An efficient two-stage form of the Crank-Nicolson scheme with application to an HDG method for poroelasticity

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Abstract. We propose an alternative form of the Crank-Nicolson scheme devised for linear space operators, though its properties, such as convergence and stability, also hold for nonlinear ones. The discretization mimics the implicit Euler scheme by adding a decoupled weighted average equation as a second stage for each time step. This creates a two-stage method in contrast to the classical one-stage one, but it also evaluates the space operator at only the middle point, therefore generating a scheme that is computationally more efficient for highly expensive space operators. The scheme also simplifies the time update of the solution, and its implementation easily extends from the implementation of the implicit Euler scheme. In order to verify the scheme's stability and convergence, we apply it to a series of time dependent one-dimensional problems, and also to a two-dimensional poroelasticity problem, also known as Biot's Consolidation problem. Our application to the Consolidation problem uses an HDG method to approximate the space operator, which exemplifies yet another feature of our scheme: it does not rely on Lagrange multipliers to update the solution at the interior of elements.

Keywords: Crank-Nicolson, Poroelasticity, Hybrid Discontinuous Galerkin

1 Introduction

Perhaps the most commonly used time integration schemes for nonlinear problems is the Implicit Euler scheme. This scheme is popular in this class of problems mostly because of two main reasons: it is unconditionally stable (A-stability) and its approximation properties are optimal in the sense that higher order schemes do not improve convergence, see LeVeque [1]. This last constraint usually appears from the presence of discontinuous traveling wave solutions which drop time converge to a first order one.

For linear problems and ODEs, however, the application of higher order schemes usually pays of, as one can get improved convergence orders in time. Examples of higher order time integration schemes include Runge-Kutta schemes, linear multi-step schemes such as Adams-Bashforth and Adams-Moulton, and compact finite difference schemes, see Gottlieb et al. [2], Deshpande et al. [3]. High order linear Runge-Kutta schemes have the advantage of being easy to implement, though they provide only conditional stability. Nonlinear Runge-Kutta schemes are better conditioned, but they become highly expensive for increasing accuracy orders. Adams-Bashforth and Adams-Moulton schemes are very attractive in the computational sense, but their stability region decreases rapidly with increasing accuracy order. These schemes also require $k-1$ initial conditions for k-th order convergence, which implies having to use other schemes to approximate the other $k-2$ initial conditions. Compact finite differences are usually complicated schemes with many floating point calculations, although they improve stability constraints of many explicit methods.

We propose in this work an alternative form of the Crank-Nicolson scheme, which is less expensive and easily extended from an implementation of an implicit Euler scheme. The first idea for the scheme is to be presented in Hoyos [4] as a solution for avoiding dealing with past solutions of Lagrange multipliers for an HDG method. We build upon the idea of the reference, using the same scheme but providing numerical evidence of its second order convergence. More details regarding numerical analysis can be found in Hoyos [4].

We apply the proposed scheme to linear and nonlinear problems in 1D, and also to approximate the time dependent terms of the poroelasticity equations. Poroelasticity equations, or elsewhere known as Biot's consolidation problem, couple linear elasticity equations with Poisson's equation in a time dependent regime, and are posed in the following form: given a body force $\mathbf{f}(t)$ and volumetric fluid source $s(t)$, find the displacement field $\mathbf{u}(t) : \Omega \to \mathbb{R}^2$ and the pore pressure $p(t) : \Omega \to \mathbb{R}$ such that for $t > 0$ there holds

$$-\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}, p) = \mathbf{f}, \qquad \frac{\partial}{\partial t}\left(c_0 p + \alpha \operatorname{div} \mathbf{u}\right) - \frac{1}{\eta} \operatorname{div}\left[\kappa\left(\nabla p - \rho \mathbf{g}\right)\right] = s, \qquad \text{in} \quad \Omega \times \mathbb{R}_*^+, \tag{1}$$

where the Cauchy stress tensor $\boldsymbol{\sigma}(\mathbf{u}, p)$ is given by $\boldsymbol{\sigma}(\mathbf{u}, p) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(tr\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} - \alpha p\mathbf{I}$. Here λ, μ are the Lammé parameters, $\mathbf{g}$ represents the gravitational acceleration vector, κ is the permeability of the porous media,

c_0 is the constrained specific storage constraint, α is the Biot-Willis coefficient and η and ρ are the viscosity and density of the pore fluid. For simplicity, we consider boundary conditions of the type

$$\mathbf{u} = \mathbf{u}_\partial, \quad -\frac{\kappa}{\eta}\nabla p \cdot \mathbf{n} = g, \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \qquad [2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(tr\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I}]\,\mathbf{n} = \mathbf{h}, \quad p = p_\partial, \quad \text{on } \Gamma_2 \times \mathbb{R}^+, \tag{2}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$. More general boundary conditions can be accommodated with small modifications. We assume an incompressible solid as the initial condition in $t = 0$, with the pore pressure field satisfying the Stokes equations

$$-\operatorname{div}(2\mu\boldsymbol{\varepsilon}(\mathbf{u}) - \alpha p\mathbf{I}) = \mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0, \qquad \text{in} \quad \Omega. \tag{3}$$

In the sections to come, we first design the alternative form of the Crank-Nicolson scheme for linear space operators, providing evidence of the second order convergence. In the subsequent section we discretize the poroelasticity equations by means of the just exposed Crank-Nicolson scheme and an HDG formulation first introduced in Ledoino [5]. The final section presents numerical evidence of second order convergence of the time integration scheme, applying it to approximate the locking-free numerical solution to Biot's consolidation problem in time.

