Validation of a new finite element formulation for unsaturated flow in porous stiff solids.

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Abstract. This paper presents the validation of a finite element formulation of Richards' equation for transient unsaturated flow. The formulation is presented and its results are compared to an exact analytical solution obtained for the model of Gardner of water retention curve and hydraulic conductivity. The results obtained with a coarse regular mesh with the element are in excellent correspondence with the analytical solution. Numerical aspects of application of this formulation for unsaturated flow in concrete slabs are commented.

Keywords: Richards' Equation, Finite Element, Analytical solution

1 Introduction

In the last few decades, a large improvement on personal computers boosted the research on numerical methods to solve problems in engineering. Complex phenomena, such as the flux in unsaturated media, have been studied by several authors [1–5]; however, due to this complexity, few applications in real engineering cases have occurred. The water flow in unsaturated porous media is very important in civil engineering due to its applicability in studies like groundwater, soil mechanics, dams, mining, embankments, soil contamination and others. The mechanism of percolation/evaporation can be considered in studies of thermal comfort of buildings using concrete like a porous media and applying the Richards' Equation.

Mannich [6] has developed an analytical solution for transient unsaturated flow using the model developed by Gardner [7] for the water retention curve hydraulic conductivity. Mannich also developed numerical models based on finite difference and finite volume methods. Although FDM and FVM results obtained by that author were good, finite element method have the appeal of being suitable for coupled stress-strain formulation. Moreover, FEM formulations usually produce more precise results when the same number of degrees of freedom are employed in FDM or FVM. These aspects motivated the FEM formulation described in this work.

2 Fundamentals and Methodology

This work has the purpose of presenting the comparison between the finite element formulation developed by Scussiato et al. [8] to an exact analytical solution for Richards' equation developed by Mannich [6]. The next subsection presents Richards' equation and Gardner formulation for unsaturated flow. The following subsection presents the analytical solution of the Richards' equation. The last subsection will present the finite element formulation.

2.1 Governing Equations

The Richards' equation is a nonlinear partial differential equation which is used to describe the water flow in unsaturated porous media. In this study, it was considered a homogeneous porous media and a transient flow. The Richard's equation form considered here was the irreducible form for one-dimensional vertical flow,

$$m_2^w \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial y}\left[k_w \frac{\partial (\psi - y)}{\partial y}\right], \tag{1}$$

where m_2^w is the storage modulus, t stands for the time, y stands for the height, k_w stands for the hydraulic conductivity, ψ stands for the matric suction.

In this study, for both analytical and numerical solution, it was used the Gardner's formulation [7], for the hydraulic conductivity and moisture, written as

$$\theta_w = \theta_r + (\theta_s - \theta_r)e^{-\alpha\,\psi} \tag{2}$$
$$k_w = k_s e^{-\alpha\,\psi}, \tag{3}$$

where θ_r stands for the residual water content, θ_s stands for the saturated water content, α stands for the air-entry parameter and k_s is the saturated hydraulic conductivity. Although there are more recent formulations which have better adherence to experimental data of θ_w and k_w, this is supposed to be the only formulation that allows exact solutions for Richards' equation, at least in "simple" equations, as shown in the next subsection.

2.2 Analytical Solution for the Richards' Equation

The analytical solution chosen for the validation was proposed by Mannich [6], in which the water flow calculated as

$$q_w = k_s\left[q_b + e^{\left(\frac{L-y}{2}\right)}(Q_a + Q_b + Q_s)\right], \tag{4}$$

where L stands for thickness of the porous media.

The coefficients Q_a, Q_b and Q_s are defined as:

$$Q_a = \frac{(q_c - q_b)\left[\frac{1}{2}\,\sinh(u\,y) + u\,\cosh(u\,y)\right]e^{-at}}{\frac{1}{2}\,\sinh(u\,L) + u\cosh(u\,L)} \qquad \text{for } a < \frac{1}{4},\ u = \left(\frac{1}{4} - a\right)^{\frac{1}{2}} \tag{5}$$

$$Q_b = -\frac{(q_c - q_b)\left[\frac{1}{2}\,\sin(\nu\,y) + \nu\,\cosh(\nu\,y)\right]e^{-bt}}{\frac{1}{2}\,\sin(\nu\,L) + \nu\,\cosh(\nu\,L)} \qquad \text{for } b > \frac{1}{4},\ \nu = \left\|\frac{1}{4} - b\right\|^{\frac{1}{2}} \tag{6}$$

