On the imposition of the local boundary conditions in the G/XFEM-gl analysis

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Abstract. The present work investigates the imposition of boundary conditions in the scope of the Generalized Finite Element Method with Global-Local Enrichment (GFEM^{gl}) in the analysis of a two-dimensional linear elastic fracture mechanics problem. In the GFEM^{gl}, the "global problem" is firstly solved with a course discretization, and a "local problem" is defined in the region containing singularities, imposing the previously obtained solution as boundary conditions. The solution of the local problem provides numerically obtained enrichment functions capable of representing the singular features. Two aspects are analyzed here: the application of Cauchy boundary conditions (displacements and stresses), as well as the influence of the local domain size. As for the Cauchy boundary conditions, the problem is simulated using average stresses and recovered stresses obtained by the ZZ-BD recovery procedure (based on the stragey of Zienkiewicz-Zhu to recover a smooth stress field, but using a block-diagonal matrix in its formulation). The results are presented in terms of the stress intensity factors and the strain energy of the enriched global problem. The numerical simulations are performed in INSANE (INteractive Structural ANalysis Environment), an open-source software developed in the Department of Structural Engineering at the Federal University of Minas Gerais.

Keywords: Finite Element Method, Generalized Finite Element Method, Global-Local Analysis, Fracture Mechanics.

1 Introduction

The Generalized Finite Element Method with Global-local Enrichments (GFEM^{gl}) was proposed by Duarte and Kim [1] to address the need of a more general enrichment function construction in the Generalized Finite Element Method (G/XFEM) (Duarte et al. [2] and Strouboulis et al. [3], Belytschko et al. [4]). The GFEM^{gl} is based on a two-scale decomposition of the solution: a coarse scale (global problem), representing the smooth component of the solution, and a fine-scale (local problem), responsible for representing special features like cracks. In the scope of Linear Elastic Fracture Mechanics (LEFM) problems, several studies confirm the accuracy of the GFEM^{gl} considering static or propagating cracks, as in Pereira et al. [5] and O'hara et al. [6,7].

In the GFEM^{gl}, the transferring of boundary conditions between global and local problems is extremely important for accuracy of the results. Kim et al. [8] proposed different types of boundary conditions to be imposed in the local problem: Dirichlet (displacements), Neumann (stresses) and Cauchy (displacements and stresses). These three strategies were applied to LEFM problems, with the Cauchy boundary conditions showing superior results.

In the present paper, the Cauchy boundary conditions are explored in the GFEM^{gl} considering three different stress fields, computed by discontinuous, average or recovered stresses. The recovered stresses are obtained by the ZZ-BD recovery procedure of Lins et.al [9].The application of recovered stresses in the GFEM^{gl} is an original contribution of this paper.

Following this introduction, formulation aspects of the GFEM^{gl} and the ZZ-BD recovery are presented in Section 2. Then, in Section 3, the numerical experiments are exposed. Finally, the main conclusions of this paper are summarized in Section 4.

2 Formulation Aspects

In this section, the key formulation aspects of the GFEM^{gl} and the ZZ-BD recovery are presented.

2.1 GFEM^{gl}

Figure 1 illustrates the three stages of the the GFEM^{gl} strategy applied to a Fracture Mechanics problem. In this case, the local problem is defined around the crack, where a high concentration in the stress field is expected.

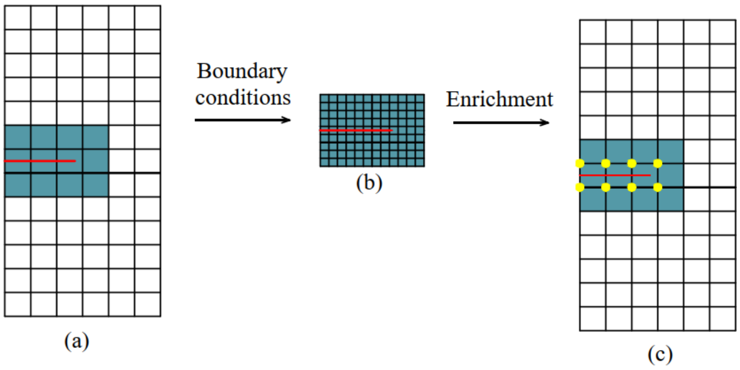

Figure 1. Illustration of the three stages of the GFEM^{gl}. (a) Initial global problem. (b) Local problem. (c) Enriched global problem, with the nodes enriched with the global-local functions highlighted in yellow.

