POST-PROCESSING OF ANISOTROPIC LAMINATES IN DYNAMIC PROBLEMS MODELED USING GENERALIZED FINITE ELEMENT METHOD

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Abstract. The paper presents a post-processing method to be applied in results obtained from dynamic analysis in thin laminated structures. The problems are modeled using the Generalized Finite Element Model (GFEM) and are based on first order shear deformation theory. The main goal of this work is to analyze the effects of the inertial and body forces on the results obtained from the stress and displacement recovery method. The method relies on values obtained from GFEM which are next are applied to the general local constitutive, kinematic and motion equations. GFEM are integrated and the values of the transverse stresses are then corrected using stress-strain relations. Also, all components of displacements can be recovered. These new corrected displacement and stress values are then compared with results obtained from analytical and numerical solutions for validation.

Keywords: Stress and displacement recovery, Post-processing, Generalized Finite Element Model, Dynamic Problem, Laminate Structures.

1 Introduction

The increased use of laminated composite materials makes the need for correct structural analysis a modern need for many engineers nowadays. Numerical analysis of laminated plates, however, still constitutes a challenge due to its complexity. One of the main problems is the accurate estimate of the stress distribution, especially its transverse components. Since transverse resistance is usually much smaller than the ones found in-plane, it is crucial that the estimates be as precise as possible for correct laminate design. Another common problem found in this types of structures are the several failure modes that occur randomly distributed over the plate, lowering the overall resistance and making difficult to predict the rupture.

In general, it can be considered that an analysis on a plate or shell is realized with the objective of finding its complete stress and displacements response. The first approach consists on using a Finite Element Model (FEM) based on solid elements to approximate the numerical solution to a standard three-dimensional solid mechanics. Although very effective, even today this kind of approach is still not regularly used due to its high computer time required. The alternative to this strategy is the use simplified models, like Reissner-Mindlin model, that reduces the problem to a two dimensions. This model is commonly found in most commercial codes due to its low cost and satisfactory approximations for thin laminates for most results such as transverse displacements, in-plane stress components and the first vibration modes.

Reissner-Mindlin model considers uniform shear strain deformations across the thickness and discontinuous transverse shear stresses constant in every layer. Analyzing the exact three-dimensional solid model one concludes that transverse shear deformations should in fact be discontinuous and the shear stresses should present variability in each layer and continuity across the whole laminate. Furthermore, the stresses found in exact solid solutions should satisfy the force boundary conditions on all of the laminate faces, a detail that in most situations is not met by the Mindlin model.

The discrepancies that follow the Reissner-Mindlin model are well known and many studies are still made to correct and improve its results. Many theories and procedures use results directly obtained from FEM or Generalized Finite Element Method (GFEM). One of the most popular is presented in Chaudhuri [], which involves using the in-plane local equilibrium equations, to integrate across the laminate thickness a continuous distribution of the transverse shear stresses. In most cases, the transverse integration starts with a null value or a prescribed nonzero value for the stresses at the bottom of the laminate. At the end of integration usually the force boundary conditions are not satisfied at the top face requiring further procedures for its correction.

A majority of problems that are solved using this stress recovery method don't consider the effects of inertia and body forces. This current paper aims to investigate the influence that these forces have on the integrated and corrected transverse stresses and displacement. These values will also satisfy the local equilibrium as well as the force boundary conditions on the faces of the laminate and interlayer continuity. In addition, approximate values for transverse normal stress and displacement, which are not directly calculated using GFEM, will be extracted.

In regard to the results used in the stress recovery process it is crucial to have a correct method to obtain them. GFEM takes in consideration this aspect in a very easy and low cost way. Doing the in-plane enrichment of the function basis grants better results due to the expansion of the solution space. This also results in an easy and direct way for obtaining the necessary differentiations while having low cost and using a fixed mesh. In most commercial codes it is used low order finite elements making it impossible to use stress recovery techniques as a consequence of not allowing the required differentiations. In this cases the strategy is to create a patch, changing the structure of the mesh, using elements of higher order, increasing the cost of the analysis.

2 Literature review

2.1 Generalized finite element method

The GFEM is a variation on the classic Finite Element Method done in order to obtain better results without increased computational const. This is done by the enrichment of the basis seen in FEM which results in an increased solution space. The main concepts necessary for understanding are Partition of Unity (PoU) and the enrichment functions. The definition of PoU is any set functions $\varphi_\alpha(\mathbf{x})$ that obey the following properties:

$$\sum_{\alpha=1}^{N} \varphi_\alpha(\boldsymbol{x}) = 1 \qquad \varphi_\alpha(\boldsymbol{x}) \in C_0^\infty \qquad \varphi_\alpha(\boldsymbol{x}) \geq 0 \ \forall \boldsymbol{x} \in \boldsymbol{\Omega} \tag{1}$$

where $\mathbf{\Omega}$ is the plate domain in $\mathbb{R}^n$.