$$Q_s = -2\sum_{n=1}^{\infty}\left[(q_b - q_a) - \frac{(q_c - q_b)(b-a)\left(4\lambda_n^2 + 1\right)}{\left(a - \lambda_n^2 - \frac{1}{4}\right)\left(b - \lambda_n^2 - \frac{1}{4}\right)}\right]\cdot$$
$$\cdot\left[\frac{\sin(\lambda_n L)\cos(\lambda_n y)[\tan(\lambda_n y) + \lambda_n]e^{-(\lambda_n^2 - \frac{1}{4})t}}{1 + \frac{L}{2} + 2\lambda_n^2 L}\right]. \tag{7}$$

In this work, it is assumed that $q_a = 1 \cdot 10^{-1}$, $q_b = 1 \cdot 10^{-1}$, $q_c = 8.33 \cdot 10^{-1}$, $a = 1.33 \cdot 10^{-1}$ and $b = 1.33$ are the infiltration function parameters. For $a \geq \frac{1}{4}$ or $b \leq \frac{1}{4}$, Mannich [6] present equations for Q_a and Q_b.

The λ_n are a series of eigenvalues to be calculated as roots of the equation

$$\tan(L\lambda_n) + 2\lambda_n = 0. \tag{8}$$

2.3 Richards' Equation in Finite Elements

For this study, it was used the finite element formulation proposed by Scussiato et al. [8]. This formulation considers the one-dimensional flow in unsaturated porous media using Richard's Equation. In that was used implicit time discretization and linear interpolation for spatial discretization. This formulation is represented by

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \psi_1^{t+\Delta t} \\ \psi_2^{t+\Delta t} \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} \tag{9}$$

where $[K]$ is the "stiffness" matrix and $\{B\}$ is the "load" vector.

The matrix elements K_{11}, K_{12}, K_{21} and K_{22}, can be expressed by the following equations,

$$\begin{aligned} K_{11} = {} & \frac{\Delta t}{\gamma_w L}\left\{\frac{\theta}{2}[k_w(\psi_{p_1}^{t+\Delta t}) + k_w(\psi_{p_2}^{t+\Delta t})]\right\} - \\ & \frac{L}{2}\left\{\left(\frac{2+\sqrt{3}}{6}\right)[(1-\theta)m_2^w(\psi_{p_1}^t) + \theta m_2^w(\psi_{p_1}^{t+\Delta t})] + \right. \\ & \left.\left(\frac{2-\sqrt{3}}{6}\right)[(1-\theta)m_2^w(\psi_{p_2}^t) + \theta m_2^w(\psi_{p_2}^{t+\Delta t})]\right\} \end{aligned} \tag{10}$$

$$\begin{aligned} K_{12} = {} & -\frac{\Delta t}{\gamma_w L}\cdot\left\{\frac{\theta}{2}[k_w(\psi_{p_1}^{t+\Delta t}) + k_w(\psi_{p_2}^{t+\Delta t})]\right\} - \\ & \frac{L}{12}\{[(1-\theta)m_2^w(\psi_{p_1}^t) + \theta m_2^w(\psi_{p_1}^{t+\Delta t})] + \\ & [(1-\theta)m_2^w(\psi_{p_2}^t) + \theta m_2^w(\psi_{p_2}^{t+\Delta t})]\} \end{aligned} \tag{11}$$

$$\begin{aligned} K_{21} = {} & -\frac{\Delta t}{\gamma_w L}\left\{\frac{\theta}{2}[k_w(\psi_{p_1}^{t+\Delta t}) + k_w(\psi_{p_2}^{t+\Delta t})]\right\} - \\ & \frac{L}{12}\{[(1-\theta)m_2^w(\psi_{p_1}^t) + \theta m_2^w(\psi_{p_1}^{t+\Delta t})] + \\ & [(1-\theta)m_2^w(\psi_{p_2}^t) + \theta m_2^w(\psi_{p_2}^{t+\Delta t})]\} \end{aligned} \tag{12}$$

