Hybrid Discontinuous Galerkin methods for elliptic problems based on a Least-Squares variational principle

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Abstract. We propose new hybrid finite element methods for elliptic problems based on a Least-Squares variational principle (LS-h). We devised the LS-h formulation considering local minimization problems in each element of the mesh, with the objective function composed of Least-Squares residual terms in each element and local interface conditions (i.e., transmission conditions on the mesh skeleton). The LS-h formulation can be rewritten in terms of independent local problems and a coupled global problem. The former consists of Least-Squares formulations and the latter is written in terms of a Lagrange multiplier – identified as the trace of the primal variable – imposing the transmission condition on the mesh skeleton. Thus, we obtain the global system by static condensation, reducing considerably the number of unknowns to be solved. For the resulting algebraic system, through Singular Value Decomposition (SVD) numerical calculations, we estimate the condition number of the LS-h using the l^2-norm. We compare the LS-h with classical Hybridizable Discontinuous Galerkin (HDG), showing that LS-h has similar condition number estimates in spite of the different block structure in its resulting system. Furthermore, we performed numerical experiments using the method of manufactured solutions to show that LS-h has optimal convergence rates – in terms of l^2-norm – for both primal and flux variables.

Keywords: Discontinuous Least-Squares, Hybridization, Condition number, Static condensation

1 Introduction

In the context of Finite Element methods, the mixed formulation is a suitable approach to solve accurately for two or more variables, frequently primal (scalar) and associated flux (vector-valued) variables, as explained in Boffi et al. [1]. However, a higher computational cost is demanded to solve mixed problems when compared to classical primal formulations, since the number of unknowns is increased. The scenario is even worse when discontinuous approximations are used in the function spaces. Nonetheless, such approaches have desirable features: local mass conservation, optimal convergence rates for both variables, and stability (see Arnold et al. [2] for details).

A particular approach to derive stabilized mixed formulations – satisfying the inf-sup condition – is based on Least-Squares (LS) variational principles (for instance, see Bochev and Gunzburger [3]). Such formulations provide a way to devise finite element methods with more flexibility concerning the choice of function spaces and polynomial degrees in the discretization. The purely Least-Squares based formulations are well explored for a wide range of applications using conforming approximations, as shown in Bochev and Gunzburger [4] and Bochev and Gunzburger [3]. However, only a few pieces of work aiming at nonconforming LS are published. For instance, Houston et al. [5] and De Sterck et al. [6] tackled hyperbolic problems, and Lin [7] worked on reaction-diffusion problems. Recently, a weak Galerkin LS formulation for second-order elliptic problems was proposed in Mu et al. [8], followed by a discontinuous LS method addressing the same problem (see Ye and Zhang [9]).

On the other hand, discontinuous Galerkin (DG) methods are in active development. One of the main subjects is the *hybrid methods*, which solve a problem for unknowns in the elements' interior and on their boundaries (for some trace function), as defined in Oden and Reddy [10]. On the light of hybridization and static condensation, Cockburn and Gopalakrishnan [11] characterized the solution of classical hybridized mixed formulations. Such ideas paved the way for Cockburn et al. [12] to propose a unified framework to obtain hybridized formulations of classical methods. The fundamental outcome of this work is the Hybridizable Discontinuous Galerkin (HDG) methods, a powerful framework to formulate hybrid methods. The main advantage of HDG is its capability to perform static condensation, drastically reducing the number of unknowns while preserving the desirable features aforementioned. Thus, the high computational cost associated to discontinuous approximations can be decreased, providing a competitive alternative for DG methods.

In this context, the main goal of the present work is to introduce new LS formulations based on HDG ideas. The new method is composed of Least-Squares local problems, with flux conservation being weakly imposed by the transmission condition on the mesh skeleton through a Lagrange multiplier. A global system depending only on the Lagrange multiplier is obtained by static condensation, reducing the number of degrees of freedom (unknowns) of the problem. To the best of our knowledge, this is the first time such LS formulations are proposed. A similar idea is presented in Mu et al. [8], but combining an LS formulation with Weak Galerkin method, which requires additional vector-valued unknowns on mesh skeleton when compared with our approach.