After choosing a set of PoU functions these are the enhanced with enrichment functions by way of multiplication. These enrichment functions can be polynomial, harmonic or any function that contains parts of the solution to the boundary value problem. In this paper these functions are the following:

$$Linear\ enrichment, p = 1: L_{i\alpha} = [1, \bar{x}, \bar{y}];$$

$$Quadratic\ enrichment, p = 2: L_{i\alpha} = [1, \bar{x}, \bar{y}, \bar{x}^2, \overline{xy}, \bar{y}^2];$$

$$Cubic\ enrichment, p = 3: L_{i\alpha} = [1, \bar{x}, \bar{y}, \bar{x}^2, \overline{xy}, \bar{y}^2, \bar{x}^3, \bar{y}\bar{x}^2, \bar{x}\bar{y}^2, \bar{y}^3];$$

$$Quartic\ enrichment, p = 4: L_{i\alpha} = [1, \bar{x}, \bar{y}, \bar{x}^2, \overline{xy}, \bar{y}^2, \bar{x}^3, \bar{y}\bar{x}^2, \bar{x}\bar{y}^2, \bar{y}^3, \bar{x}^4, \bar{y}\bar{x}^3, \bar{x}^2\bar{y}^2, \bar{x}\bar{y}^3, \bar{y}^4]; \tag{2}$$

where

$$\bar{x} = \frac{x - x_\alpha}{h_\alpha}, \ \bar{y} = \frac{y - y_\alpha}{h_\alpha} \tag{3}$$

where x_α, y_α are coordinates of node $\mathbf{x}_\alpha$ and h_α is a representative radius of the cloud ω_α.

The versatility of the different possibilities for the enrichment functions has given rise in its popularity. GFEM over the years has become the preferred method for solving many complex problems, such as crack opening studies, crack discontinuity movement, discontinuity associated with internal boundaries in multi-phase materials, etc.

2.2 Reissner-Mindlin model

Also known as First-Order Shear Deformation Theory (FSDT), the Reissner-Mindlin model is an adaption of the Classic Lamination Theory (CLT). In this model, the adaptation used in CLT is based on a hypothesis stating that any segment which is initially normal and straight to the non-deformed reference surface will remain straight and non-stretched, however no longer necessarily normal to the deformed reference surface Mendonça [3]. This change allows the estimation for the transverse shear stress.

First proposed by Reissner [5] and Mindlin [4], later extended to composite materials by Yang [6] and modified by Whitney, this theory is a classic and widely used in many commercial software and academic research. This is due to its requirement of only C^0 approximation functions, which reduce computational implementation and cost. And yet it suffers, when using FEM implementation, from some numerical pathologies such as shear locking, found in certain thin plate problems.

The Reissner-Mindlin kinematic model condensed by the displacement relations, the strain-displacement relations and the local equilibrium equations for infinitesimal deformations.

$$
\begin{aligned}
&u(x,y,z,t) = u^0(x,y,t) + z\theta_x(x,y,t), && \varepsilon_x = \frac{\partial u}{\partial x}, && \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, && \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho^k b_x = \rho^k \frac{\partial^2 u}{\partial t^2},\\
&v(x,y,z,t) = v^0(x,y,t) + z\theta_y(x,y,t), && \varepsilon_y = \frac{\partial v}{\partial y}, && \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, && \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho^k b_y = \rho^k \frac{\partial^2 v}{\partial t^2},\\
&w(x,y,z,t) = w(x,y,t), && \varepsilon_z = \frac{\partial w}{\partial z}, && \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, && \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho^k b_z = \rho^k \frac{\partial^2 w}{\partial t^2}
\end{aligned}
\tag{4}
$$

The innovation explored in the present paper consists in the introduction of the inertia terms in motion equations to perform the stress recovery process.

2.3 Stress recovery post-processing

Using the Reissner-Mindlin model means that all transverse stress is calculated from the constitutive equations resulting in incorrect values making it necessary to use stress recovery techniques in post-processing so that a better failure prediction can be made and consequently a better laminate sizing.