$$\begin{aligned} K_{22} = {} & \frac{\Delta t}{\gamma_w L}\cdot\left\{\frac{\theta}{2}[k_w(\psi_{p_1}^{t+\Delta t}) + k_w(\psi_{p_2}^{t+\Delta t})]\right\} - \\ & \frac{L}{2}\left\{\left(\frac{2-\sqrt{3}}{6}\right)[(1-\theta)m_2^w(\psi_{p_1}^t) + \theta m_2^w(\psi_{p_1}^{t+\Delta t})] + \right. \\ & \left.\left(\frac{2+\sqrt{3}}{6}\right)[(1-\theta)m_2^w(\psi_{p_2}^t) + \theta m_2^w(\psi_{p_2}^{t+\Delta t})]\right\} \end{aligned} \tag{13}$$

where Δt stands for the time interval, γ_w stands for the specific weight water, L stands for the element length and θ is the "implicity" weighting factor

The elements B_1 and B_2 of the vector $\{B\}$ can be represented by the following equations,

$$\begin{aligned} B_1 = {} & -\frac{L}{2}\left\{\left(\frac{1+\frac{1}{\sqrt{3}}}{2}\right)\psi_{p_1}^t[(1-\theta)m_2^w(\psi_{p1}^t) + \theta m_2^w(\psi_{p1}^{t+\Delta t})] + \right. \\ & \left.\left(\frac{1-\frac{1}{\sqrt{3}}}{2}\right)\psi_{p_2}^t[(1-\theta)m_2^w(\psi_{p2}^t) + \theta m_2^w(\psi_{p2}^{t+\Delta t})]\right\} + \\ & \frac{\Delta t}{\gamma_w L}(\psi_2^t - \psi_1^t)\left\{\frac{(1-\theta)}{2}[k_w(\psi_{p_1}^t) + k_w(\psi_{p_2}^t)]\right\} - \Delta t[(1-\theta)q_w^t + \theta q_w^{t+\Delta t}] - \\ & \frac{\Delta t}{2}(1-\theta)[k_w(\psi_{p_1}^t) + k_w(\psi_{p_2}^t)] - \frac{\Delta t}{2}\theta[k_w(\psi_{p_1}^{t+\Delta t}) + k_w(\psi_{p_2}^{t+\Delta t})] \end{aligned} \tag{14}$$

$$B_2 = -\frac{L}{2}\left\{\left(\frac{1-\frac{1}{\sqrt{3}}}{2}\right)\psi_{p_1}^t[(1-\theta)m_2^w(\psi_{p1}^t) + \theta m_2^w(\psi_{p1}^{t+\Delta t})] + \right.$$

$$\left(\frac{1+\frac{1}{\sqrt{3}}}{2}\right)\psi_{p_2}^t[(1-\theta)m_2^w(\psi_{p2}^t)+\theta m_2^w(\psi_{p2}^{t+\Delta t})]\Bigg\}-$$
$$\frac{\Delta t}{\gamma_w L}(\psi_2^t-\psi_1^t)\left\{\frac{(1-\theta)}{2}[k_w(\psi_{p_1}^t)+k_w(\psi_{p_2}^t)]\right\}+\Delta t[(1-\theta)q_w^t+\theta q_w^{t+\Delta t}]+$$
$$\frac{\Delta t}{2}(1-\theta)[k_w(\psi_{p_1}^t)+k_w(\psi_{p_2}^t)]+\frac{\Delta t}{2}\theta[k_w(\psi_{p_1}^{t+\Delta t})+k_w(\psi_{p_2}^{t+\Delta t})] \tag{15}$$

The solution of the formulation presented above can be performed by solving the global equation system

$$[K]\{\psi^{t+\Delta t}\} \;=\; \{B\}. \tag{16}$$

The vector $\{B\}$ is dependent of $\{\psi^{t+\Delta t}\}$ and this equation is, in fact, nonlinear. This equation should be iterated, and $\{B\}$ should be re-calculated with the previous value of $\{\psi^{t+\Delta t}\}$ up to some convergence criterion is attained.

3 Results and Discussion

In order to validate the element presented in the last subsection, it was intended to use two different problems: the third problem of Mannich [6] and a different problem with extreme parameters of unsaturated porous media for a concrete mixture with water content of 0.55 provided by Leech [9]. These parameters are presented in Table 1. However, as the hydraulic conductivity of the concrete is too low of the order of 10^{-12}, one came across

Table 1. Properties of materials used in this study.