The formulation of the GFEM^{gl} can be found in Kim et. al [8]. In the local problem, the formulation includes the boundary conditions from the global problem ($\tilde{\boldsymbol{u}}_G^0$) through the spring stiffness parameter κ:

$$\text{Find } \tilde{\boldsymbol{u}}_{\boldsymbol{L}} \in \chi_L(\Omega_L) \subset H^1(\Omega_L)\ \forall\, \boldsymbol{v}_L \in \chi_L(\Omega_L):$$

$$\begin{aligned}&\int_{\Omega_L} \boldsymbol{\sigma}(\tilde{\boldsymbol{u}}_{\boldsymbol{L}}) : \boldsymbol{\varepsilon}(\boldsymbol{v}_L) d\boldsymbol{x} + \kappa \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \partial\Omega_G)} \tilde{\boldsymbol{u}}_{\boldsymbol{L}} \cdot \boldsymbol{v}_L d\boldsymbol{x} \\ &= \kappa \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \partial\Omega_G)} \bar{\boldsymbol{t}}\, \boldsymbol{v}_L\, ds + \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \partial\Omega_G)} \left(t(\tilde{\boldsymbol{u}}_G^0) + \kappa \tilde{\boldsymbol{u}}_G^0\right) \cdot\ \boldsymbol{v}_L\, ds\end{aligned} \tag{1}$$

where Ω_L is the local domain, $\partial\Omega_G$ and $\partial\Omega_L$ are the boundary of the global and local domains, respectively, $\boldsymbol{\sigma}$ is the stress tensor, $t(\tilde{\boldsymbol{u}}_G^0)$ is the traction vector from the global solution, $\boldsymbol{v}_L$ is the test function and $\chi_L(\Omega_L)$ is the discretization of $H^1(\Omega_\mathrm{L})$, a Hilbert space built with the G/XFEM shape functions in the local problem.

In Birner and Schweitzer [10], an automatic scheme of computing an optimal parameter in Cauchy boundary conditions is proposed. This strategy is applied in this paper, as will be shown in Section 3.

After computing the local solution $\tilde{\boldsymbol{u}}_{\boldsymbol{L}}$, a new analysis of the global problem is performed using $\tilde{\boldsymbol{u}}_{\boldsymbol{L}}$ as enrichment functions, defining the so called enriched global problem.

Another key aspect of the GFEM^{gl} is the use of multiple global-local iterations, as proposed initially by O'hara et al. [11] and expanded by Gupta et al. [12]. This strategy consists of using the solution of the enriched global problem as boundary conditions in a new analysis of the local problem. The effect of global-local iterations will also be studied in Section 3.

2.2 ZZ-BD recovery

The ZZ-BD recovery procedure was proposed by Lins et. al [9] for the G/XFEM and its stable version (SGFEM), considering 2-D LEFM problems. The procedure is presented as a computationally efficient and robust strategy for computing a recovered stress tensor, based in the stress recovery method adopted in the classic ZZ error estimator (Zienkiewickz and Zhu [13]).