The method used in this work, proposed by Chaudhuri [1], uses the first two equilibrium equations to calculate de transverse shear stresses distribution by plugging in the in-plane stress values obtained using the FEM model. The integrated equations result in:

$$
\begin{aligned}
\tau^i_{xz}(z) - \tau^i_{xz}(-H/2) &= -\int_{-\frac{H}{2}}^{z} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho^k b_x - \rho^k \frac{\partial^2 u}{\partial t^2} \right) dz,\\
\tau^i_{yz}(z) - \tau^i_{yz}(-H/2) &= -\int_{-\frac{H}{2}}^{z} \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho^k b_y - \rho^k \frac{\partial^2 v}{\partial t^2} \right) dz,\\
\sigma^i_{z}(z) - \sigma^i_{z}(-H/2) &= -\int_{-\frac{H}{2}}^{z} \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho^k b_z - \rho^k \frac{\partial^2 w}{\partial t^2} \right) dz,
\end{aligned}
\tag{5}
$$

By examining the equations, we can deduce that all of the transverse stresses are functions of the in-plane stresses and that the transverse normal stress can be calculated using transverse shear stresses obtained from the FEM model or alternatively we can use the integrated transverse shear stresses proposed by the Chaudhuri method.

3 Methodology

3.1 Correction of the integrated transverse stresses

As mentioned before when dealing with most problems with methods that result in approximations such as FEM or GFEM most likely the force boundary conditions in the superior surface are not met. This means that a procedure must be made in order to improve stress and consequently displacement distribution.

The method presented in this work consists in simply creating a function for each of the integrated transverse stresses and adding it to them. First it is considered a cubic polynomial function of z such as $f(z)=a+bz+cz^2+dz^3$ as shown in Fig. 1.

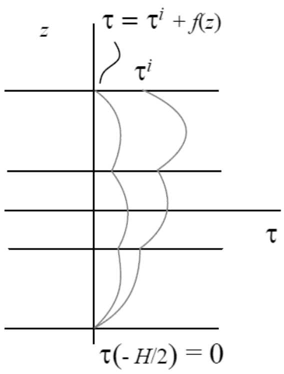

Figure 1. Correction of the Integrated Transverse Shear Stresses

In order to only correct the one boundary condition, the following restrictions are imposed to this function: f(-H/2) = 0 and f(H/2) = A. Taking in these restrictions the polynomial turns to:

$$f(\boldsymbol{x}, z) = A(\boldsymbol{x})\left(\frac{1}{2} + \frac{z}{H}\right) + c\left(z^2 + \frac{H^2}{4}\right) \tag{6}$$

where A(**x**) $A(\boldsymbol{x})$ is the term dependent on the **x** = (x,y) position of the reference surface. The correction procedure is summarized in a few steps. Firstly, given the in-plane stresses, obtained by GFEM, and the integrated transverse stresses, find the following functions:

$$f_x(z) = A_x(\boldsymbol{x})\left(\frac{1}{2} + \frac{z}{H}\right) + c_x\left(z^2 + \frac{H^2}{4}\right),\ f_y(z) = A_y(\boldsymbol{x})\left(\frac{1}{2} + \frac{z}{H}\right) + c_y\left(z^2 + \frac{H^2}{4}\right),\ f_z(z) = A_z(\boldsymbol{x})\left(\frac{1}{2} + \frac{z}{H}\right) \tag{7}$$

that adjusts the integrated transverse stresses using the subsequent equations:

$$\tau_{xz}(z) = \tau_{xz}^i(z) + f_x(z) \qquad \tau_{yz}(z) = \tau_{yz}^i(z) + f_y(z) \qquad \sigma_z(z) = \sigma_z^i(z) + f_z(z) \tag{8}$$

Next step is to impose the following boundary conditions of the top surface of the laminate:

$$\begin{cases} \tau_{xz}(H/2) = q_x^s, \\ \tau_{yz}(H/2) = q_y^s, \\ \sigma_z(H/2) = q_z^s, \\ \int_{-H/2}^{H/2}\{\tau_{xz}, \tau_{yz}\}dz = \{Q_x, Q_y\}, \\ \left.\frac{\partial \tau_{xz}}{\partial x}\right|_{H/2} = \frac{\partial q_x^s}{\partial x}, \\ \left.\frac{\partial \tau_{yz}}{\partial y}\right|_{H/2} = \frac{\partial q_y^s}{\partial y}, \end{cases} \tag{9}$$

where q_x^i, q_y^i e q_z^i are known values of the distributed loads at the cartesian directions applied on the upper surface of the laminate. The presented shear forces Q_x and Q_y are calculated using the GFEM results with the constitutive relations. This means that the adjusted functions in eq. (8) are defined by only five constants, A_x, A_y, A_z, c_x e c_y.