2 Problem Statement

Throughout this work, we consider a scalar second-order elliptic problem with Dirichlet boundary condition. The model problem is rewritten as an equivalent first-order system, as usual for classical mixed finite element methods that obtain a solution for both primal and flux variables. Additionally, a curl equation is considered as did in Bochev and Gunzburger [3, 4] and Cai et al. [13], which is a well known rearrangement of the elliptic system due to its the irrotational characteristic.

The problem is stated as follows: Given an open and bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) with a Lipschitz boundary $\partial\Omega$, find the pair $\{\mathbf{u}, p\}$ in Ω satisfying:

$$\mathbf{A}\mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = f, \quad \nabla \times (\mathbf{A}\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega; \quad p = p_D \quad \text{on } \partial\Omega, \tag{1}$$

in which we call $\mathbf{u}$ and p as the flux and primal variable, respectively. Moreover, we have $\mathbf{A} \equiv \mathbf{K}^{-1}$, assuming that $\mathbf{K}$ is an $n \times n$ symmetric uniformly positive definite tensor of functions in $L^2(\Omega)$, as considered in Cai et al. [14], such that $\lambda\xi^T\xi \leq \xi^T\mathbf{K}\xi \leq \Lambda\xi^T\xi$ for all $\xi \in \mathbb{R}^n$ with $0 < \lambda \leq 1 \leq \Lambda$, and $\lambda, \Lambda \in \mathbb{R}$.

3 Weighted Least-Squares formulations

Combining the ideas presented in Bochev and Gunzburger [3, 4] for Finite Element methods based on Least-Squares variational principle with the Hybridizable Discontinuous Galerkin (HDG) framework proposed in Cockburn et al. [12], we devise new hybridizable L^2-Weighted Least-Squares formulations (LS-h, for short).

3.1 Preliminaries and Notation

Hereafter – for simplicity, although easily extensible to general cases – we adopt a polygonal domain $\Omega \subset \mathbb{R}^2$. Let $\mathcal{T}_h = \{K\} :=$ union of all elements K be a regular partition composed by non-overlapping K (triangles or quadrilaterals). We denote $\mathcal{E}_h^\circ$ and $\mathcal{E}_h^\partial$ the sets of interior and boundary edges, respectively, defined on the skeleton of the partition $\mathcal{T}_h$, which is represented as $\mathcal{E}_h$.

For scalar functions use the standard $L^2(D)$ inner product as

$$(u, v)_D = \int_D uvdx, \quad \langle u, v\rangle_S = \int_S uvds, \tag{2}$$

for $D \subset \mathbb{R}^n$ and $S \subset \mathbb{R}^{n-1}$, with appropriate extensions to vector or tensor functions. On broken function spaces we define

$$(u, v)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \int_K uvdx, \quad \langle u, v\rangle_{\mathcal{E}_h} = \sum_{e\in\mathcal{E}_h} \int_e uvds. \tag{3}$$

The associated $L^2(\omega)$-norms are given as $\|v\|_\omega := \sqrt{(v, v)_\omega}$, where $v \in L^2(\omega)$ with ω being $D, S, \mathcal{T}_h$ or $\mathcal{E}_h$.

3.2 Conforming Least Squares formulation

Before introducing the hybrid formulation, we recall the $H^1(\Omega)$ conforming formulation based ont the following weighted least squares functional

$$\mathcal{L}_{\mathrm{HC}}(\mathbf{v}, q) := \frac{1}{2}\left(\delta_1(\mathbf{A}\mathbf{v} + \nabla q, \mathbf{v} + \mathbf{K}\nabla q)_\Omega + \delta_2 \|\nabla \cdot \mathbf{v} - f\|_\Omega^2 + \delta_3 \|\nabla \times (\mathbf{A}\mathbf{v})\|_\Omega^2\right) \tag{4}$$

