix $[A]$, it is possible to obtain the approximation of the derivatives of $\{Df\}$ as a linear combination of the values of the functions f_i of the neighborhood of f_o and itself.

2.2 Choice of stencil nodes

As for the choice of the 8 nodes in the neighborhood of each balance point, there are criteria that aim to capture closer nodes from different directions, improving the effectiveness of the approximation, in addition to avoiding the alignment of nodes, which can cause singularity in the inversion of the matrix $[A]$. Among these criteria, the eight octants used by Perrone and Kao [6] and the four quadrants used by Liszka and Orkisz [1] stand out. However, since the perturbed grid in the examples in this work was obtained through the perturbation of a grid initially mapped in rectangles, the closest 8 nodes of each *star* were used as a criterion when the grid was in its uniform version, which in itself already prevented the alignment trend of 4 or more nodes.

2.3 Numerical Integration

The classical FDEM associates each point of the grid to a respective integration subdomain, where the unknown function value of each point is considered constant in its respective integration area, which differentiates FDEM from other methods meshless which, when applied in the *weak* form, usually use other numerical integration methods, such as the *Gauss* quadrature. Determining these areas of integration is quite intuitive in classical FDEM, in which, as it has a grid of points structured and mapped into rectangles, the areas end up being rectangular. However, determining these areas is not so intuitive when working with arbitrary grids. The method of determining the integration areas used in this work is through the generation of *Voronoi* polygons (Milewski [7]), where each point is associated with a polygon that is obtained as a dual consequence of the *Delaunay* triangulation, widely used in mesh generators, both in Finite Volume, Finite Element modeling and in Geoprocessing software.

When generating *Voronoi* areas in situations involving grids with points that are very close or very distant from each other, it is common for the points to be located far from the centroid of the generated area. Since the MDFE is based on considering constant the value of the function within an integration area, it is intuitive that a point centralized in each area has better representation than a point located far from the centroid. Thus, two situations were tested: (1) domain points with grid of maintained points, that is, the coordinates of each point are the same used in the generation of the *Voronoi* areas; (2) domain points with coordinates shifted to the centroids of the *Voronoi* areas. In this second situation, from the *Voronoi* polygons generated by the grid of points, the centroids of the areas are obtained, using these points as the new grid, keeping the area generated by the *Voronoi* polygons. In Fig. 1 it is possible to notice the difference in the location of these points for the two situations.

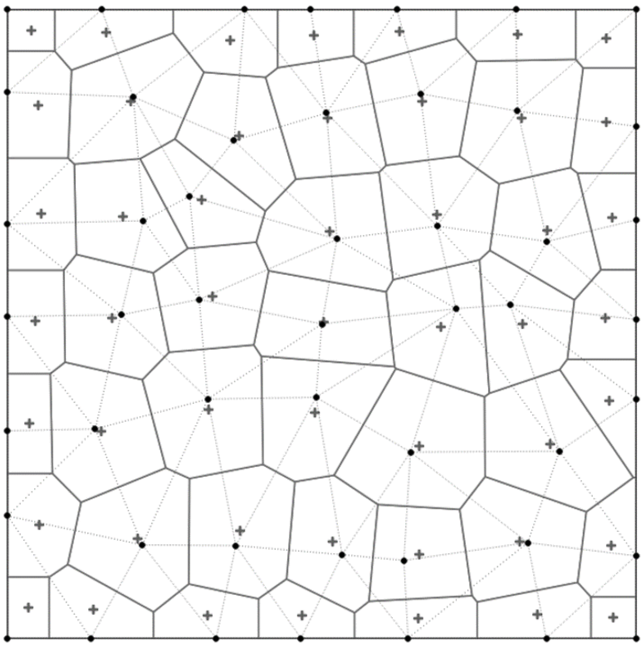

Figure 1. Generating points of *Voronoi* areas (circles) and centroid domain points of *Voronoi* areas (crosses) for an 6x6 nonuniform grid

3 The Model Problem

In Garcia [8] it is possible to find a formulation of geometric nonlinearity in the scope of moderate rotations, which was used by Graça [3] applying it in thin plates bending (*Kirchhoff plates*). The formulation presented here is based on that of Graça [3], but excluding the portions referring to membrane efforts, and disregarding the geometric nonlinearity.

The use of the *weak* form of this formulation reduces the order of derivatives and dispense with the need to prescribe *Neumann* boundary conditions, unlike of the biharmonic differential equation of the strong form.