Parameter	Symbol	Concrete	Analytical Solution	Unit
Saturated hydraulic conductivity	k_s	$5\cdot10^{-12}$	$3\cdot10^{-6}$	m/s
Saturated water content	θ_s	0.131	0.5	-
Residual water content	θ_r	0	0.1	-
Air-entry parameter	α	$1.39\cdot10^{-3}$	10^{-1}	m^{-1}

with numerical problems related with floating point operations. Some calculations in analytical solution just could not be calculated as double precision numbers, as some exponential produced numbers greater than $1.8\cdot10^{308}$. According to Kincaid and Cheney [10] this is largest number that can be represented in double precision. This problem will be subject of future research in quadruple precision.

The problem presented by Mannich was implemented with the following boundary conditions,

$$\psi(y=0) \;=\; 0 \tag{17}$$
$$q(y=L) \;=\; q_b+(q_c-q_b)\left[e^{-a\,t}-e^{-b\,t}\right], \tag{18}$$

which has eq4 as exact solution for Richards' equation.

For the finite elements method, it was used the Crank-Nicolson scheme ($\theta=0.5$)[10]. In this scheme, the consistent "load" is given by the volume of water that enters in $y=L$ over time span Δt, and is represented by:

$$\Delta t[(1-\theta)q_w^t+\theta q_w^{t+\Delta t}]=\int_t^{t+\Delta t}(u^*\cdot q_w)\,dt \tag{19}$$

where u^* is the weighting function. For the Crank-Nicolson method, u^* is a uniform function, that is,

$$u^*(t)=1. \tag{20}$$

Fig. 1 shows the evolution of the water flow over time. In this chart, it was used equation (16) for the dashed curve (finite elements) and equation (4) for continuous curve (analytical solution). One can notice that the water

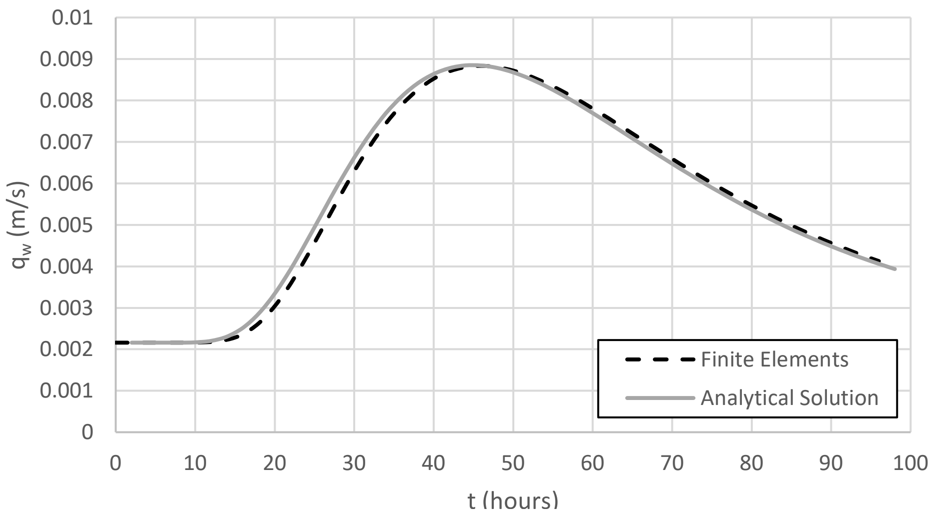

Figure 1. Variation of water flow over time for finite element and analytical solution.

flow increase until time equal 45 hours than decrease and tends to stabilize after 100 hours. The comparison between analytical and numerical solution shows an excellent agreement.

The Fig. 2 shows the water flow at the top and base of a soil with thickness of 1 m for the analytical solution.