with $(\mathbf{v}, q) \in \mathbf{U}(\Omega) \times V(\Omega)$ where $\mathbf{U} = H^1(\Omega) \times H^1(\Omega)$ and $V = H^1_0(\Omega)$. The least squares minimization problem

$$(\mathbf{u}, p) = \arg\min \mathcal{L}_{\text{HC}}(\mathbf{v}, q), \qquad \forall (\mathbf{v}, q) \in \mathbf{U}(\Omega) \times V(\Omega). \tag{5}$$

is equivalent to solving the following weak problem: Find $(\mathbf{u}, p) \in \mathbf{U} \times V$ such that

$$B((\mathbf{u}, p), (\mathbf{v}, q)) = F(\mathbf{v}), \qquad \forall (\mathbf{v}, q) \in \mathbf{U} \times V, \tag{6}$$

with

$$B((\mathbf{u}, p), (\mathbf{v}, q)) := \delta_1(\mathbf{A}\mathbf{u} + \nabla, \mathbf{v} + \mathbf{K}\nabla q)_\Omega + \delta_2(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v})_\Omega + \delta_3(\nabla \times (\mathbf{A}\mathbf{u}_h), \nabla \times (\mathbf{A}\mathbf{v}))_\Omega \tag{7}$$

$$F(\mathbf{v}) := \delta_2(f, \nabla \cdot \mathbf{v})_\Omega \tag{8}$$

Finite element approximations of this $H^1(\Omega)$ conforming least squares formulation are analyzed in Cai et al. [13].

3.3 Hybrid Least Squares finite element formulation

To present the hybrid least squares finite element method, we introduce the following broken finite-dimensional polynomial spaces on the partition $\mathcal{T}_h$

$$\boldsymbol{\mathcal{U}}^k_h = \{\mathbf{v}_h \in [L^2(\Omega)]^2; \mathbf{v}_h|_K \in [\mathcal{P}^k_h(K)]^2, \ \forall K \in \mathcal{T}_h\}, \tag{9}$$

$$\mathcal{V}^k_h = \{q_h \in L^2(\Omega); q_h|_K \in \mathcal{P}^k_h(K), \ \forall K \in \mathcal{T}_h\}, \tag{10}$$

where $\mathcal{P}^k_h(K)$, $k \geq 1$, is the space of polynomials of degree k. On the edges $e \in \mathcal{E}_h$ we introduce the polynomial space

$$\mathcal{M}^n_h = \{\mu \in L^2(\mathcal{E}_h) : \mu|_e = \mathcal{P}^n_h(e), \ \forall e \in \mathcal{E}_h\}, \tag{11}$$

where $\mathcal{P}^n_h(e)$ is the space of discontinuous polynomials of degree less or equal to n on each edge e.

Our hybrid finite element formulation is constructed based on least following square functional

$$\begin{aligned}\mathcal{L}_{\text{lsh}}(\mathbf{v}, q, \mu) :=& \frac{1}{2}\left(\delta_1(\mathbf{A}\mathbf{v} + \nabla q, \mathbf{v} + \mathbf{K}\nabla q)_{\mathcal{T}_h} + \delta_2 \left\|\nabla \cdot \mathbf{v} - f\right\|^2_{\mathcal{T}_h} + \delta_3 \left\|\nabla \times (\mathbf{A}\mathbf{v})\right\|^2_{\mathcal{T}_h}\right) \\ &+ \sum_{K \in \mathcal{T}_h} \left(\frac{\delta_4}{2}\int_{\partial K} (q - \hat{q})^2 ds + \int_{\partial K} \mu \widehat{\mathbf{v}} \cdot \mathbf{n} ds\right),\end{aligned} \tag{12}$$