Being a plate with domain Ω and boundary Γ, with mean plane contained in the plane xy, and w the displacement in the direction z, the virtual work of internal forces to linear bending problem of plates it is given by:

$$\delta W_{int} = \int_{\Omega} (D(w_{,xx} + \nu\, w_{,yy})\delta w_{,xx} + D(w_{,yy} + \nu\, w_{,xx})\delta w_{,yy} + 2D(1-\nu) w_{,xy} \delta w_{,xy})\; dx\; dy; \tag{4}$$

where ν is the *Poisson*'s ratio, D is the flexural stiffness of the plate and $q(x, y)$ is the transverse force per unit area applied to Ω, with boundary conditions of *Dirichlet* applied to Γ_u and of *Neumann* in Γ_t, such that $\Gamma_u \cup \Gamma_t = \Gamma$ and $\Gamma_u \cap \Gamma_t = \varnothing$.

The virtual work of external forces can be written as

$$\delta W_{ext} = \int_{\Omega} q \delta w\; dxdy + \int_{\Gamma} (\bar{F}_{nz} \delta w - \bar{M}_n \delta w_{,n}) d\Gamma; \tag{5}$$

where:

$\bar{F}_{nz}$ is the force per unit of length applied to Γ, along the normal direction to the contour, with direction oriented in the direction z; and $\bar{M}_n$ is the bending moment per unit length applied to Γ, along the direction normal to the contour.

Thus, by equaling the internal virtual work to the external virtual work, the integro-differential equation for linear bending of plates in weak form is obtained:

$$\delta W_{int} = \delta W_{ext}; \tag{6}$$

with boundary conditions of *Dirichlet* applied to Γ_u.

Thus, it is possible to transform the continuous problem in the domain Ω into a discrete problem of n scattered points, having each point $\mathbf{x}_i$ an integration area A_i, and m contour integration stretches L_i, obtaining:

$$\begin{aligned} \sum_{i=1}^{n} D[(w_{,xx(i)} + \nu w_{,yy(i)})\delta w_{,xx(i)} + (w_{,yy(i)} + \nu w_{,xx(i)})\delta w_{,yy(i)} - 2(1-\nu) w_{,xy(i)} \delta w_{,xy(i)}] A_i \quad &= \\ = \sum_{i=1}^{n} q_i \delta w_i A_i + \sum_{i=1}^{m} (F_{nz(i)} \delta w_i - \bar{M}_{n(i)} \delta w_{,n(i)}) L_i. \end{aligned} \tag{7}$$

where the derivatives $w_{,xx(i)}$, $w_{,yy(i)}$, $w_{,xy(i)}$ and $w_{,n(i)}$ in the point $\mathbf{x}_i$ are replaced by the finite difference approximations, that are given by the linear combinations of the approximated values of w in the selected neighborhood points.

4 Applications

The previously presented formulation was computationally implemented in *FORTRAN* language and applied to a case of square plates, simply supported on the 4 edges, with load uniformly distributed on the surface. So for analysis, 4 grid configurations were used (Fig. 2): regular grid (0% perturbation), and irregular grids with up to 10%, 20% and 30% of perturbation. This perturbation percentage threshold is random and generated by reference to the distance between two neighboring points in the regular configuration.

So for each grid configurations, were used in each direction 10, 20, 30 and 40 subdivisions, generating 121, 441, 961 and 1681 points, respectively, at each discretization level (Fig. 3). Furthermore, within these situations,

