Analysis of suspended cables by the Generalized Finite Element Method using trigonometric enrichment functions

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Abstract. Presenting light weight, cost-effective construction and the possibility of pre-tensioning, cables have been used as structural elements in suspension bridges, mooring lines, transmission lines, guyed towers, marine and off-shore constructions, cable trusses and roof structures. The analysis of cable structures by the Finite Element Method (FEM) using straight elements usually requires a high number of degrees of freedom in order to obtain acceptable results for the cable shape and its properties, such as cable tension and length. In this paper the Generalized Finite Element Method (GFEM) is studied with the use of trigonometric enrichment functions, considering a linear and inextensible cable analysis. The results obtained by the GFEM using the proposed trigonometric enrichment functions are compared to linear solutions provided by the Hierarchical Finite Element Method (HFEM). The computational cost is analyzed in terms of the total number of degrees of freedom and the program execution time. The condition number of the stiffness matrix is also discussed.

Keywords: Generalized Finite Element Method, Linear Analysis, Cables.

1 Introduction

Cables can be used as structural elements in several engineering branches. Among its constructive advantages, one can mention its low weight and ease of assembly. One of the most notable uses of cables in structures is in suspension and cable-stayed bridges. According to Ren and Peng [1], cable-stayed bridges still have advantages over other types of bridges, including suspension bridges, and these are: better efficiency in the use of materials, greater stiffness and smaller size of structural elements, facilitating their manufacture/construction. In addition to the high architectural appeal, these bridge typologies have enabled increasingly larger spans, reaching up to 1 km (see Karoumi [2]). Another essential use of cables is in transmission lines. Costa [3] highlights that, in large countries, such as Brazil, the energy generated in plants needs to travel large distances in transmission lines to reach the final consumer. From an economic point of view, it is important that cables span great distances without compromising their structural behaviour, because, as in bridges, such cables are subject to static, thermal and dynamic loads.

Another use of cables consists in suspension roofs. In 1950, the State Fair Arena project in Raleigh, North Carolina, promoted further study and construction of suspended roofs, as an advantage of this type of coverage is the ability to cover large areas with little material, according to Tibert [4]. However, all of the cited uses for cables require a rigorous structural analysis. An initial difficulty in the study of cables is that its geometry is load dependent, that is, its shape varies according to the applied forces (see Irvine [5]). Loads distributed along the arc length of the cable, such as its own weight, make it assume the form of a catenary. On the other hand, loads distributed along the span produce the shape of a parabolic curve.

The Finite Element Method (FEM) is widely used in the study of cables in computational mechanics. In the case of cable analysis, straight elements require a high number of degrees of freedom in order to obtain satisfactory results for the cable profile and its properties, such as the length and the stresses developed in the cable itself. One can overcome this problem using the FEM itself and the *h*, *p* and *hp* refinements. The *h* refinement consists of refining the mesh by increasing the number of elements, which can demand a high computational cost. The p refinement, on the other hand, consists in increasing the degree of the shape functions, however, the formulation of elements in the p refinement is not always simple, making its computational implementation difficult (see Proença and Torres [6]). In this context, enriched methods were developed, including the Generalized Finite Element Method (GFEM), which includes known information (a priori) of the problem in order to improve the solution with lower computational cost than the FEM. The Generalized Finite Element Method (GFEM) has been successfully used in several problems of computational mechanics, such as in linear dynamic analysis (see Arndt et al. [7]), nonlinear dynamic analysis (see Piedade Neto and Proença [8]) and three-dimensional nonlinear analysis (see Proença and Torres [6]). Therefore, this paper intends to apply the GFEM in the analysis of cable structures.

2 Cables

In Figure 1 a cable submitted to distributed loads and an infinitesimal segment of the cable are shown.

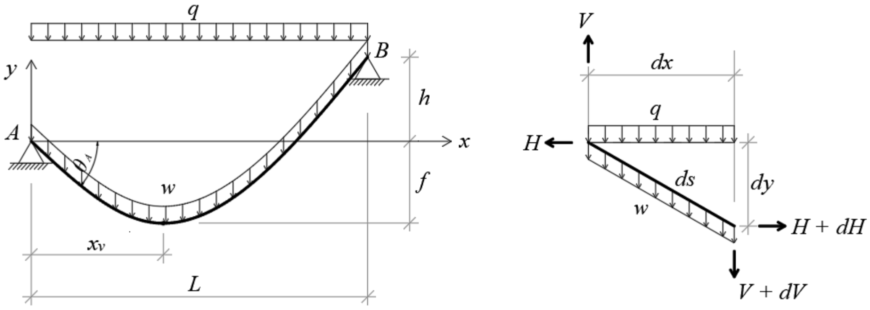

Figure 1. General cable and infinitesimal cable segment.