with $(\mathbf{v}, q, \hat{q}) \in \boldsymbol{\mathcal{U}}^k_h \times \mathcal{V}^k_h \times \mathcal{M}^n_h$, where $\widehat{q}$ is the trace of primal variable on the mesh skeleton, $\widehat{\mathbf{v}} := \mathbf{v} + \tau\,(q - \widehat{q})\,\mathbf{n}$, which is frequently called as the flux numerical trace and, τ is an edge stabilizing parameter and $\mathbf{n}$ the outward unit normal vector on ∂K. The δ_i parameters are weights whose values should be conveniently set. Through the functional defined in eq. (12), it is clear that $\mu \in \mathcal{M}^n_h$ is a Lagrange multiplier that weakly imposes the numerical flux conservation locally, enforcing the normal component of numerical flux to be zero on inter-element boundaries. We identify the trace of the primal variable as the Lagrange multiplier, i.e., $\hat{q} = \mu$ and $\hat{p} = \lambda$.

The minimization problem

$$(\mathbf{u}_h, p_h, \lambda_h) = \arg\min \mathcal{L}_{\text{lsh}}(\mathbf{v}, q, \mu), \qquad \forall (\mathbf{v}, q, \mu) \in \boldsymbol{\mathcal{U}}^k_k \times \mathcal{V}^k_h \times \mathcal{M}^n_h. \tag{13}$$

is equivalent to the following weak problem: Find $(\mathbf{u}_h, p, \lambda) \in \boldsymbol{\mathcal{U}}^k_h \times \mathcal{V}^k_h \times \mathcal{M}^n_h$ such that

$$B_{lsh}((\mathbf{u}_h, p, \lambda), (\mathbf{v}, q, \mu)) = F_{lsh}(\mathbf{v}), \qquad \forall (\mathbf{v}, q, \mu) \in \boldsymbol{\mathcal{U}}^k_h \times \mathcal{V}^k_h \times \mathcal{M}^n_h, \tag{14}$$

with

$$\begin{aligned}B_{lsh}((\mathbf{u}_h, p_h, \lambda_h), (\mathbf{v}, q, \mu)) := \sum_{K \in \mathcal{T}_h} [&\delta_1(\mathbf{A}\mathbf{u}_h + \nabla p_h, \mathbf{v} + \mathbf{K}\nabla q)_K + \delta_2(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v})_K \\ &+ \delta_3(\nabla \times (\mathbf{A}\mathbf{u}_h), \nabla \times (\mathbf{A}\mathbf{v}))_K + \delta_4 \left\langle p_h - \lambda_h, q - \mu\right\rangle_{\partial K} + \left\langle \widehat{\mathbf{u}}_h \cdot \mathbf{n}, \mu\right\rangle_{\partial K}]\end{aligned} \tag{15}$$

$$F_{lsh}(\mathbf{v}) := \sum_{K \in \mathcal{T}_h} \delta_2(f, \nabla \cdot \mathbf{v})_K \tag{16}$$

where $\widehat{\mathbf{u}}_h := \mathbf{u}_h + \tau\,(p_h - \lambda_h)\,\mathbf{n}$.

To solve eq. (14) two further developments can be considered. One possible approach is evaluating the inner product on δ_1-terms and applying the Green's identity, resulting:

$$\begin{aligned}\delta_1(\mathbf{A}\mathbf{u}_h + \nabla p_h, \mathbf{v} + \mathbf{K}\nabla q)_K = \delta_1(\mathbf{A}\mathbf{u}_h, \mathbf{v})_K - \delta_1(q, \nabla\cdot\mathbf{u}_h)_K - \delta_1(p_h, \nabla\cdot\mathbf{v})_K + \delta_1(\mathbf{K}\nabla p_h, \nabla q)_K \\ + \delta_1\left\langle q, \widehat{\mathbf{u}}_h\cdot\mathbf{n}\right\rangle_{\partial K} + \delta_1\left\langle \lambda_h, \mathbf{v}\cdot\mathbf{n}\right\rangle_{\partial K}\end{aligned} \tag{17}$$

which leads to a non-symmetric formulation when substituting into eq. (15).

