Enriched Modified Local Green's Function Method for singular Poisson problem

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Abstract. The Modified Local Green's Function Method (MLGFM) is a hybrid method that couples the Finite Element Method (FEM) and the Boundary Element Method (BEM). The method presents high convergence rates for both potential and flux in the problem boundary and does not require a priori knowledge of the fundamental solution of the problem or a Green Function. In fact, the method automatically obtains an approximation of Green tensor by solving an auxiliary problem. On the other hand, some improvements have been made in conventional FEM to expand its approximation space, mainly by the Generalized Finite Element Method (GFEM) and its stable version, the Stable Generalized Finite Element Method (SGFEM). The GFEM uses the Partition of Unity Method idea to bring previous knowledge about the solution of the problem to enrich the traditional FEM approximation space with appropriate functions. Here the MLGFM will be enriched with GFEM and SGFEM to obtain an approximated Green tensor projection, using singular functions as enriched functions. This Enriched Modified Local Green's Function Method will be applied to singular Poisson problem and the potential and flux will be compared with reference results.

Keywords: Green Function, Generalized Finite Element Method, Boundary Element Method

1 Introduction

The Modified Local Green's Function Method (MLGFM) was firstly proposed in the late 80s, by Barcellos and Silva [1]. The method is a hybrid that couples the Finite Element Method (FEM) and the Boundary Element Method (BEM). Using the advantages of these two methods, the MLGFM tends to presents high convergence rates for both potential and flux in the problem boundary.

One of the advantages of MLGFM over BEM is the fact that it does not need an explicit knowledge of the fundamental solution or a Green Function to the problem. In fact, a projection of the Green tensor was automatically obtained solving an auxiliary domain problem with the FEM (or GFEM in this paper).

The mathematical background of the MLGFM was formally presented in Barbieri et al. [2] and some other researches have been studying the method over the years, for example, Barcellos and Barbieri [3], Barbieri and Barcellos [4], Machado et al. [5], Barbieri and Muñoz [6], Muñoz-Rojas and Vaz Junior [7], Machado et al. [8] and Barbieri and Machado [9].

Mainly since the 90s, new FEM enrichment techniques have risen to expand its approximation space, amongst them the Generalized Finite Element Method (GFEM). The GFEM was developed by several authors, as Babuška et al. [10], Melenk and Babuška [11], Duarte et al. [12], Dolbow et al. [13], Strouboulis et al. [14]. The GFEM is based on the Partition of Unity approach, expanding the FEM approximation space enriching this space with functions that represent the solution local behavior.

Since its proposal, the GFEM presents numerical ill-conditioning issues caused by the almost linear dependence of the base functions. Over the years some strategies have been applied to try to avoid these issues. In Babuška and Banerjee [15], the Stable Generalized Finite Element Method (SGFEM) was proposed. This version consists of a stable version of the GFEM that tries to eliminate the linear dependence of the base functions.

With the advent of the GFEM enrichment technique, the idea of incorporating these techniques into the MLGFM arises, and was firstly explored in Silva [16] and Silva et al. [17]. In this researches, the MLGFM was enriched mainly with Hierarchical Finite Element Method (HFEM) and presented good results.

In this paper, the MLGFM is enriched with the GFEM and its stable version, the SGFEM, and applied to singular Poisson problem know as the Motz problem. The approximation space is enriched with singular functions.

2 The Modifield Local Green's Funcion Method Formulation

Assuming there is a differential operator L and two associated operators D and N related to Dirichlet and Neumann boundary conditions, respectively. The method consists of finding the Green tensor projections solving an auxiliary problem in domain related to adjunct operators L^* and N^*. To solve this auxiliary problem is necessary to define an additional operator N' (for more details see Barbieri et al. [2]) and define a new variable:

$$f(p) = (N + N')\, u(p), \tag{1}$$

where $u(p)$ is the potential in boundary.

The additional operator allows writing the system of integral equations as:

$$u(Q) = \int_\Omega \mathbf{G}^T(P,Q)\, b(P)\, d\Omega_P + \int_\Gamma \mathbf{G}^T(p,Q)\, f(p)\, d\Gamma_p; \quad Q, P \in \Omega \text{ and } p \in \Gamma, \tag{2}$$

where $\mathbf{G}(\cdot,\cdot)$ is the Green's tensor, $b(P)$ is the domain source, $f(p)$ is the flux in boundary, $u(Q)$ is the potential in domain and P and Q are a source point and a field point in domain, respectively.