In Figure 1, L is the span of the cable, f is its sag, the vertical distance between supports A and B is h and θ_A is the angle measured between the cable and the horizontal at support A. The loads q and w are distributed along the span and the cable length, respectively, and forces V and H are the vertical and horizontal components of the axial force in the cable, respectively. By equilibrium of the infinitesimal segment, one obtains

$$H\frac{d^2y}{dx^2} = q + w\sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{1}$$

which is the governing differential equation. In the following, the simplification

$$\frac{dy}{dx} = \sinh\left(\frac{wx}{H} - \frac{wL}{2H}\right) \tag{2}$$

is used. Substituting Eq. (2) in Eq. (1), one obtains

$$H\frac{d^2y}{dx^2} = q + w\sqrt{1+\sinh^2\left(\frac{wx}{H} - \frac{wL}{2H}\right)} = q + w\cosh\left(\frac{wx}{H} - \frac{wL}{2H}\right) \tag{3}$$

which can be used with Galerkin Method.

2.1 Linear Finite Element Method

The cable element hereby used for the linear analysis is shown in Figure 2. In this element, firstly introduced by Przybysz *et al*. [9], ξ is the local coordinate, L_e is the element length and y_1 and y_2 are the vertical displacements of nodes 1 and 2, respectively.

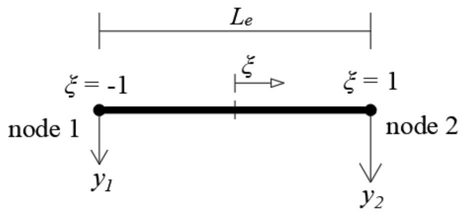

Figure 2. Linear cable element.

According to the element presented in Figure 2, the displacement field y is

$$y(\xi) = \boldsymbol{N}^T \boldsymbol{y} = [N_1 \quad N_2] \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = N_1 y_1 + N_2 y_2, \qquad N_1 = \frac{1-\xi}{2}, \qquad N_2 = \frac{1+\xi}{2} \tag{4}$$

where $\boldsymbol{N}^T$ is the transpose of the vector containing the linear shape functions N_i. Replacing Eq. (4) in Eq. (3) and applying the Galerkin Method yields

$$\int_{-1}^{1} H \frac{d\boldsymbol{N}}{d\xi} \frac{d\boldsymbol{N}^T}{d\xi} \frac{2}{L_e} \boldsymbol{y} \, d\xi - \int_{-1}^{1} \boldsymbol{N}^T \left[q + w \cosh\left(\frac{wx}{H} - \frac{wL}{2H} \right) \right] \frac{2}{L_e} d\xi = \boldsymbol{0} \tag{5}$$

where it has been considered that

$$\frac{d\boldsymbol{N}}{dx} = \frac{d\boldsymbol{N}}{d\xi} \frac{d\xi}{dx} = \frac{d\boldsymbol{N}}{d\xi} \frac{2}{L_e} \tag{6}$$

Considering $\boldsymbol{B} = dN/d\xi$ and replacing it into Eq. (5), one obtains

$$\int_{-1}^{1} H \boldsymbol{B} \boldsymbol{B}^T \frac{2}{L_e} \boldsymbol{y} \, d\xi - \int_{-1}^{1} \boldsymbol{N}^T \left[q + w \cosh\left(\frac{wx}{H} - \frac{wL}{2H} \right) \right] \frac{2}{L_e} d\xi = \boldsymbol{0} \tag{7}$$

From Eq. (7), it is possible to define

$$\boldsymbol{K}_e = \int_{-1}^{1} H \boldsymbol{B} \boldsymbol{B}^T \frac{2}{L_e} \, d\xi, \qquad \boldsymbol{F}_e = \int_{-1}^{1} \boldsymbol{N}^T \left[q + w \cosh\left(\frac{wx}{H} - \frac{wL}{2H} \right) \right] \frac{2}{L_e} d\xi \tag{8}$$

where $\boldsymbol{K}_e$ is the element stiffness matrix and $\boldsymbol{F}_e$ is the element external load vector.