Another approach to solve the system presented in eq. (14) is based on the incorporation of terms that provide a solution for the same strong local problem, avoiding the evaluation of the inner product in δ_1-terms. This technique is commonly employed to stabilize mixed finite element methods to circumvent the inf-sup condition, providing more flexibility to choose approximation function spaces, as proposed in Correa and Loula [15] and Masud and Hughes [16] for Darcy problems. In this way, we propose the inclusion of a Galerkin-type term related to the equation $\mathbf{A}\mathbf{u} + \nabla p = 0$ (flux residual), obtaining the following problem:

Find $(\mathbf{u}_h, p, \lambda) \in \boldsymbol{\mathcal{U}}_h^k \times \mathcal{V}_h^k \times \mathcal{M}_h^n$, such that

$$\begin{aligned}B_{lsh}((\mathbf{u}_h, p, \lambda), (\mathbf{v}, q, \mu) + S(\mathbf{u}_h, p_h, \lambda_h; \mathbf{v}) = F(\mathbf{v}) + \delta_1\left\langle \mathbb{P}_{M_h}(p_D), \mathbf{v}\cdot\mathbf{n}\right\rangle_{\partial K\cap\partial\Omega}, \\ \forall\mathbf{V} \in \boldsymbol{\mathcal{U}}_h^k \times \mathcal{V}_h^k \times \mathcal{M}_h^n \text{ and } \forall\mathbf{v} \in \boldsymbol{\mathcal{U}}_h^k,\end{aligned} \tag{18}$$

where

$$S(\mathbf{u}_h, p_h, \lambda_h; \mathbf{v}) := \delta_1(\mathbf{A}\mathbf{u}_h, \mathbf{v})_K - \delta_1(p_h, \nabla\cdot\mathbf{v})_K + \delta_1\left\langle \lambda_h, \mathbf{v}\cdot\mathbf{n}\right\rangle_{\partial K} \tag{19}$$

and $\mathbb{P}_{\mathcal{M}_h}(p_D)$ is the projection of p_D into $\mathcal{M}_h^n$.

Remark 1. In the above formulation, the parameter δ_1 is not a L^2-weight, but a generic stabilizing parameter. It is worth mentioning the close relation with the Stabilized Dual Hybrid Mixed (SDHM) proposed by Núñez et al. [17], which we can retrieve by adding a Galerkin-type term related to flux residual and choosing its stabilizing parameter value as one.

Remark 2. The stabilization with the Galerkin-type in eq. (18) is necessary for the associated discrete linear system solvability. Some clarifications in this regard are provided in the next section.

4 Algebraic System and Condition Number estimation

The formulations proposed in the previous section introduced local problems and a general variational formulation involving three unknown fields. Due to the hybridization, a global problem can be written only in terms of the Lagrange multiplier after manipulating the resulting systems with static condensation, as demonstrated in Cockburn and Gopalakrishnan [11]. This section's goal is to analyze the resulting algebraic system structure and its spectral condition number estimation to shed light in solvability aspects.

4.1 Algebraic System

We consider a general linear system structure of the form:

$$\mathbf{M}\mathbf{x} = \mathbf{f}, \tag{20}$$

where $\mathbf{M}$ is the discretization resulting matrix (LHS), $\mathbf{x}$ is the vector of unknowns (degrees of freedom), and $\mathbf{f}$ is the "load" vector (RHS). For the Hybrid Mixed system, as described in Gibson et al. [18], the following block structure is obtained:

$$\mathbf{M} = \begin{bmatrix}\mathbf{M}_{00} & \mathbf{M}_{01} & \mathbf{M}_{02} \\ \mathbf{M}_{10} & \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{20} & \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix}\mathbf{u} \\ \mathbf{p} \\ \boldsymbol{\lambda}\end{bmatrix}; \quad \mathbf{f} = \begin{bmatrix}\mathbf{f}_u \\ \mathbf{f}_p \\ \mathbf{f}_\lambda\end{bmatrix}, \tag{21}$$

where $\mathbf{u}$, $\mathbf{p}$ and $\boldsymbol{\lambda}$ are the unknowns' vectors related to $\mathbf{u}_h$, p_h, and λ_h, respectively. The load vector is partitioned analogously, and $\mathbf{M}_{ij}$ denotes block-matrices related to each unknown.