In the ZZ-BD, special enrichment functions are used to represent singular and discontinuous recovered stress fields in the vicinity of cracks. The components of the recovered stress field are written as.

$$\begin{Bmatrix} \sigma_{xx}^* \\ \sigma_{yy}^* \\ \sigma_{xy}^* \end{Bmatrix} = \sum_{\beta=1}^{NN} \boldsymbol{N_\beta} \begin{pmatrix} \boldsymbol{a_{\beta,0}^1} \\ \boldsymbol{a_{\beta,0}^2} \\ \boldsymbol{a_{\beta,0}^3} \end{pmatrix} + \sum_{\beta \in I} \sum_{n=1}^{2} \boldsymbol{N_\beta} \begin{pmatrix} \boldsymbol{a_{\beta,n}^1\, g_n^1(r,\theta)} \\ \boldsymbol{a_{\beta,n}^2\, g_n^2(r,\theta)} \\ \boldsymbol{a_{\beta,n}^3\, g_n^3(r,\theta)} \end{pmatrix} \tag{2}$$

where NN is the total number of nodes in the mesh, $\boldsymbol{N_\beta}$ is the partition of unity associated with the node β, *r* and θ are polar coordinates from the crack tip and I is the set of nodes enriched with branch functions $\boldsymbol{g_n^d}$g, with d=1,2,3 representing the 3 stress components of the 2D elasticity problem. The expressions of the branch functions can be found in Lins et. al [9].

In Equation (2), $\boldsymbol{a_{\beta,n}^d}\,(d = 1,2,3)$ are unknown coefficients to be found by solving L^2 projection problems, defined by the minimization of the following functional:

$$\boldsymbol{\Pi} = |(\boldsymbol{\sigma}^* - \tilde{\boldsymbol{\sigma}})^{\boldsymbol{T}}(\boldsymbol{\sigma}^* - \tilde{\boldsymbol{\sigma}})| \tag{3}$$

where $\tilde{\boldsymbol{\sigma}}$ is the stress field computed by the G/XFEM solution. The ZZ-BD procedure consists of using an locally weighted L^2 projection to compute the system matrix of the minimization problem, resulting in a symmetric, positive-definite block diagonal matrix.

More details of the formulation can be found in Lins et. al [9]. In that paper, the accuracy of the recovered solutions is confirmed for both G/XFEM and SGFEM, showing very good agreement with the exact solutions in singular problems. In addition, the results show that the error of the recovered stresses converges at a higher rate than the error of the computed stresses, which guarantees that the recovered stresses are asymptotically exact.

In the present paper, the ZZ-BD procedure is employed in the global problem to compute recovered stresses to be imposed as boundary conditions in the local problem, as discussed in the next session.

3 Numerical Experiments

Figure 2 illustrates the problem studied in this work, which consists of a plate under plane strain containing an edge crack, subjected to shear stresses. The linear elastic constitutive model is adopted, with $E=1.0x10^5$and ν=0.3. The plate is submitted to a constant horizontal stress τ = 1.0. The values obtained by Wilson [14], KI=34.00 and KII=4.55 are adopted as reference solution for the stress intensity factors. For the strain energy, the reference solution is $2{,}467211 \times 10^{-2}$ (numerically obtained by Fonseca [15] with ANSYS Academical Mechanical APDL). All the values are given in consistent units.

In all cases studied in this work, the transfer of information from the initial global model to the local model was carried out through the imposition of Cauchy boundary conditions (stresses and displacements), with the spring stiffness parameter being calculated in each global-local iteration as proposed by Birner and Schweitzer [10]. Two local domain sizes are proposed, as discussed in sections 3.1 and 3.2 below.

The global mesh used in all analyzes is composed by 105 quadrilateral elements (Q4), with dimensions of 1.0 x 1,067. First order polynomial enrichments are used in all nodes of the global model. In addition, global-local iterations are performed until the difference between the current value (iteration *t*) and the previous value (iteration *t*-1) of K_I and K_{II} was less than 1%.

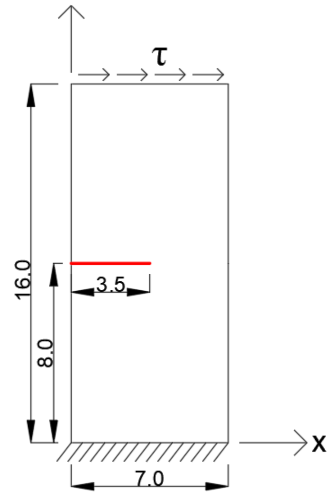

Figure 2. Geometry and boundary conditions of the plate studied in this section.