2.2 Iterative scheme

The element stiffness matrix presented in Eq. (8) depends on the horizontal thrust of the cable H. By using an estimate for H, called H_i, it is possible to evaluate the stiffness matrix of the elements, obtain the displacement vector and, therefore, the sag f_i, based on the estimate H_i. Since the value of the sag f is known, it is possible to use an iterative process so that the difference $|f_i-f|$ is smaller than the given tolerance when approaching better values for H_i. Using the Secant Method, as used by Przybysz *et al*. [9], one obtains

$$H_{i+1} = \frac{H_{i-1}(f_i - f) - H_i(f_{i-1} - f)}{(f_i - f) - (f_{i-1} - f)}, \quad i = 1, 2, \dots \tag{9}$$

The iterative process is halted when the difference $|f_i-f|$ is less than a given tolerance, that is, $|f_i-f| \leq \epsilon_{tol}$. Especially in the case of enriched analyses, to find the value of the sag f_n it is not necessary that a nodal point of an element coincides with the abscissa where the sag *f* occurs, once f_n can be found through interpolation. For every enriched analysis presented in this work, H_0 was considered

$$H_0 = \frac{pl^2}{8f} \tag{10}$$

where *p* is the distributed load acting on the cable. The value for H_1 is taken as $H_1 = 1.01H_0$.

2.3 Generalized Finite Element Method

The Generalized Finite Element Method (GFEM), was independently proposed by Melenk and Babuška [10], and by Duarte and Oden [11], and originated from the Partition of Unity Method.

The shape functions of the GFEM are built by the product between the partition of unity function and enrichment functions, where the enrichment functions are not necessarily polynomial (see Kim *et al*. [12]). These functions are then assigned to the element nodes, expressed as a function of the system global coordinates (see Schwebke and Holzer [13]).

A key feature of the GFEM is the use of previously known information about the solution of the differential equation that describes the system for the construction of the enrichment functions, presenting good local and global results. GFEM has been successfully used in various fields such as crack analysis (see Sukumar *et al*. [14]; O'Hara *et al*. [15]) and structural dynamics (see Torii [16])

For the two-node element showed in Figure 2, the solution provided by the GFEM for the element displacements y^e can be written as

$$y^e(\xi) = y^e_{FEM} + y^e_{ENRICH} = \sum_{i=1}^{2} \eta_i\, y_i + \sum_{i=1}^{2} \eta_i \left(\sum_{j=1}^{n_l} \gamma_j\ a_{ij} \right) \tag{11}$$

where y^e_{FEM} is the displacement obtained by the FEM, y^e_{ENRICH} is obtained through the enrichment functions, η_i are the linear partition of unity functions, y_i are the nodal displacements, n_l is the level of enrichment, γ_j are the enrichment functions and a_{ij} are the degrees of freedom related to those functions.

In this paper, two sets of trigonometric functions obtained using the Fourier Series Theory (see Monteiro [17]) are used for enrichment. The set of sine functions are

$$Fs_i = \sin\left(\frac{\pi}{2}(i-2)(\xi+1)\right), \qquad i = 3, 4, 5, 6 \tag{12}$$

Differently from eq. (12), the set of cosine functions are

$$Fc_i = \cos\left(\frac{\pi}{2}(i-3)(\xi+1)\right) - Fc_1 + (-1)^i Fc_2 \quad i = 3, 4, 5, 6 \tag{13}$$

where

$$Fc_1 = \frac{1}{4}(3\xi+1)(\xi-1), \qquad Fc_2 = \frac{1}{4}(3\xi-1)(\xi+1) \tag{14}$$

In order to maintain numerical stability, the cosine functions in eq. (13) are divided by

$$\|Fc_i\| = \sqrt{\int_{-1}^{1} Fc_i^2\, d\xi} \tag{15}$$

The functions in eq. (12) and eq. (13) will be used as the γ_i enrichment functions in eq. (11), and the results provided by the GFEM are compared to those obtained by the Hierarchical Finite Element Method (HFEM), where the following Lobatto functions are used for refinement

$$L_2 = \frac{1}{2}\sqrt{\frac{3}{2}}(\xi^2 - 1), \qquad L_3 = \frac{1}{2}\sqrt{\frac{5}{2}}(\xi^2 - 1)\xi, \qquad L_4 = \frac{1}{8}\sqrt{\frac{7}{2}}(\xi^2 - 1)(5\xi^2 - 1),$$
$$L_5 = \frac{1}{8}\sqrt{\frac{9}{2}}(\xi^2 - 1)(7\xi^2 - 3)\xi \tag{16}$$