To extend this integral equation to boundary, the trace operator is applied:

$$u(q) = \int_\Omega \mathbf{G}^T(P,q)\, b(P)\, d\Omega_P + \int_\Gamma \mathbf{G}^T(p,q)\, f(p)\, d\Gamma_p; \quad P \in \Omega \text{ and } p, q \in \Gamma, \tag{3}$$

were $u(q)$ is the potential in boundary and p and q are a source point and a field point in boundary, respectively.

Approximating domain and boundary variables using domain $\mathbf{\Psi}$ and boundary $\mathbf{\Phi}$ basis functions, as shown next:

$$\begin{array}{ll} u(Q) = \mathbf{\Psi}(Q)\, \mathbf{u}_\mathrm{D} & u(q) = \mathbf{\Phi}(q)\, \mathbf{u}_\mathrm{B} \\ b(P) = \mathbf{\Psi}(P)\, \mathbf{b} & f(q) = \mathbf{\Phi}(p)\, \mathbf{f} \end{array}. \tag{4}$$

The basis functions are the FEM and BEM traditional functions. Here these functions are the bilinear FEM shape for domain and linear functions for boundary, both enriched with singular functions presented in the next section. Note that the boundary functions need to be a trace of the domain functions, or:

$$\mathbf{\Phi}(q) = \lim_{Q \to q} \mathbf{\Psi}(Q); \qquad q \in \Gamma \text{ and } Q \in \Omega. \tag{5}$$

The equations (2) and (3) can be rewritten as:

$$\mathbf{A}\, \mathbf{u}_\mathrm{D} = \mathbf{B}\, \mathbf{f} + \mathbf{C}\, \mathbf{b}, \tag{6}$$

$$\mathbf{D}\, \mathbf{u}_\mathrm{B} = \mathbf{E}\, \mathbf{f} + \mathbf{F}\, \mathbf{b}, \tag{7}$$

where:

$$\mathbf{A} = \int_{\Omega} \mathbf{\Psi}^T(Q)\, \mathbf{\Psi}(Q)\, d\Omega_Q; \tag{8}$$

$$\mathbf{B} = \int_{\Omega} \mathbf{\Psi}^T(Q)\, \mathbf{G}_{\mathrm{b}}(Q)\, d\Omega_Q; \tag{9}$$

$$\mathbf{C} = \int_{\Omega} \mathbf{\Psi}^T(Q)\, \mathbf{G}_{\mathrm{d}}(Q)\, d\Omega_Q; \tag{10}$$

$$\mathbf{D} = \int_{\Gamma} \mathbf{\Phi}^T(q)\, \mathbf{\Phi}(q)\, d\Gamma_q \tag{11}$$

$$\mathbf{E} = \int_{\Gamma} \mathbf{\Phi}^T(q)\, \mathbf{G}_{\mathrm{b}}(q)\, d\Gamma_q \tag{12}$$

$$\mathbf{F} = \int_{\Gamma} \mathbf{\Phi}^T(q)\, \mathbf{G}_{\mathrm{d}}(q)\, d\Gamma_q \tag{13}$$

where $\mathbf{G}_{\mathrm{b}}(Q)$, $\mathbf{G}_{\mathrm{d}}(Q)$, $\mathbf{G}_{\mathrm{b}}(q)$ and $\mathbf{G}_{\mathrm{d}}(q)$ are the Green's function projections over the boundary Γ and the domain Ω, evaluated on the points Q and q. The Green's projections can be written as:

$$\mathbf{G}_{\mathrm{b}}(Q) = \int_{\Gamma} \mathbf{G}^T(p, Q)\, \mathbf{\Phi}(q)\, d\Gamma_p, \tag{14}$$

$$\mathbf{G}_{\mathrm{d}}(Q) = \int_{\Omega} \mathbf{G}^T(P, Q)\, \mathbf{\Psi}(P)\, d\Omega_p, \tag{15}$$

$$\mathbf{G}_{\mathrm{b}}(q) = \int_{\Gamma} \mathbf{G}^T(p, q)\, \mathbf{\Phi}(q)\, d\Gamma_p, \tag{16}$$

$$\mathbf{G}_{\mathrm{d}}(q) = \int_{\Omega} \mathbf{G}^T(P, q)\, \mathbf{\Psi}(P)\, d\Omega_p. \tag{17}$$