2 A two-stage version of the Crank-Nicolson scheme

We now introduce the alternative form of the Crank-Nicolson scheme. To fix ideas, we will be focusing on the time discretization of the following problem:

$$\partial_t u + L(u, x, t) = 0, \tag{4}$$

where ∂_t^n and ∂_x^n denote n-th order time and space derivatives respectively, and $L(\cdot,\cdot,\cdot)$ is the space operator. We shorten $u(x_i, t_j)$ to u_i^j where $x_i = i\Delta x$ and $j = j\Delta t$ for $\Delta x > 0$ and $\Delta t > 0$ being the space and time steps respectively. We may as well write L_i^j rather than $L(u_i^j, x_i, t_j)$, and when the subscript $[\cdot]_i$ is omitted, it means that we do not treat space discretization.

The Crank-Nicolson scheme proposes time stepping from t_j to t_{j+1} by means of an evaluation of the PDE in (4) at the midpoints $t_{j+\frac{1}{2}}$: $(\partial_t u)^{j+\frac{1}{2}} + L^{j+\frac{1}{2}} = 0$. For the time derivative, one expands u^j and u^{j+1} into Taylor series about $u^{j+\frac{1}{2}}$, obtaining $(\partial_t u)^{j+\frac{1}{2}} = \frac{u^{j+1}-u^j}{\Delta t} + O(\Delta t^2)$. Being able to solve (4) in time for u^{j+1} still requires getting rid of the superscript $t_{j+\frac{1}{2}}$ in L. The classical way of doing so is to use the second order average approximation given by $\frac{L^{j+1}+L^j}{2} = L^{j+\frac{1}{2}} + O(\Delta t^2)$. The final approximation in time reads

$$\frac{u^{j+1} - u^j}{\Delta t} + \frac{L^{j+1} + L^j}{2} = O(\Delta t^2). \tag{5}$$

There is, however, another way of implementing the Crank-Nicolson scheme, which we introduce next. The idea comes from an attempt to avoid evaluating the space operator (here represented by L) at more than one point in time. Let us suppose that the space operator L is linear. This means that the scheme (5) can be written as

$$\frac{u^{j+1} - u^j}{\Delta t} + L\left(\frac{u^{j+1} + u^j}{2}\right) = O(\Delta t^2). \tag{6}$$

Now we notice that the input value of L in the equation above is actually a second order average approximation for $u^{j+\frac{1}{2}}$. That is, we can rewrite the equation above as the system

$$\frac{u^{j+1} - u^j}{\Delta t} + L^{j+\frac{1}{2}} = O(\Delta t^2), \qquad \frac{u^{j+1} + u^j}{2} - u^{j+\frac{1}{2}} = O(\Delta t^2). \tag{7}$$

Since the second order average approximation is linear, it can be solved for u^{j+1} and replaced in the first equation. The final method has two stages, given by

$$2\frac{u^{j+\frac{1}{2}} - u^j}{\Delta t} + L^{j+\frac{1}{2}} = 0, \tag{8}$$

$$u^{j+1} = 2u^{j+\frac{1}{2}} - u^j. \tag{9}$$

The idea is that one solves for $u^{j+\frac{1}{2}}$ in the first stage (8), which is the expensive one, and then uses the solution in the second stage (9), therefore calculating u^{j+1}. The space operator is now evaluated at one single point in time ($L^{j+\frac{1}{2}}$) rather than two points (L^j and L^{j+1}), as in the classical implementation. This reduces computational cost considerably for highly expensive space operators L. Notice that the form of equation (8) differs from that of an

implicit Euler scheme by a multiplication of a constant (the 2 multiplying the time derivative approximation). This fact makes it very easy to extend an implementation of implicit Euler to Crank-Nicolson.