Each analysis labeled by the prefix "GFEM" indicates that the solution is enriched by GFEM, and the prefix "HFEM" denotes a refined solution obtained by HFEM. The following letters indicate the enrichment function used, e.g. "GFEM Fs_3" stands for the solution provided by the GFEM considering the sine function Fs_3, as presented in Eq. (12), while "GFEM Fs_5" is the enriched solution obtained using Fs_3, Fs_4 and Fs_5 as enrichment functions, that is, it includes Fs_5 and all other sine function whose index i, as shown in eq. (12), is lower than 5. The notation is similar regarding the set of cosine enrichment functions.

3 Numerial Results

The cable shown in Figure 3 is analyzed. The cable weights w = 0,005 kN/m, its sag is f = 6 m and it spans a distance of L = 40 m.

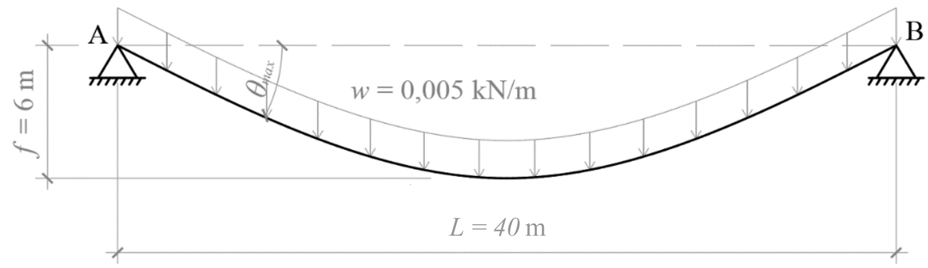

Figure 3. Cable under self-weight load.

The results are shown on Table 1, where the percentages in parentheses are the relative errors with respect to the linear analytic solution according to the catenary equations of the cable. The variables analyzed are the cable horizontal thrust H, the cable's length S, the traction at support A T_A and the angle with the horizontal θ_{max}, as shown in Figure 3. The number of degrees of freedom is N_{dof} and N_{cond} is the condition number of the stiffness matrix obtained in the last iteration. All the results of the GFEM and HFEM linear analyses were obtained using only one element.

According to Table 1, the results obtained by GFEM were overall good. When enriched with the cosine functions Fc_i, the GFEM presented all errors less than 2.2412%, regardless of the number of degrees of freedom. Considering GFEM Fc_4, Fc_5 and Fc_6, all errors were smaller than 0.1169%, with GFEM Fc_6 achieving the best results for every studied variable when compared to the solutions of the standard FEM, HFEM and GFEM.

The Fc_i outperformed the Fs_i analyses when comparing solutions with the same number of degrees of freedom, that is, GFEM Fc_3 results were better than GFEM Fs_3, GFEM Fc_4 results were better than GFEM Fs_4 and so forth. However, GFEM Fc_4, for example, did not outperform HFEM L_5 and HFEM L_4, when both the hierarchical solutions had not only fewer or equal number of degrees of freedom, but also lower condition number of the stiffness matrix.

Table 1. Results. Cable subjected to self-weight.