The Green's tensor projection can be also projected in the space formed by the domain and the boundary shape functions:

$$\begin{array}{ll} \mathbf{G}_{\mathrm{d}}(Q) = \mathbf{\Psi}(Q)\, \mathbf{G}^{\mathrm{DQ}} & \mathbf{G}_{\mathrm{d}}(q) = \mathbf{\Phi}(q)\, \mathbf{G}^{\mathrm{Dq}} \\ \mathbf{G}_{\mathrm{b}}(Q) = \mathbf{\Psi}(Q)\, \mathbf{G}^{\mathrm{BQ}} & \mathbf{G}_{\mathrm{b}}(q) = \mathbf{\Phi}(q)\, \mathbf{G}^{\mathrm{Bq}} \end{array}. \tag{18}$$

where $\mathbf{G}^{\mathrm{DQ}}$, $\mathbf{G}^{\mathrm{Dq}}$, $\mathbf{G}^{\mathrm{BQ}}$ and $\mathbf{G}^{\mathrm{Bq}}$ are the nodal coefficients of the Green Tensors project in the space formed by domain and boundary shape functions. Its values can be determined by the minimization of an appropriate functional (see Barbieri et al. [2]), resulting in the expression:

$$[\mathbf{K}_{\mathrm{GFEM}} + \mathbf{K}'] \left[\mathbf{G}^{\mathrm{DQ}} \,|\, \mathbf{G}^{\mathrm{BQ}}\right] = [\mathbf{A} \,|\, \mathbf{D}]\,, \tag{19}$$

where $\mathbf{K}_{\mathrm{GFEM}}$ is the GFEM stiffness matrix, $\mathbf{K}'$ is a diagonal matrix related to additional operator N', $\mathbf{A}$ and $\mathbf{D}$ are the matrix defined in (8) e (11). The values of $\mathbf{G}^{\mathrm{Dq}}$ and $\mathbf{G}^{\mathrm{Bq}}$ can be determined by the application of the trace operator in $\mathbf{G}^{\mathrm{DQ}}$ and $\mathbf{G}^{\mathrm{BQ}}$.

3 GFEM enrichment functions

The Generalized Finite Element Method can be described as the combination of the FEM with the Partition of Unity Method, so the approximate potential $u_{\mathrm{h}}(\boldsymbol{x})$ in domain Ω can be written as:

$$u_{\mathrm{h}}(\boldsymbol{x}) = \sum_{j=1}^{n} \mathcal{N}_j(\boldsymbol{x}) \left\{ u_j + \sum_{i=1}^{q_j} S_{ji}(\boldsymbol{x})\, b_{ji} \right\} \tag{20}$$

where $\mathcal{N}_j(\boldsymbol{x})$ is the partition of unity function, b_{ji} are the GFEM degrees of freedom and $S_{ji}(\boldsymbol{x})$ are the enrichment functions defined, in this paper, as:

$$\mathbf{S}(r,\theta) = \sqrt{r}\, \left\{ \cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\, \sin(\theta)\,, \cos\left(\frac{\theta}{2}\right)\, \sin(\theta) \right\}, \tag{21}$$

where r and θ are coordinates of the polar system centered on singularity point.

The stable version of the GFEM, the Stable Generalized Finite Element Method (SGFEM), proposed in Babuška and Banerjee [15] suggests a simple modification of the enrichment functions extracting its piecewise linear interpolant:

$$S_{ji}^{\mathrm{mod}} = S_{ji} - I_{\Omega_j}(S_{ji}), \tag{22}$$

were $I_{\Omega_j}(S_{ji})$ is the piecewise linear interpolant S_{ji} on path Ω_j.

4 Numerical Results

The Motz's problem is a classical problem and its scheme is present in Fig. 1.

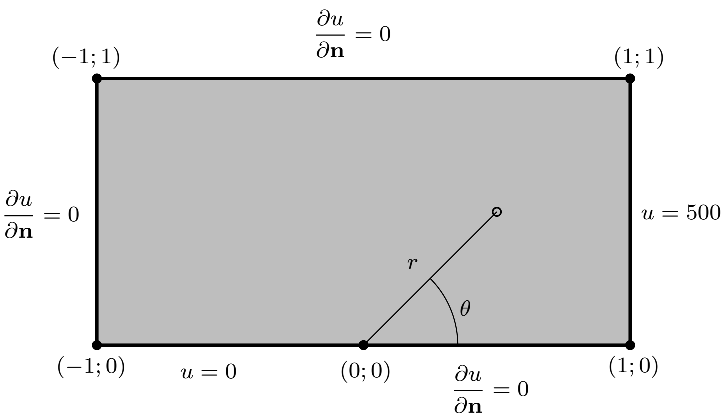

Figure 1. Motz's problem scheme.