One question that arises during the development of this version of the Crank-Nicolson scheme is whether or not it is first order convergent rather than second order one. This suspicion comes from the fact that if one carries out the error of equation (9) into equation (8), an order 1 term appears into the right hand side of the latter. Given that it is in fact second order convergent (not only for linear operators L, but also for nonlinear ones, as shown in the numerical experiments below), another thought that comes to mind is whether the choice $j+\frac{1}{2}$ is mandatory for having such property. Since we want to time step from t_j to t_{j+1}, evaluating the PDE at any other midpoint $t_{j+\theta}$ with $0<\theta<1$ would be equally acceptable, given that the outcome were still a second order approximation. Aiming to investigate these questions, let us now consider evaluating the PDE at the midpoint $t_{j+\theta}$: $(\partial_t u)^{j+\theta}+L^{j+\theta}=0$. From the Taylor expansions $u^j = u^{j+\theta}+(\partial_t u)^{j+\theta}(-\theta)\Delta t+(\partial_t^2 u)^{j+\theta}(-\theta)^2\frac{\Delta t^2}{2}+O(\Delta t^3)$ and $u^{j+1} = u^{j+\theta}+(\partial_t u)^{j+\theta}(1-\theta)\Delta t+(\partial_t^2 u)^{j+\theta}(1-\theta)^2\frac{\Delta t^2}{2}+O(\Delta t^3)$, we get

$$(\partial_t u)^{j+\theta} = \frac{\theta}{(1-\theta)\Delta t}u^{j+1}+\frac{1-2\theta}{(1-\theta)\Delta t}u^{j+\theta}-\frac{1-\theta}{\theta\Delta t}u^j+O(\Delta t^2). \tag{10}$$

From the above equation we see why the choice $\theta=\frac{1}{2}$ is made in the classical Crank-Nicolson scheme: it is the only one that removes the term with $u^{j+\theta}$ from the expression. However, since we actually want to solve a system for u^{j+1} and $u^{j+\theta}$, we proceed to find the second order average approximation. We use again the Taylor expansions of u^j and u^{j+1} about $t_j+\theta\Delta t$, which allow us to write

$$\theta u^{j+1}-u^{j+\theta}+(1-\theta)u^j = \theta(1-\theta)(\partial_t^2 u)^{j+\theta}\frac{\Delta t^2}{2}+O(\Delta t^3). \tag{11}$$

Let us now turn to one of the simplest differential equations one can get: we fix $L(u,x,t)=-p(t)$, with $p(t)$ being a polynomial in t. The original PDE then turns to the simple ODE $\partial_t u = p(t)$, which can be solved by direct integration. As a result, the system we must solve in order to time step from t_j to t_{j+1} is

$$\begin{cases} \frac{\theta}{(1-\theta)\Delta t}u^{j+1}+ & \frac{1-2\theta}{\theta(1-\theta)\Delta t}u^{j+\theta} & = p^{j+\theta}+\frac{1-\theta}{\theta\Delta t}u^j+O(\Delta t^2), \\ \theta u^{j+1}- & u^{j+\theta} & = (\theta-1)u^j+\theta(1-\theta)(\partial_t^2 u)^{j+\theta}\frac{\Delta t^2}{2}+O(\Delta t^3). \end{cases} \tag{12}$$

These equations are solved exactly, leading to the following solution for u^{j+1}:

$$u^{j+1} = u^j+p^{j+\theta}\Delta t+(1-2\theta)(\partial_t^2 u)^{j+\theta}\frac{\Delta t^2}{2}+\theta(1-\theta)(\partial_t^3 u)^{j+\theta}\frac{\Delta t^3}{3!}+O(\Delta t^4). \tag{13}$$

From this equation we see that only the choice $\theta=\frac{1}{2}$ could cancel out the second order term in the expansion. Now we take $\theta=\frac{1}{2}$ and we rewrite (13) in the following form:

$$\int_{t_j}^{t_{j+1}}\partial_t u\,dt = \int_{t_j}^{t_{j+1}}p(t)\,dt = p^{j+\frac{1}{2}}\Delta t+(\partial_t^2 p)^{j+\frac{1}{2}}\frac{\Delta t^3}{24}+O(\Delta t^4). \tag{14}$$

Notice that if the polynomial $p(t)$ is a first order one, then this version of the Crank-Nicolson scheme gives an exact approximation to the ODE. That is, with one point evaluation we can integrate exactly all polynomials $p(t)$ of order up to 1. Obviously, this can only be true if the midpoint $t_{j+\frac{1}{2}}=t_j+\frac{\Delta t}{2}$ is a Gaussian quadrature point, and the coefficient Δt of the polynomial evaluation is a Gaussian quadrature weight. In fact, these values are the point and weight of the one-point Gaussian quadrature, but scaled to the interval $[t_j,t_{j+1}]$.