Solution	H (N)	S (m)	T_A (N)	θ_{max} (°)	N_{dof}	N_{cond}	Iterations
Linear analytic	171.445713	42.306960	201.445713	-31.671051	-	-	-
FEM (2 elements)	171.445712 (0.0000%)	41.761265 (1.2898%)	178.994746 (11.1449%)	-16.699423 (47.2723%)	1	1.00E+00	4
FEM (10 elements)	171.445712 (0.0000%)	42.285503 (0.0507%)	195.642091 (2.8810%)	-28.798030 (9.0714%)	9	3.99E+01	4
HFEM L_2	172.345656 (0.5249%)	42.284638 (0.0528%)	200.982527 (0.2299%)	-30.961229 (2.2412%)	1	1.00E+00	5
HFEM L_3	172.345656 (0.5249%)	42.284638 (0.0528%)	200.982527 (0.2299%)	-30.961229 (2.2412%)	2	1.00E+00	5
HFEM L_4	171.443268 (0.0014%)	42.307077 (0.0003%)	201.431692 (0.0070%)	-31.665910 (0.0162%)	3	1.00E+00	4
HFEM L_5	171.443268 (0.0014%)	42.307077 (0.0003%)	201.431692 (0.0070%)	-31.665910 (0.0162%)	4	1.00E+00	4
GFEM Fc_3	172.345656 (0.5249%)	42.284638 (0.0528%)	200.982527 (0.2299%)	-30.961229 (2.2412%)	2	1.67E+00	5
GFEM Fc_4	171.416348 (0.0171%)	42.307816 (0.0020%)	201.330971 (0.0570%)	-31.634017 (0.1169%)	4	2.86E+03	4
GFEM Fc_5	171.447171 (0.0009%)	42.306970 (0.0000%)	201.434525 (0.0056%)	-31.665102 (0.0188%)	6	4.16E+05	4
GFEM Fc_6	171.445592 (0.0001%)	42.307014 (0.0001%)	201.439258 (0.0032%)	-31.668140 (0.0092%)	8	3.52E+07	4
GFEM Fs_3	177.251174 (3.3862%)	42.135844 (0.4045%)	195.946054 (2.7301%)	-25.231652 (20.3321%)	2	1.87E+00	5
GFEM Fs_4	170.534648 (0.5314%)	42.330968 (0.0567%)	197.210808 (2.1023%)	-30.147795 (4.8096%)	4	3.03E+02	4
GFEM Fs_5	171.568499 (0.0716%)	42.303626 (0.0079%)	200.832825 (0.3042%)	-31.319150 (1.1111%)	6	7.42E+03	4
GFEM Fs_6	171.426363 (0.0113%)	42.307538 (0.0014%)	201.260864 (0.0918%)	-31.596162 (0.2365%)	8	2.71E+05	4

The performance of the GFEM and HFEM were also satisfatory in relations to those of the standard FEM. Except for the horizontal thrust H, which the FEM analyses aproximated best, every other variable had better numerical results in GFEM and HFEM, especially when considering that the enriched and refined analyses had, at most, 8 degrees of freedom, and the best FEM mesh had 9 degrees of freedom.

In Figure 4, the program time of execution is compared with the number of degrees of freedom.

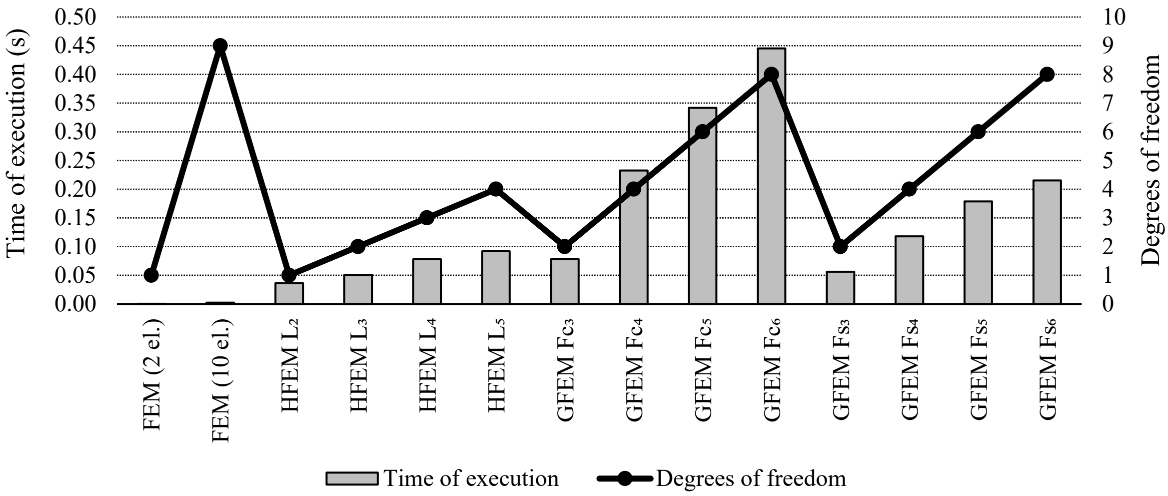

Figure 4. Program execution time.

As observed in Figure 4, the solution with the best presented results GFEM Fc_6 also took the longest time to run. In general, GFEM Fc_{i} took longer duration times when compared to GFEM Fs_{i} and HFEM L_{i}, among solutions with the same number of degrees os freedom, while the standard FEM solution were the fastest ones.