3.2 In-plane and transverse displacement trough thickness

To obtain an estimate for the displacements, both in-plane and transverse, the 3D stress-strain relations for anisotropic layer, ε = **S**σ, are used given that the stress vector is constructed using the in-planes stresses, obtained by GFEM, in conjunction with the integrated transverse stress. Below are presented these relations:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \overline{\boldsymbol{S}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz}^i \\ \tau_{xz}^i \\ \tau_{xy} \end{Bmatrix} \tag{10}$$

where the matrix $\bar{\mathbf{S}}$, is the material flexibility matrix, which for orthotropic material is $\bar{\mathbf{S}} = \mathbf{T}^{\mathbf{T}}\mathbf{S}\mathbf{T}$. With the deformations calculated the strain-displacement relations in eq. (4) are used respectively for each displacement and the integrated resulting in:

$$u^i(z) = \int_{-H/2}^{z}\left(\gamma_{xz}^i - \frac{\partial w^i}{\partial x}\right)dz, \qquad v^i(z) = \int_{-H/2}^{z}\left(\gamma_{yz}^i - \frac{\partial w^i}{\partial y}\right)dz, \qquad w^i(z) = \int_{-H/2}^{z}\varepsilon_z\, dz, \quad (11)$$

After the integration to the top surface, the next step consists in the translation of each integrated displacement function by taking each respective value at reference surface $u_0 = u^i(0)$, $v_0 = v^i(0)$ e $w_0 = w^i(0)$. The equation for the translation process are:

$$\delta u = u_0 - u_{GFEM}, \qquad \delta v = v_0 - v_{GFEM}, \qquad \delta w = w_0 - w_{GFEM}, \quad (12)$$

$$u^i(z) = u^i(z) - \delta u, \qquad u^i(z) = u^i(z) - \delta u, \qquad u^i(z) = u^i(z) - \delta u \quad (13)$$

where u_{GFEM}, v_{GFEM} e w_{GFEM} are the GFEM value of displacement associated to the reference surface.

4 Results

The analyzed problem consists of a simply supported anisotropic plate with length and width a=b=1 m, thickness H=0.2 m, made with layers of orthotropic materials with the following material constants: E1=172.5GPa, E2=6.89GPa, G12=G13=0.5E2, G23=0.2E2 and v12=v23=0.25. A transverse harmonic load with q0=1N/m2 is applied with a double sinusoidal variation in the **x** plane:

$$q(x, y) = q_0\, sin\left(\frac{\pi x}{a}\right) sin\left(\frac{\pi y}{b}\right) e^{i\Omega t} \quad (14)$$

where Ω is the load frequency. The studied plate is made of a symmetric orthotropic laminate orientated to the [0º,90º,0º] configuration.

The GFEM harmonic analysis was made using a mesh M=6 with a p=4 enrichment functions made with triangular elements shown in Fig. 2.

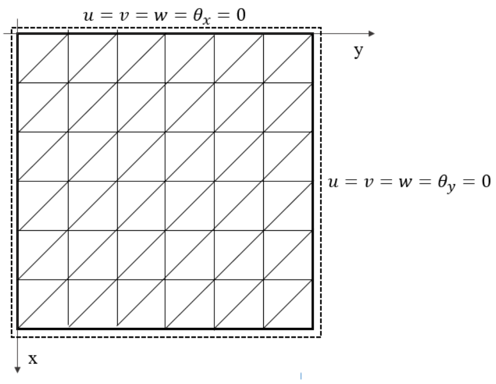

Figure 2. Mesh arrangement example with boundary conditions.

The same type of analysis was made using three different approaches. An analytical solution was made using the Reissner-Mindlin model, the harmonic solution presented by Dobyns [2] and uses a post-processing correcting stress recovery method. Another solution was obtained using commercial software ANSYS through the APDL language. And the last approach was made using a university made program that uses GFEM to obtain the in-plane stresses and the same correcting stress recovery post-procedure applied to the first approach, symbolized by the GFEMc marker.

Fig. 3 shows the transverse displacement spectrum graph made in an interval between 90% and 97.5% of the first natural frequency. This frequency was obtain through analytical calculation resulting in Ω_{FM}=536.105HZ, making the interval Ω ∈ [482.4945;522.7024].