3 SMHB: A locking-free HDG method for poroelasticity

Numerical methods for solving poroelasticity equations can be strongly affected by the presence of two nonphysical scenarios: spurious *pressure oscillations* and *volumetric locking*. Pressure oscillations, or sometimes referred to as poroelasticity locking, happen either in the case of a time discretization with a time step too small, or in the case of incompressible fluid with small permeability of the media. Volumetric locking, on the other hand, is an issue related to numerical methods for elasticity. It happens in cases of nearly incompressible solids, making the original elasticity system approach a Stokes problem. A spatial HDG approximation that solves both numerical issues has been proposed and analyzed in Ledoino [5]. To solve the elasticity locking, the authors proposed altering the original primal problem (1) to a mixed one in the unknowns $\mathbf{u},\phi,p$, where the hydrostatic pressure

$\phi = -\lambda \operatorname{div} \mathbf{u}$ contains the compressibility of the solid[1]. To solve the poroelasticity locking, the authors proposed using a hybrid-mortar HDG method to approximate the Darcy equations. Since the method is discontinuous, it captures well changes in the gradient of the pore pressure field.

We present now the SMHB method for approximating model problem (1). For the time discretization we use the scheme presented in the previous section, and for the space discretization, we use the HDG method presented and analyzed in Ledoino [5]. The final approximation, which inherits the locking properties of Ledoino [5], reads: for each $j = 0, \ldots, J$, and given the previous solution $[\mathbf{u}_h^j, p_h^j]$, find the displacement approximation $\mathbf{u}_h^{j+\frac{1}{2}} \in \mathcal{V}_h^k$, the hydrostatic pressure approximation $\phi_h^{j+\frac{1}{2}} \in \mathcal{Q}_h^l$, the pore pressure approximation $p_h^{j+\frac{1}{2}} \in \mathcal{P}_h^r$, and the discontinuous Lagrange multipliers $[\hat{\mathbf{u}}_h^{j+\frac{1}{2}}, \hat{p}_h^{j+\frac{1}{2}}] \in \mathcal{M}_h^m$ such that

$$
\left\{
\begin{aligned}
a_h^E([\mathbf{u}_h^{j+\frac{1}{2}}, \hat{\mathbf{u}}_h^{j+\frac{1}{2}}], [\mathbf{v}_h, \hat{\mathbf{v}}_h]) + b_h(\phi_h^{j+\frac{1}{2}}, [\mathbf{v}_h, \hat{\mathbf{v}}_h]) + \alpha b_h(p_h^{j+\frac{1}{2}}, [\mathbf{v}_h, \hat{\mathbf{v}}_h]) &= (\mathbf{f}^{j+\frac{1}{2}}, \mathbf{v}_h)_{\mathcal{T}_h} + \langle \mathbf{h}, \hat{\mathbf{v}} \rangle_{\mathcal{E}_h^2}, \\
a_h^S(\phi_h^{j+\frac{1}{2}}, \varphi_h) + b_h(\varphi_h, [\mathbf{u}_h^{j+\frac{1}{2}}, \hat{\mathbf{u}}_h^{j+\frac{1}{2}}]) &= 0, \\
a_{h,1}^P(p_h^{j+\frac{1}{2}}, q_h) + \alpha b_{h,1}(q_h, \mathbf{u}_h^{j+\frac{1}{2}}) + \frac{\Delta t}{2} a_{h,2}^P([p_h^{j+\frac{1}{2}}, \hat{p}_h^{j+\frac{1}{2}}], [q_h, \hat{q}_h]) &= -\frac{\Delta t}{2}(s^{j+\frac{1}{2}}, q_h)_{\mathcal{T}_h} \\
-\frac{\Delta t}{2}\frac{\rho}{\eta}(\mathbf{g}, q_h)_{\mathcal{T}_h} + \frac{\Delta t}{2}\langle g, \hat{p}_h \rangle_{\mathcal{E}_h^1} + a_{h,1}^P(p_h^j, q_h) &+ \alpha b_{h,1}(q_h, \mathbf{u}_h^j)
\end{aligned}
\right. \tag{15}
$$

for every $\mathbf{v}_h \in \mathcal{V}_h^k$, $\varphi_h \in \mathcal{Q}_h^l$, $q_h \in \mathcal{P}_h^r$, and $[\hat{\mathbf{v}}_h, \hat{q}_h] \in \mathcal{M}_h^m$, and then, using the just calculated interior solutions $\mathbf{u}_h^{j+\frac{1}{2}}$, and $p_h^{j+\frac{1}{2}}$, find the solutions at next step $\mathbf{u}_h^{j+1}$, and p_h^{j+1} through the second order average equation

$$
\mathbf{u}_h^{j+1} = 2\mathbf{u}_h^{j+\frac{1}{2}} - \mathbf{u}_h^j, \qquad\qquad p_h^{j+1} = 2p_h^{j+\frac{1}{2}} - p_h^j. \tag{16}
$$