The analytical solution of the problem is written as:

$$u(r,\theta) = \sum_{i=0}^{\infty} M_i\, r^{i+\frac{1}{2}} \cos\left[\left(i + \frac{1}{2} \right) \theta \right]. \tag{23}$$

where the 35 M_i first terms are found with high precision in Lu et al. [18] and are used here as reference.

The singularity exists at the coordinate point $(0, 0)$, due to the discontinuity of the boundary conditions. So it was defined the enriched region with 0.36 of radius around the singularity, as shown in Fig. 2. The MLGFM enriched with GFEM was call $\text{MLGFM}_{\text{GFEM}}$ and, the one enriched with SGFEM was called $\text{MLGFM}_{\text{SGFEM}}$.

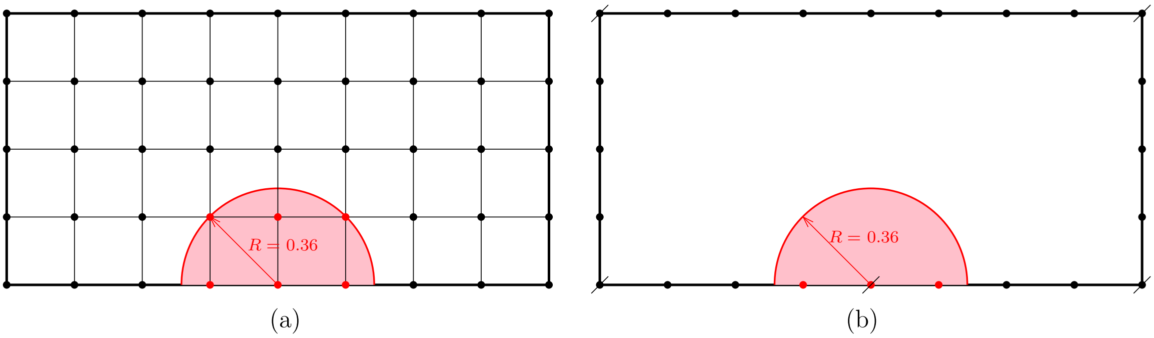

Figure 2. MLGFM mesh: (a) Finite element mesh and (b) Boundary element mesh.

The potential results for an 8x16 mesh are presented in Fig 3. The results of the two enriched MLGFM show good agreement with the semi-analytical solution (reference solution). For the primary variable (potential), the results don't present any numerical disturbance.

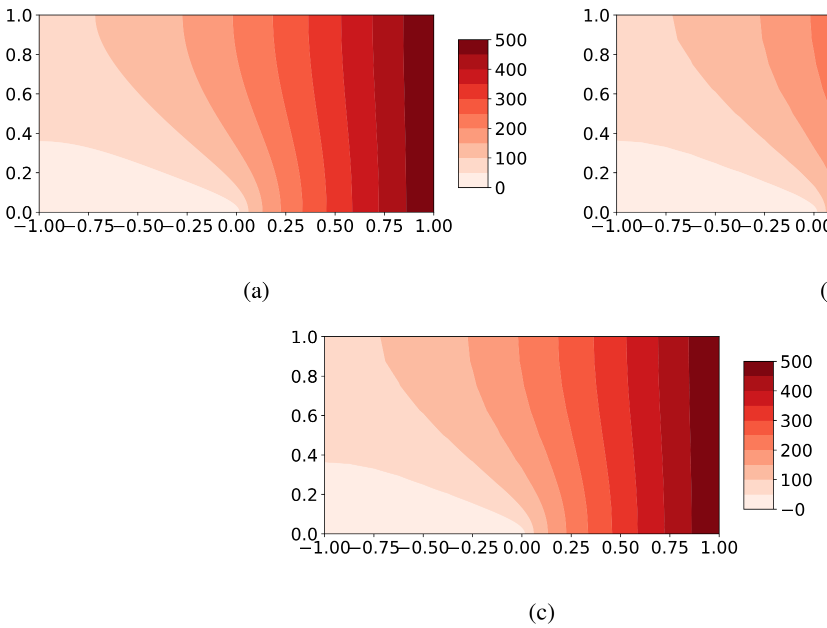

Figure 3. Potential solution for 8x16 mesh: (a) Semi-analytical solution; (b) $\text{MLGFM}_{\text{GFEM}}$ results and (c) $\text{MLGFM}_{\text{SGFEM}}$ results.