3.1 Local Model I

The Local Model I is composed of 135 quadrilateral elements (Q4), obtained from 15 global elements by dividing each edge into 3 parts. This local model is illustrated in Figure 2, where the global nodes highlighted in blue are the ones enriched with the local functions.

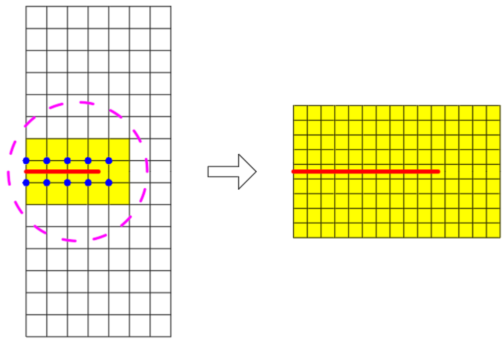

Figure 3. Local Model I (The local domain indicated in the right of the figure is out of scale).

Three types of stress fields are tested in the Cauchy boundary conditions. First, the stresses are obtained without any treatment, that is, they consist of the discontinuous GFEM stresses field obtained in the global elements located at the boundary of the local domain. Next, the stress field is defined by an average of the stresses obtained in each global element that shares the edge located at the local domain boundary. Finally, the stress field is obtained by the ZZ-BD recovery procedure in the same global elements where the first stress field (the discontinuous one) is calculated.

The results obtained for the stress intensity factors and the strain energy, considering the three types of stress fields and the reference solutions described in the beginning of this section, are presented in Figure 4.