The bilinear forms in the systems above are given by the following expressions:

$$
\begin{aligned}
a_h^E([\mathbf{u}_h, \hat{\mathbf{u}}_h], [\mathbf{v}_h, \hat{\mathbf{v}}_h]) &= 2\mu(\boldsymbol{\varepsilon}(\mathbf{u}_h), \boldsymbol{\varepsilon}(\mathbf{v}_h))_{\mathcal{T}_h} - 2\mu\langle \boldsymbol{\varepsilon}(\mathbf{u}_h)\,\mathbf{n}, (\mathbf{v}_h - \hat{\mathbf{v}}_h)\rangle_{\mathcal{E}_h} \\
&\quad - 2\mu\langle \boldsymbol{\varepsilon}(\mathbf{v}_h)\,\mathbf{n}, (\mathbf{u}_h - \hat{\mathbf{u}}_h)\rangle_{\mathcal{E}_h} + \mathbf{s}_{SMH}([\mathbf{u}_h, \hat{\mathbf{u}}_h], [\mathbf{v}_h, \hat{\mathbf{v}}_h]),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
a_{h,2}^P([p_h, \hat{p_h}], [q_h, \hat{q_h}]) &= -\frac{1}{\eta}(\kappa\nabla p_h, \nabla q_h)_{\mathcal{T}_h} + \frac{1}{\eta}\langle \kappa\nabla p_h \cdot \mathbf{n}, (q_h - \hat{q}_h)\rangle_{\mathcal{E}_h} \\
&\quad + \frac{1}{\eta}\langle \kappa\nabla q_h \cdot \mathbf{n}, (p_h - \hat{p}_h)\rangle_{\mathcal{E}_h} - s_{SHM}([p_h, \hat{p_h}], [q_h, \hat{q_h}]),
\end{aligned} \tag{18}
$$

$$
b_h(\phi_h, [\mathbf{v}_h, \hat{\mathbf{v}}_h]) = -(\phi_h, \operatorname{div} \mathbf{v}_h)_{\mathcal{T}_h} + \langle \phi_h, (\mathbf{v}_h - \hat{\mathbf{v}}_h)\cdot\mathbf{n}\rangle_{\mathcal{E}_h}, \tag{19}
$$

$$
a_h^S(\phi_h, \varphi_h) = -\frac{1}{\lambda}(\phi_h, \varphi_h)_{\mathcal{T}_h}, \qquad a_{h,1}^P(p_h, q_h) = -c_0(p_h, q_h)_{\mathcal{T}_h}, \qquad b_{h,1}(q_h, \mathbf{v}_h) = (q_h, \operatorname{div} \mathbf{v}_h)_{\mathcal{T}_h}, \tag{20}
$$

where $\mathcal{T}_h$ is a mesh of polytopal elements, $\mathcal{E}_h$ is the collection of element interfaces, and $\mathcal{E}_h^1$ and $\mathcal{E}_h^2$ are element interfaces that live at the two boundaries Γ_1 and Γ_2. In the expressions above, we used the classical inner products in L^2: $(p, q)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\int_K p\,q\,d\mathbf{x}$, $\langle p, q\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\int_{\partial K} p\,q\,ds$, and $\langle p, q\rangle_{\mathcal{E}_h} := \sum_{e\in\mathcal{E}_h}\int_e p\,q\,ds$. Essential for obtaining coercivity of the symmetric HDG formulation, the stabilization bilinear forms are given by

$$
\mathbf{s}_{SMH}([\mathbf{u}_h, \hat{\mathbf{u}}_h], [\mathbf{v}_h, \hat{\mathbf{v}}_h]) = 2\mu\frac{\beta_1}{h}\langle \hat{\pi}_h^m(\mathbf{u}_h - \hat{\mathbf{u}}_h), \hat{\pi}_h^m(\mathbf{v}_h - \hat{\mathbf{v}}_h)\rangle_{\mathcal{E}_h}, \tag{21}
$$

$$
s_{SHM}([p_h, \hat{p_h}], [q_h, \hat{q_h}]) = \frac{|\kappa|}{\eta}\frac{\beta_2}{h}\langle \hat{\pi}_h^m(p_h - \hat{p}_h), \hat{\pi}_h^m(q_h - \hat{q}_h)\rangle_{\mathcal{E}_h}, \tag{22}
$$

with $\hat{\pi}_h^m$ being the L^2 orthogonal projector at the space $\mathbb{P}_m$ at the interfaces of the mesh. The presence of this projector operator allows one to choose the space of the Lagrange multipliers one order lower than usually needed: $m = k - 1$, as first discussed in Oikawa [8]. This property is appealing computationally because, when allied to static condensation of local inner degrees of freedom, it reduces expressively the size of the main linear system of the problem.