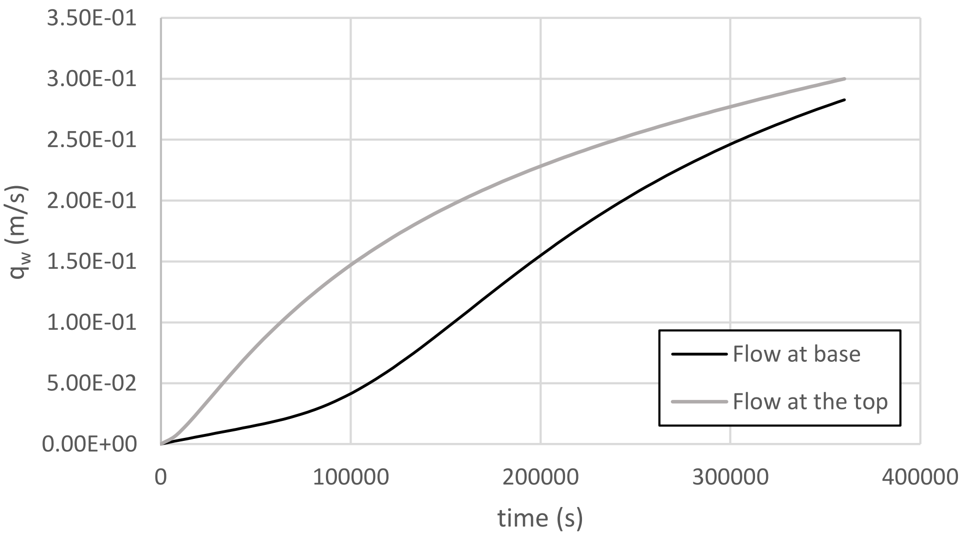

Figure 2. The water flow for analytical solution at the top and base of the element.

The Fig. 3 shows the results of variation of matric suction in relation to depth for some time instants (0, 10, 20, 30, 50, 75-76 and 100-98) for analytical and numerical analysis. The dashed lines was obtained by Mannich [6] and the points are the results of the finite elements method with 8 elements. It should be noted that the results are almost the same. Mannich [6] needed 100 nodes in finite differences to get the same results, this suggests that the finite elements proposed by Scussiato et al. [8] is very robust, as it produced a remarkable result with a very coarse mesh.

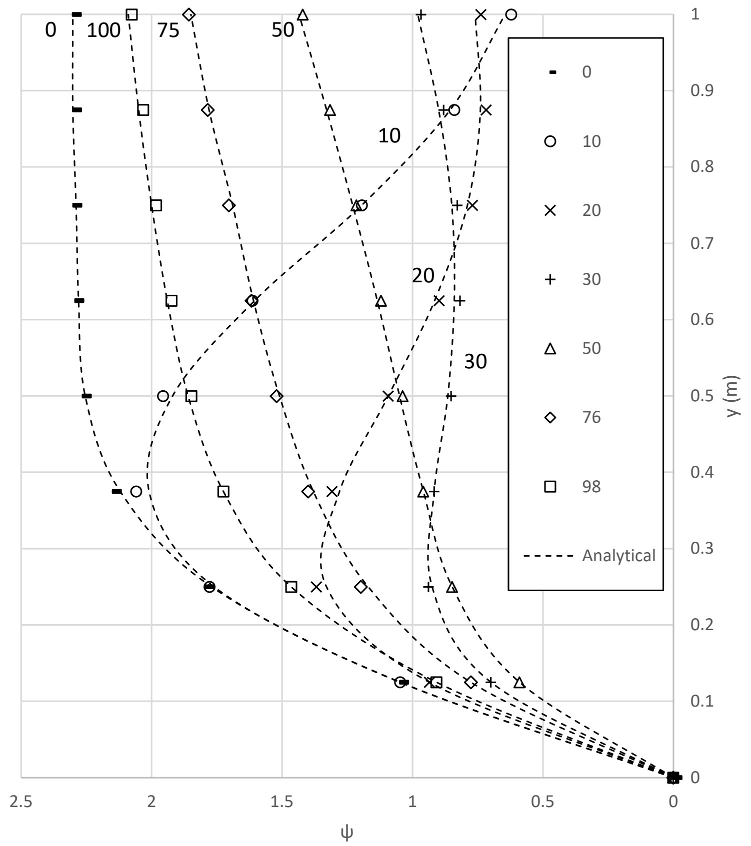

Figure 3. Variation of matric suction on the depth comparison between finite elements and analytical solution.

4 Conclusions

This study presented the validation of a finite element formulation for Richard's equation proposed by Scussiato et al. [8]. Herein, it was compared the results of the numerical analysis, with 8 nodes, for water flow and matric suction, with the analytical solution proposed by Mannich [6].

The results of the validation of the finite element presented excellent correspondence with analytical solution.

Future works will address the convergence and error analysis of this formulation. The final purpose of this research is evaluation of thermal comfort in the buildings taking account the evaporation process, heat flow and water flow in unsaturated porous media using the Richards' Equation in concrete slabs.

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