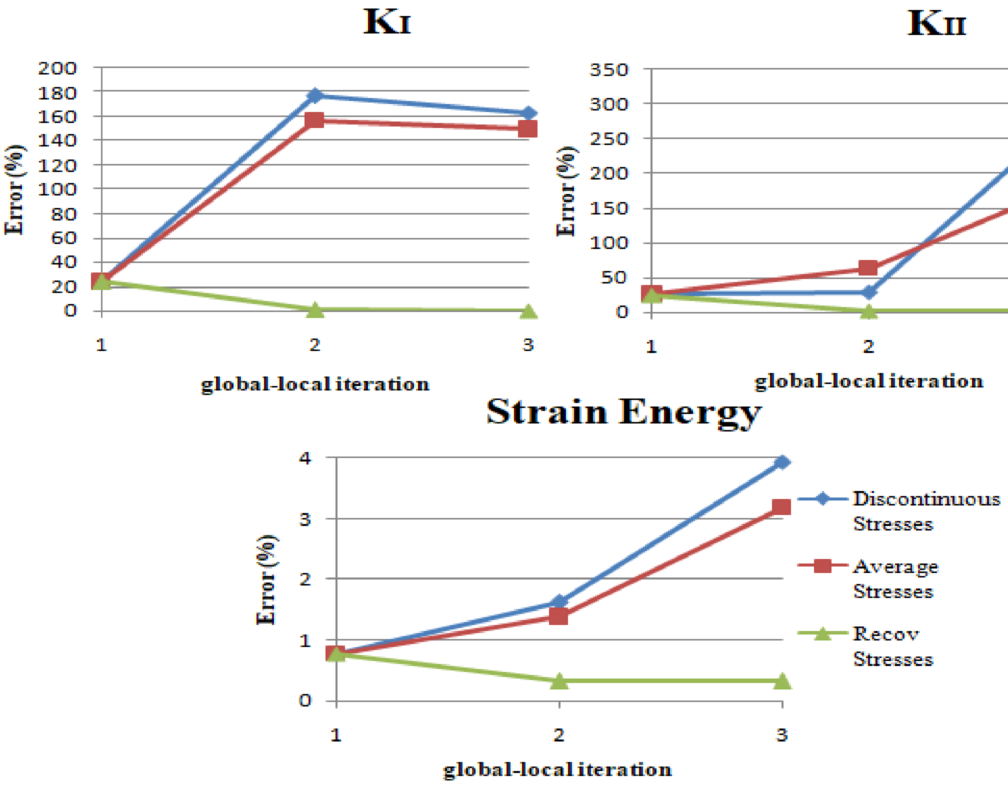

Figure 4. Stress intensity factors and strain energy obtained with Local Model I.

In the analyzes where discontinuous and average stresses were used, 12global-local iterations were needed until the convergence of K_I and K_{II} was reached, and the values in Figure 4 are presented only until the third iteration due to the lack of accuracy of the results. As can be seen in the figure, the results obtained with the discontinuous and the average stresses are very poor, especially for the stress intensity factors. Considering the recovered stresses, however, only 3 global-local iterations where required until K_I and K_{II} convergence, and the accuracy of the results is much better (errors smaller than 1% in all cases).

3.2 Local Model II

Aiming to improve the results and verify the effectiveness of discontinuous and average stresses, a larger local domain is tested. This case is identified as Local Model II, composed by 270 quadrilateral (Q4) elements, as shown in Figure 5. In this figure, the global elements highlighted in green represent the extension of Local Model II in comparison with Local Model I. Global nodes enriched with the global-local functions are highlighted in blue. The other parameters used in section 3.1 remained unchanged.

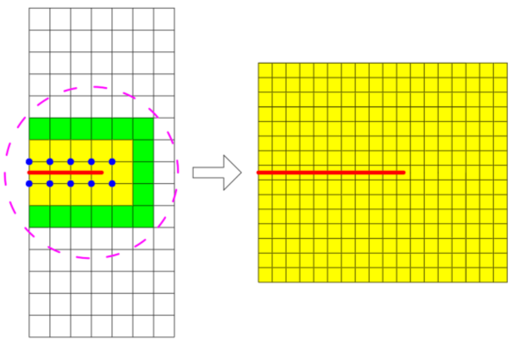

Figure 5. Local Model II (The local domain indicated in the right of the figure is out of scale).

Similarly to the analyses of section 3.1, discontinuous, average and recovered stresses where tested in Local Model II. The results are presented in Figure 6.