The SMHB formulation is applicable to general shaped polytopes because, since it does not need to be continuous along element interfaces, it can use physical-frame polynomials rather than reference-frame ones: $\mathcal{V}_h^k = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \mathbf{v}|_K \in [\mathbb{P}_k(K)]^d,\ \forall K \in \mathcal{T}_h\}$, $\mathcal{Q}_h^l = \{\varphi \in L^2(\Omega); \varphi|_K \in \mathbb{P}_l(K),\ \forall K \in \mathcal{T}_h\}$, $\mathcal{P}_h^r = \{q \in L^2(\Omega); q|_K \in \mathbb{P}_r(K),\ \forall K \in \mathcal{T}_h\}$. Since our Lagrange multipliers are discontinuous, they can also be approximated by physical-frame polynomials. However, we have used reference-frame polynomials to approximate them in our simulations:

$$
\mathcal{M}_h^m = \{\hat{\mathbf{v}}_h \in \mathbf{L}^2(\mathcal{E}_h) : \hat{\mathbf{v}}_h|_e \in [\mathbb{P}_m^{d-1}(\sigma)\circ\boldsymbol{\Psi}_e^{-1}]^d,\ e = \boldsymbol{\Psi}_e(\sigma),\ \forall e \in \mathcal{E}_h^0;\ \hat{\mathbf{v}}_h|_e = \mathbf{0},\ \forall e \in \mathcal{E}_h^\partial\}, \tag{23}
$$

with $\mathbb{P}_m^{d-1}(\sigma)$ denoting the $(d-1)$-variate space of polynomial functions of degree at most m on the reference facet σ, mapped to the physical facet e through the mapping $\boldsymbol{\Psi}_e$.

[1]The addition of an extra scalar unknown in order to solve volumetric locking is not a new idea: see for example Hughes [6], §4.2, in the context of continuous Galerkin methods, or Hansbo and Larson [7], §4.1, in the context of DG methods

Problem 1			Problem 2			Problem 3			Problem 4		
Δt	$\log(Err)$	Rate	Δt	$\log(Err)$	Rate	Δt	$\log(Err)$	Rate	Δt	$\log(Err)$	Rate
0.6250	0.056		2.500	-1.728		2.500	0.406		1.500	-0.996	
0.3125	-0.542	1.988	1.250	-2.365	1.868	1.250	-0.085	1.631	0.750	-1.558	1.868
0.1563	-1.144	1.997	0.625	-2.978	1.999	0.625	-0.685	1.995	0.375	-2.160	1.999
0.0781	-1.745	1.999	0.313	-3.583	2.008	0.313	-1.312	2.082	0.188	-2.764	2.008
0.0391	-2.347	2.000	0.156	-4.186	2.002	0.156	-1.913	1.997	0.094	-3.367	2.002
0.0195	-2.950	2.000	0.078	-4.788	2.001	0.078	-2.515	1.998	0.047	-3.969	2.001

Table 1. Errors and convergence rates for the application of the two-stage version of the Crank-Nicolson scheme to ODEs and 1D PDEs

4 Numerical Experiments

We now provide numerical evidence that the alternative form of the Crank-Nicolson scheme presented in section 2 is second order convergent not only for linear problems, but also for nonlinear ones.

4.1 The two-stage Crank-Nicolson scheme for ODEs and 1D PDEs

Let us suppose that the operator L is given by $L(u,x,t) = -g(u,t)$, so that problem (4) turns to the ODE

$$\frac{du}{dt} = g(u,t), \quad 0 < t < t_f; \qquad\qquad u(0) = u_0, \quad t = 0. \tag{24}$$

As a first test, we consider $g(u,t)$ to be an infinite order polynomial function, given by $g(u,t) = e^t$, with $u_0 = 1$, $t_f = 5$. This ODE is linear and easily integrated, and the application of the proposed time integration scheme should with no doubt provide second order convergence. This is in fact what we observe in Table 1, where the results are labeled "Problem 1". As a second test, we verify whether or not the convergence rates hold for a nonlinear problem. We now take the source function to be $g(u,t) = -u^2 e^{-\frac{1}{u}}$, with $u_0 = 1$, $t_f = 5$. Notice that the solution to this problem can be calculated to be $u(t) = \frac{1}{\ln(t+e)}$, which has infinite power series expansion. We observe from Table 1, results labeled "Problem 2", that convergence rates once more approach 2. Although the problem is nonlinear, we verify that the properties of the Crank-Nicolson scheme are still maintained, in particular, its stability and convergence.

The next test considers the transient problem to be a 1D convection-diffusion-reaction PDE in the form

$$\partial_t u + \partial_x f(u) = \partial_x (b(u)\partial_x u) + s(u), \quad x_L < x < x_R; \quad 0 < t < t_f; \tag{25}$$