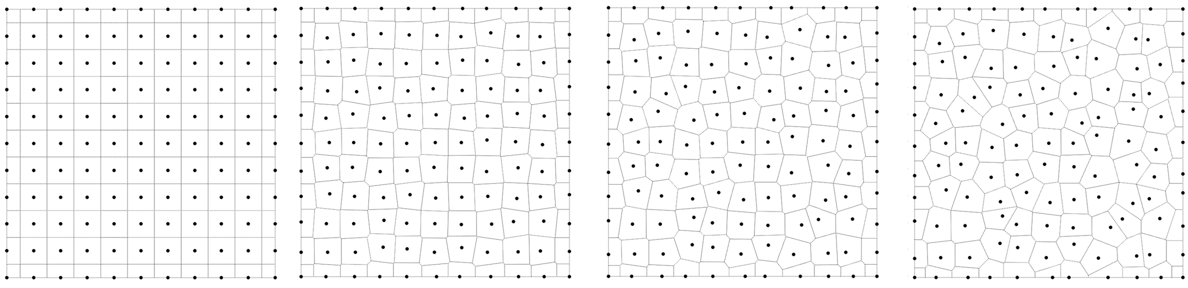

Figure 2. Implemented perturbation for a grid and the respective *Voronoi* areas

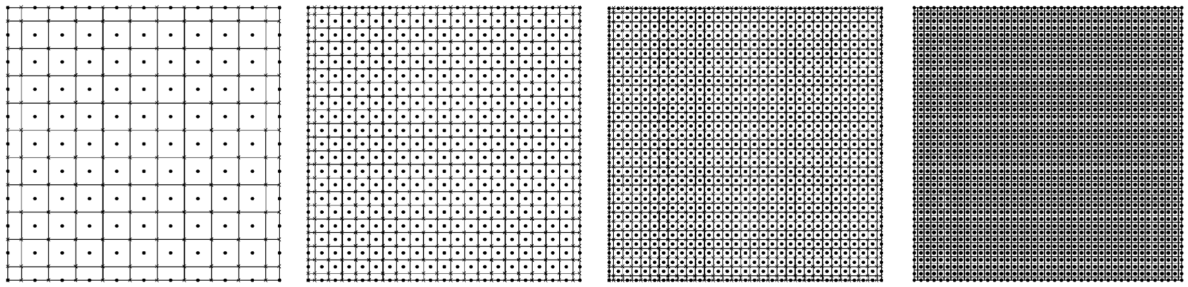

Figure 3. Regular grids implemented and the respective *Voronoi* areas

two different considerations were taken into account: the domain points being the points themselves that generated the *Voronoi* areas, and the domain points being the centroid points of the *areas Voronoi* (Fig. 1).

5 Results

Having *Navier's* analytical solution for the square plate analyzed, it is possible to calculate the error between the numerical result obtained from the formulation and the analytical solution. For the implemented problem, the vertical displacement w of the central node of the board was analyzed. For this, even in the disturbed grids, the coordinates of this central point were kept fixed. Fig. 4 shows the errors of each situation analyzed as a function of the grid refinement.

Using the domain points with their coordinates maintained, although presenting small and convergent errors for perturbations lower than 30%, it is possible to notice a trend of loss of convergence as the perturbation of the grid increases. This can be explained by the possible loss of representativeness of the function value since in this consideration the point may be on the periphery of the integration area (*Voronoi*), as discussed above.

Using the domain points with their coordinates being the centroids of the *Voronoi* areas, it is possible to see that, in addition to the errors being smaller, the convergence trend is maintained even with 30% of perturbation. In addition to the fact that in the centroids of the *Voronoi* areas the points tend to be more representative in the integration, the improvement in the results can also be explained by the "disruption reduction" caused by the displacement of the points to the centroids of the areas, making the grid "less disturbed", as you can see in Fig. 1.

6 Conclusion

The results indicate a good convergence for the analyzed case, especially when using the points in the centroid of the Voronoi area. It can be said that the FDEM has application potential for problems involving arbitrary grids.

The use of *Voronoi* areas as an integration region proved to be a good numerical integration strategy for the FDEM, which assumes that each point is directly associated with an exclusive integration area and has its constant value in this subdomain, making the integration process easier. However, it is necessary to be careful with points that are very close or very distant from each other, which can lead to integration regions with points that are very distant from their centroid, which can reduce the accuracy of numerical integration.

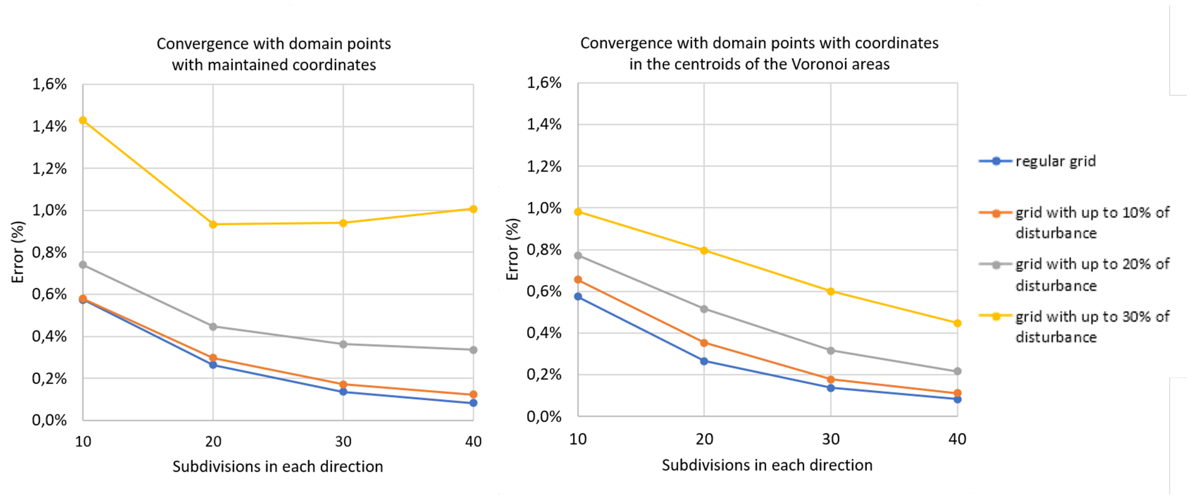

Figure 4. Errors for the analyzed cases

Although the formulation indicates potential use, in order to achieve the objective of generalizing FDEM for linear analysis of plates, other analyzes are necessary, such as application in non-rectangular contours, different loading configurations and setting boundary conditions.

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