4 Conclusions

The enrichment functions used in the GFEM presented in this work were able to achieve satisfactory results while showing some advantages over the standard FEM. In general, most of the enriched solutions presented better results than those provided by the FEM with 9 DOFs, considering that the enriched analyses used only one element with, at most, 8 DOFs.

While GFEM Fc_6 achieved the best numerical results, it was also the solution with the longest time of execution and also the highest condition number. Considering only the HFEM and GFEM, every analysis with at least three DOF took four iterations to converge the horizontal thrust H, when those with 2 or less DOF took 5 iterations.

Overall, the program time of execution increased according to the number of DOF, as expected, noting that the standard FEM analysis ran much faster than the enriched and refined ones.

References

[1] W.-X. Ren and X.-L. Peng, "Baseline finite element modeling of a large span cable-stayed bridge through field ambient vibration tests", *Computers & Structures*. v. 83, p. 536-550, 2005

[2] R. Karoumi, "Some modelling aspects in the nonlinear finite element analysis of cable supported bridges". *Computers and Structures*, v.71, pp. 397-412, 1999.

[3] R. S. Costa, Formulation for the advanced analysis of cables and spatial trusses for application in cable-stayed towers and transmission lines (in Portuguese). Doctoral Dissertation, Graduate Program in Structural Engineering, Federal University of Minas Gerais, 2014.

[4] G. Tibert, Numerical analyses of cable roof structures. Doctoral Dissertation, Royal Institute of Technology, Stockholm, 1999.

[5] H. M. Irvine. *Cable structures*. The MIT Press Series, 1981.

[6] S. P. B. Proença and I. F. R. Torres, "Generalized finite element method for nonlinear three-dimensional analysis of solids". *International Journal of Computational Methods*, v. 5, n. 1, p.37-62, 2007.

[7] M. Arndt, R. D. Machado and A. Scremin, "Accurate assessment of natural frequencies for uniform and non-uniform Euler-Bernoulli beams and frames by adaptive generalized finite element method". *Engineering Computations*, p. 1586-1609, 2016.

[8] D. Piedade Neto and S. P. B. Proença, "Generalized Finite Element Method in linear and nonlinear structural dynamic analyses". *Engineering Computations,* v. 33, n. 3, p.806-830, 2015.

[9] J. C. Przybysz, R. D. Machado, P. O. Weinhardt and M. Arndt, "Static analysis of inextensible suspended cables by the GFEM", *XL CILAMCE Ibero-Latin american congress on computational methods in engineering*, 2019.

[10] J. M. Melenk and I. Babuška, "The Partition of Unity Finite Element Method: Basic Theory and Applications". *Seminar für Angewandte Mathematik*, p. 1-31, 1996.

[11] C. A. Duarte and J. T. Oden, "An *h-p* adaptative method using clouds". *Computer Methods in Applied Mechanics and Engineering*, v. 139, p. 237-262, 1996.

[12] D. J. Kim, C. A. Duarte and S. P. Proença, "Generalized Finite Element Method with global-local enrichments for nonlinear fracture analysis". *Brazilian Society of Mechanical Sciences and Engineering*, p. 317-330, 2009.

[13] K. G. Schwebke and S. M. Holzer, "Some Remarks on Generalized Finite Element Methods (GFEM) in Solid Mechanics", *Fifth World Congress on Computational Mechanics*, 2002.

[14] N. Sukumar, N. Möes, B. Moran and T. Belytschko, "Extended finite element method for three-dimensional crack modelling". *International Journal for Numerical Methods in Engineering*, p. 1549-1570, 1999.

[15] P. O'Hara, C. A. Duarte and T. Eason, "A two-scale generalized finite element method for interaction and coalescence of multiple crack surfaces". *Engineering Fracture Mechanics*, v. 163, p.274-302, 2016.

[16] A. J. Torii, Structural dynamics analysis using the Generalized Finite Element Method (in Portuguese). Doctoral Dissertation, Federal University of Paraná, 2012.

[17] F. A. C. Monteiro, Systematic construction of sets of hierarchical C^m functions according to the Fourier Series Theory (in Portuguese). Doctoral Dissertation, Graduate Program in Aeronautical Infrastructure Engineering, Aeronautical Technology Institute, 2017.