The flux results in the y-direction in the singularity region for some mesh for both enriched MLGFM are presented in Fig. 4. Here it is possible to notice that the response of the $\text{MLGFM}_{\text{GFEM}}$ presents a large oscillation which is reduced with the more stable version of the enriched MLGFM.

The oscillation of $\text{MLGFM}_{\text{GFEM}}$ flux results seems to be related to the method being sensitive to the numerical ill-conditioning of the GFEM. With the enrichment using the stable version of GFEM (SGFEM), the problem decreases. It can also be noted that as the mesh is refined, the oscillation is shifted closer to the singularity.

The convergence of the response using the energy norm is shown in Fig. 5. The convergence of the $\text{MLGFM}_{\text{GFEM}}$ and $\text{MLGFM}_{\text{SGFEM}}$ are compared to pure MLFGM. In this result, it is possible to notice that the error of the enriched versions of MLGFM varies between 3 and 6 times smaller than its non-enriched version.

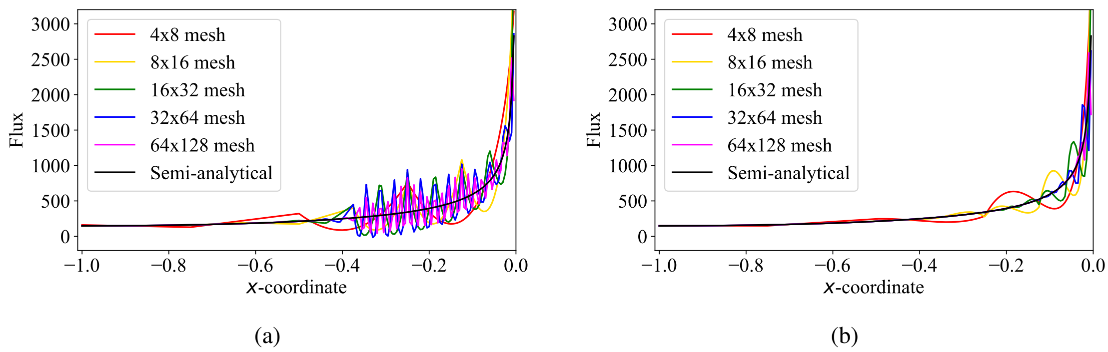

Figure 4. Flux in y-direction in singular region: (a) $\text{MLGFM}_{\text{GFEM}}$ results and (b) $\text{MLGFM}_{\text{SGFEM}}$ results.

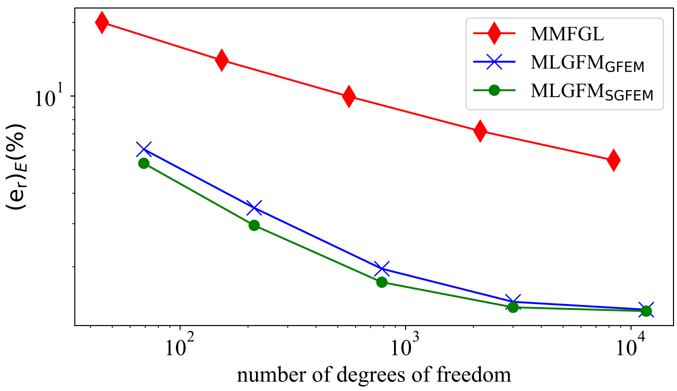

Figure 5. Energy norm convergence.

5 Concluding Remarks

This paper contains preliminary results of the enriched MLGFM for this kind of problem. The method shows potential approximating the primary variables (domain potential) once the results presented good agreement with the reference solution. Energy norm convergence shows great improvement obtained by the enriched approachs, with errors between 3 and 6 times smaller over MLGFM without enrichment.

The MLGFM seems to present sensibility for the GFEM ill-conditioning. Especially for the second variable (flux), the method shows a sensible improvement when enriched with the SGFEM. The idea is to test other partitions of unity (as the flat-top or Shepard, for example) to improve the stability of the SGFEM.

The main aim for the future is to use the sub-regions idea of the BEM to develop a local approach to the enriched MLGFM, using each finite element as a sub-region (or Green's cell). The idea is to direct obtain potential and flux results in all mesh nodes (see Barbieri and Machado [9]). This values would be obtained without the necessity of any derivative operation.

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