Through Schur complement computations, $\mathbf{u}$ and $\mathbf{p}$ can be eliminated and an equation depending only on the Lagrange multiplier is obtained:

$$\mathbf{S}\boldsymbol{\lambda} = \mathbf{f}_c \tag{22}$$

in which $\mathbf{S}$ is the global condensed system and $\mathbf{f}_c$ is the condensed load vector. It is worth noting that $\mathbf{S}$ can be assembled element-wisely, since it requires block contributions associated to local problems only. For hybrid methods, only the global system in eq. (22) has to be solved, and the solution for flux and primal variables can be easily recovered using the solution for $\boldsymbol{\lambda}$. We refer to Gibson et al. [18] for further details.

Remark 3. The condensed system in eq. (22) is symmetric for classical HDG formulations, as demonstrated by Cockburn et al. [12]. However, for the proposed LS-h formulations, the condensed system is not symmetric. This feature poses an open challenge for such formulations. Nonetheless, the global system is still structurally symmetric and sparse.

Remark 4. The general block-partitioned structure in eq. (21) shows the requirements to build a solvable global system properly coupled to the degrees of freedom (DoFs) internal to the elements. At least one of the block matrices related to the Lagrange multiplier ($\mathbf{M}_{02}$ or $\mathbf{M}_{12}$) should be a nonzero block. The equivalent counterpart ($\mathbf{M}_{20}$ or $\mathbf{M}_{21}$) in the $\boldsymbol{\lambda}$ equation should also be nonzero in order to couple the variables. Moreover, the block matrices related to the internal DoFs ($\mathbf{M}_{01}$ or $\mathbf{M}_{10}$) should be nonzero matrices.

4.2 Condition Number estimation

The conditioning of a linear system can indicate if the problem is unstable (or ill-posed). Although a poor conditioning (large condition number) is not a sufficient condition, as stated in Saad [19], it is well known that FEM discretizations with smaller condition numbers may require less iterations when using iterative methods to solve the resulting linear system. For instance, see Mardal and Winther [20] for mixed formulations on the light of inf-sup condition and Kirby [21] for a Lax-Milgram lemma perspective.

In this work, we estimate the "spectral" condition number of the condensed systems $\mathbf{S}$, which is defined as

$$\kappa_2(\mathbf{S}) = \|\mathbf{S}\|_2 \left\|\mathbf{S}^{-1}\right\|_2 = \frac{\sigma_{\max}(\mathbf{S})}{\sigma_{\min}(\mathbf{S})}, \tag{23}$$

in which $\|\bullet\|_2$ is a consistent matrix l^2-norm (euclidean norm), $\sigma_{\max}(\mathbf{S})$ is the largest singular value of $\mathbf{S}$, and $\sigma_{\min}(\mathbf{S})$ is the smallest singular value. To estimate the singular values, we used the Locally Optimal Block Preconditioned Conjugate Gradient Method (LOBPCG) proposed in Knyazev [22].