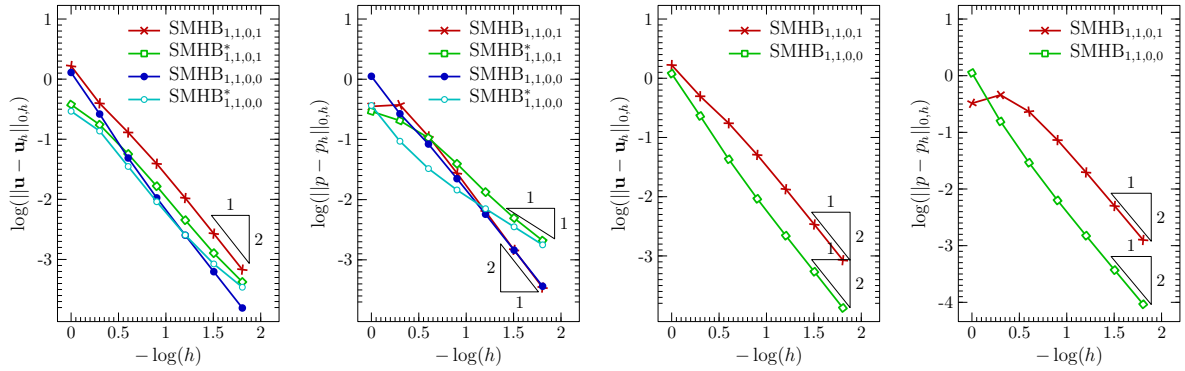
$$u(x_L,t) = u_L(t), \quad u(x_R,t) = u_R(t), \quad t > 0; \quad u(x,0) = u_0(x), \quad x_L < x < x_R;$$

Following Lambert et al. [9], we apply a second order space discretization to this problem, as follows

$$(\partial_t u)_i + \frac{f_{i+1} - f_{i-1}}{2\Delta x} = \frac{(b_{i+1} + b_i)(u_{i+1} - u_i) - (b_i + b_{i-1})(u_i - u_{i-1})}{2\Delta x^2} + s_i, \tag{26}$$

where $\Delta x > 0$ is the space step size, given by $\Delta x = \frac{x_R - x_L}{M_x}$, with M_x being a positive integer that defines the $M_x + 1$ equally spaced points $x_L = x_0, x_1, \ldots, x_{M_x - 1}, x_{M_x} = x_R$ where we want to approximate the solution. A classical model to investigate properties of numerical methods int this class of problems is the one-way wave equation given by $\partial_t u + a\partial_x u = 0$, where $a \neq 0$ is the speed parameter. The solution to this PDE is given by $u(x,t) = u_0(x - at)$, which simply translates the initial condition $u_0(x)$ in a units along the time axis. For the purpose of u having infinite many derivatives, we choose $u_0(x) = \sin(x)$, $a = 1$, $x_L = -\frac{\pi}{2}$, $x_R = \frac{\pi}{2}$, $t_f = 5$, and boundary conditions given by the analytical solution at the boundary points. For this problem, which is linear, the numerical solution calculated from the application of the two-stage form of the Crank-Nicolson scheme to the PDE should be second-order convergent, although dispersive. This is in fact the outcome shown in Table 1, results labeled "Problem 3". Notice that despite the error being high for the first simulation, the remaining ones converge with numerical rates that approach 2. For our final test for this section, we apply the scheme to the Burgers-Huxley equation, whose analytical solution has been established in Wang et al. [10]. The Burgers-Huxley equation takes advantage of the full form of (25), where $f(u) = \alpha \frac{u^{1+\delta}}{1+\delta}$, $b(u) = 1$, $s(u) = \beta u(1 - u^\delta)$. The analytical solution to this problem is

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2} \tanh\left(-\frac{\alpha\delta}{2(\delta+1)} \left(x - \left(\frac{\alpha}{\delta+1} + \beta \frac{1+\delta}{\alpha} \right) t \right) \right) \right)^{1+\delta}. \tag{27}$$



(a) Conv. rates for displacement, $\nu = 0.3$ (b) Conv. rates for pressure, $\nu = 0.3$ (c) Conv. rates for displacement, $\nu = 0.499999999$ (d) Conv. rates for pressure, $\nu = 0.499999999$

Figure 1. Convergence rates for the application of the implicit Euler and two-stage Crank-Nicolson scheme in the SMHB formulation. In the simulations, $\Delta t \approx h$.

which we use to set the initial condition and boundary conditions. We have used $\alpha = \beta = 1$ and $\delta = 2$, and also $x_L = -10$, $x_R = 10$, and $t_f = 3$. Notice that despite the PDE being nonlinear, Table 1 assures us that the scheme is still second order convergent.