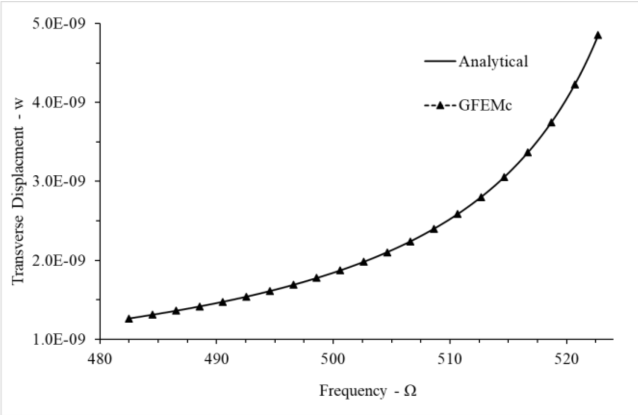

Figure 3. Results for the transverse displacement spectrum graph for (a/2;b/2;H/2) position.

Fig. 4 shows the transverse displacement distribution through the laminate's thickness. The chosen frequency used for this and all following results was 95% of the first vibration mode frequency which results in Ω=509.2998Hz.

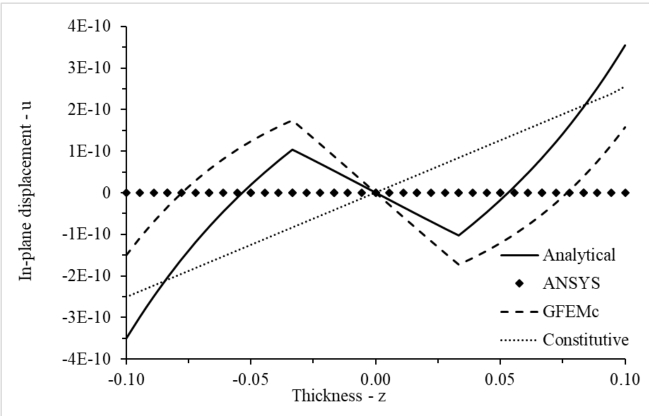

Figure 4. Results for in-plane displacement u(0;b/2;H/2) along the thickness.

The last figures, Fig. 5 and Fig. 6, show the distribution of the transverse shear stress in the YZ plane and the transverse normal stress. It can be noted the strange behavior of the Ansys response. It is obtained by the simple transverse integration of the static local equilibrium equations, while the others include the inertia terms.

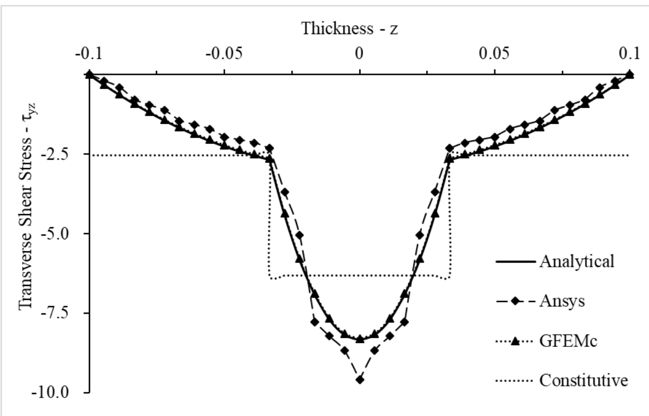

Figure 5. Results for transverse shear stress τ_{yz}(a/2;0;0) along the thickness.

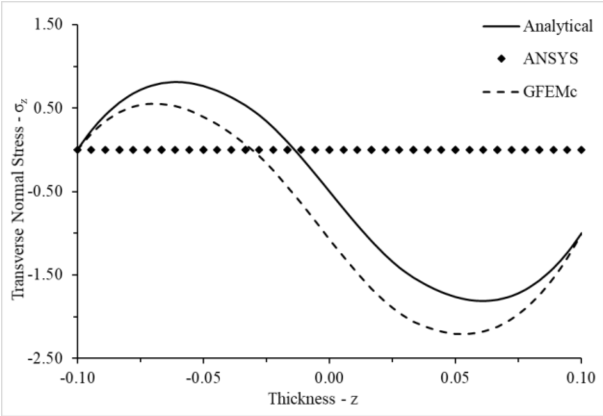

Figure 6. Results for transverse normal stress $\sigma_z(4/2;b/2;0)$ along the thickness.

5 Conclusions

The preliminary results shown for one classic case of symmetric laminate, show that the recovery procedures proposed show good behavior. This is considered to be related to the inclusion of the inertia terms in the local motion equations used in the transverse integration process. Usually this integration is performed, even in commercial codes, without these inertia terms. Their absence reduces the accuracy of the extracted values when the load frequency is close to one of the natural frequencies of the plate. Additionally, other results show that the recovery procedures being developed are capable of producing reliable values of in-plane and transverse displacement components across the thickness, with good qualitative proximity with those obtained from three-dimensional solutions.

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