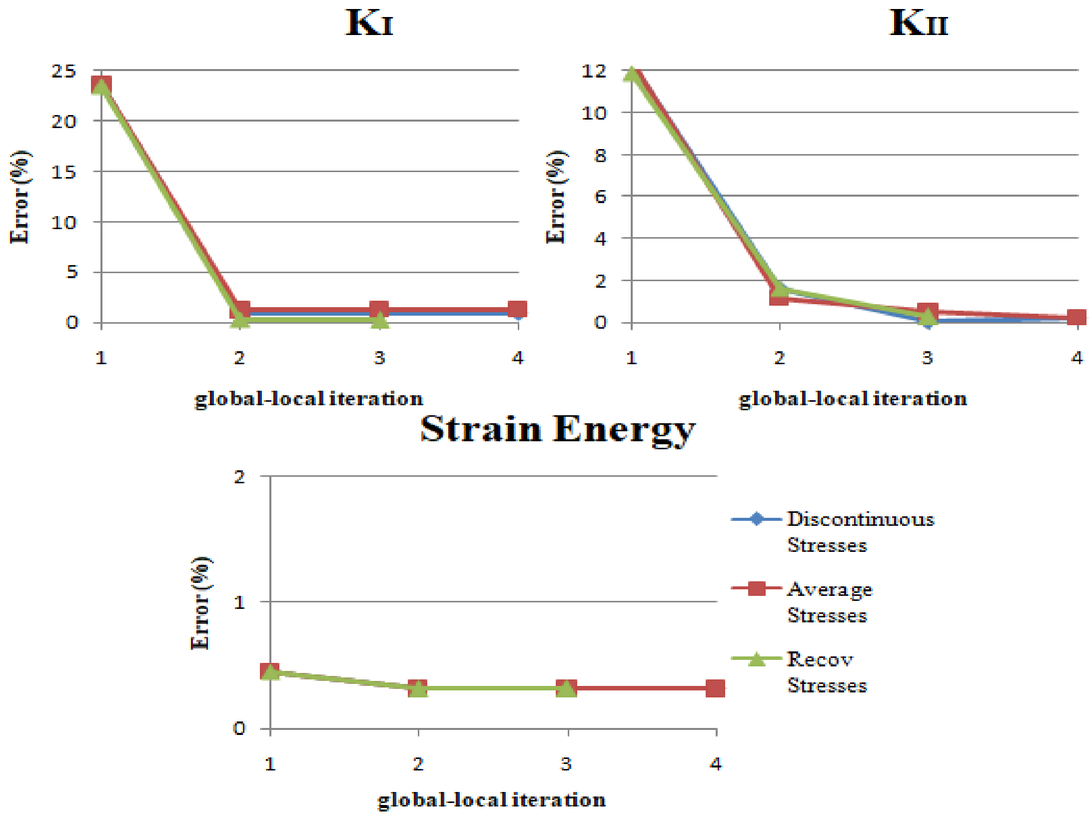

Figure 6. Errors associated with the results obtained with the Local Model II

In the analyses with discontinuous and average stresses, 4 global-local iterations where required for K_I and K_{II} convergence. When the recovered stresses where used, only 3 iterations where needed, as in the case of Local Model I. The results obtained with discontinuous and average stresses are much closer to the reference solutions when compared with Local Model I. Indeed, differently from the Model I, the global solution is smoother (and, therefore, better represented by the course global mesh) in the boundary of the local problem of the Model II as it is far from the crack tip. This improvement in the global solution implies a better approximation in the local model, which improves the accuracy of the stress intensity factors. In the case of the strain energy, the errors are significantly smaller since the first global-local iteration, as it is a global quantity with lower dependence on the local solution. As in the case of Local Model I, the results obtained with the recovered stresses are very accurate.

4 Conclusions

A summary of the results discussed in sections 3.1 and 3.2 is presented in Table 1.

Table 1. Comparison between the results obtained with Local Model I and Local Model II.

	Local Model I				Local Model II			
Stress field	K_I	K_{II}	Strain Energy	Number of global-local iterations	K_I	K_{II}	Strain Energy	Number of global-local iterations
Discontinuous Stresses	84.25	48.31	2.326E-2	12	34.31	4.54	2.459E-2	4
Average Stresses	76.82	35.41	2.321E-2	12	34.43	4.56	2.459E-2	4
Recovered Stresses	34.01	4.49	2.459E-2	3	33.93	4.54	2.459E-2	3

Considering the strategy of Local Model I, only the strategy of recovered stresses was capable to improve the quality of the results. In this case, the use of average stresses was not enough to control the error of the boundary conditions in the local model. This can be associated with the fact that the recovered stresses are not only continuous throughout the global domain but also capable of describing the asymptotic solution in the vicinity of the crack tip. The performance of the recovered stresses is also verified in terms of the number of global-local iterations required for K_I and K_{II} convergence, which was reduced by 75% compared with the results of average and discontinuous stresses.

Considering the results from Local Model II, it can be concluded that the expansion of the local domain improves the quality of the results associated with discontinuous and average stresses. In these cases, the level of accuracy is the same as the one obtained with the recovered stresses in the smaller local domain. This result confirms the effectiveness of the proposed approach in controlling the error of the boundary conditions in the $GFEM^{gl}$. Therefore, from a computational point of view, it is more interesting to use the recovered stresses and a smaller local domain than a larger local domain with average or discontinuous stresses. This topic is under investigation by our research group and will be reported in future.

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