4.2 The two-stage Crank-Nicolson scheme in the SMHB formulation

We now assess the convergence, stability, and locking properties of the SMHB formulation introduced in Section 3. For that purpose we need to know the solution to the problem prior to approximating it, and therefore we apply the SMHB formulation to a problem whose solution is given by

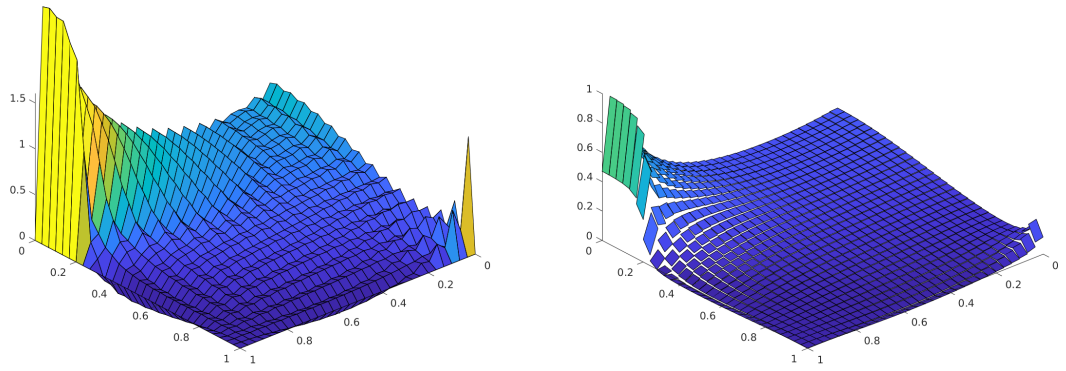
$$\mathbf{u} = e^{-t}\begin{pmatrix} 2\nu\sin(\pi x)\cos(\pi y) \\ 2(\nu-1)\cos(\pi x)\sin(\pi y) \end{pmatrix}, \qquad \phi = e^{-t}\frac{2E\nu\pi\cos(\pi x)\cos(\pi y)}{1+\nu}, \qquad p = e^{-t}\sin(\pi x)\sin(\pi y), \tag{28}$$

with $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$. We use $E = 1$, $\kappa = 1$, $c_0 = 1$, $\alpha = 1$ and $\eta = 1$. We choose the unit square in 2D as domain, with boundary conditions given by the exact solution (28). To obtain the optimal values for the stabilization parameters, we vary β_1 and β_2 in a fixed grid of 64 equally sized quadrilateral elements, collecting the errors $err := \log(\|\mathbf{u} - \mathbf{u}_h\|) + \log(\|p - p_h\|)$ for a Δt sufficiently small. The optimal values are then chosen to be the ones that provide the minimum of err.

Figures 1-(a), (b) show the results of the application of the two-stage Crank-Nicolson (and implicit Euler) scheme to the SMHB$_{k,r,l,m}$ (SMHB$^*_{k,r,l,m}$) formulation presented in section 3 with lowest-order elements. Notice that the implict Euler implementation differs from the Crank Nicolson implementation by the absence of the second stage and by the factor 2 in the equations only. Nevertheless, figure 1-(b) shows how the second order convergence can impact the pressure solution. From the figure, we notice that the green and cyan curves, which correspond to backward euler implementation, do start more accurate. Despite that fact, both curves end up showing only first order convergence as the grid is refined. Notice also that the computationally more efficient reduced order methods, those with $m = 0$ (SMHB$_{1,1,0,0}$ and SMHB$^*_{1,1,0,0}$), are even more accurate than those with $m = 1$.

Figures 1-(c), (d), show the application of the SMHB method in the incompressible limit scenario, with Poisson's ratio really close to half: $\nu = 0.499999999$. Notice that a continuous Galerkin approximation with linear interior approximations would, with no doubt, diverge in the same setting. In contrast, the SMHB formulation displays a perfect second order convergence.

In Figure 2 we compare the behavior of the SMHB formulation in the context of poroelasticity locking with the behavior of the continuous Galerkin formulation (CG) of Murad and Loula [11] in the same setting. We have reproduced the experiment of the reference Murad and Loula [11], §5.2, which consists of a bidimensional half-space with fixed impervious base being subjected to a uniform load over part of the domain. In this problem, the pore pressure lives at L^2 in $t = 0$, and satisfies a Stokes problem. For $t > 0$, the pressure field lives at H^1 and changes abruptly where the load starts acting. This test is known to generate pressure oscillations in the CG's numerical solution, as we observe from Figure 2-(a). However, the SMHB formulation does not suffer from the same deficiency, as shown in Figure 2-(b). From the plot we verify that despite the natural oscillatory behavior of the Crank-Nicolson scheme, the solution stays well controlled in space. In particular, close to where one finds an abrupt change in the pressure gradient, the discontinuities of the formulation allow for a smooth change.



(a) Pressure at $t = 5.0 \times 10^{-3}$ after 50 time steps, CG

(b) Pressure at $t = 5.0 \times 10^{-3}$ after 50 time steps, SMHB

Figure 2. Decay behavior of pressure for the CG formulation using linear approximation spaces and the $\text{SMHB}_{1,1,0,1}$ formulation

5 Conclusions

We have proposed a two-stage version of the Crank-Nicolson scheme that is easily extendable from an implicit Euler implementation, and that is also computationally more effective than the traditional implementation for expensive space operators of PDEs. We provided analytical evidence that the scheme is in fact second order convergent, and we demonstrated so via numerical experiments with ODEs, 1D convection-diffusion reaction equations and a system of poroelasticity equations in 2D.

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