5 Results

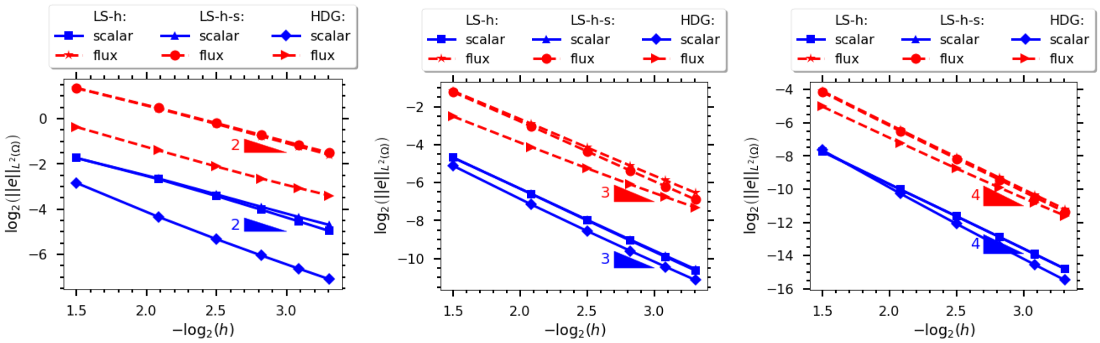
In the following, we provide the computed convergence rates using the method of manufactured solutions (MMS). Furthermore, the "spectral" condition number is numerically estimated using the LOBPCG freely available in Python's SciPy library (Virtanen et al. [23]). We implemented the formulations using Firedrake, an automated system to solve partial differential equations using FEM, described in Rathgeber et al. [24]. The resulting linear system solution is obtained through the LU direct sparse solver provided in MUMPS (see Amestoy et al. [25]). Throughout this section, we denote as LS-h the formulation presented in eq. (14) with the inner product expansion from eq. (17), and as LS-h-s the one presented in eq. (18).

Regarding the setup of the numerical experiments, for both convergence rates and condition numbers, we consider a uniform 2D unit square mesh, i.e., $\Omega = [0, 1] \times [0, 1]$ partitioned in triangles. The mesh characteristic size h is the element's circumscribed diameter. The polynomial degrees for all variables are chosen with equal order. We set $\delta_i = h^2$ for both methods, except for LS-h-s, which has $\delta_1 = 1$. To demonstrate the potential of the new formulations, we compare them with HDG results.

5.1 Convergence rates

To numerically estimate convergence rates, we employ the MMS. The exact solution is given as:

$$p(x, y) = \sin(2\pi x)\sin(2\pi y) \tag{24}$$



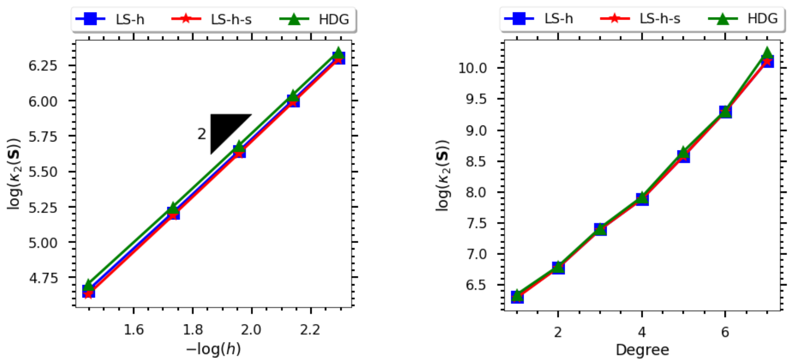
(a) $k = 1$. (b) $k = 2$. (c) $k = 3$.

Figure 1. Convergence rates in L^2-norm for LS-h formulations for different polynomial degrees k.

The convergence rates – for scalar and flux variables – are computed for $1 \leq k \leq 3$. Fig. 1 shows the results. With the exception for LS-h-s when $k = 1$, all convergence rates are optimal for both variables.

5.2 Condition number

The condition number is estimated on the condensed matrix $\mathbf{S}$ presented in eq. (22). Fig. 2a shows how condition number varies according to different characteristic mesh sizes while holding the polynomial degrees fixed ($k = 1$). Fig. 2b depicts how the condition number changes due to increases in the polynomial degrees when the mesh is fixed. For both LS-h and LS-h-s, the results are slightly better than those obtained with HDG.



(a) Varying mesh size. (b) Varying polynomial degrees.

Figure 2. Condition number results (in log scale) varying (a) the mesh size with a fixed polynomial degree ($k = 1$), and (b) the polynomial degrees for a given mesh with 14×14 triangles.

6 Conclusions

In this work, we proposed new finite element formulations based on Least-Squares variational principles, namely, LS-h and LS-h-s. The results show that both formulations are competitive with a classical HDG method when comparing convergence rates and condition numbers. However, the study of such formulations is still in progress, demanding further analysis (to be addressed in future work).

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