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A comparison among Vanka, Uzawa and Fixed-Stress smoothers for the one-dimensional poroelasticity problem using Multigrid Time-Stepping

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Abstract. The poroelasticity equations mathematically model the interaction between the deformation of a porous elastic material and the fluid flow inside it. The mathematical model that describes this theory, in its most simplified version, considers the variables displacement, pressure and time, related to each other by a system of partial differential equations. The importance of deepening the knowledge about this problem is related to the difficulty of obtaining a numerical solution, due to the presence of saddle points that generates instability in the numerical analysis. In this work, the problem of 1D poroelasticity is solved, whose boundary conditions assume a left boundary without displacement variation and with free drainage, and a rigid right border without pressure variation. For the discretization of differential equations, the Finite Volume Method is used for spatial discretization and the implicit Euler method with Time-Stepping sweep for temporal discretization are used. The linear systems from discretization are solved using Vanka, Uzawa and Fixed-Stress smoothers. The results obtained demonstrate that the different smoothers are equivalent with respect to accuracy. However, there are differences regarding the convergence factor, computational time and complexity.

Keywords: Biot's model, Finite Volume Method, convergence factor.

1 Introduction

Poroelasticity is the term used to describe the interaction between fluid flow and solid deformation within the porous medium, which is composed of solid structures with fluid-filled pores. When an external load is applied to the porous medium, the entire system is affected. Fluid-filled pores undergo a pressure variation which in turn implies in fluid movement. Consequently, the solid material elastically deforms. This combination is found in nature, and Engineering has many associated studies, such as Geomechanics, Hydrogeology, Biomechanics, and others.

The mathematical modeling of the poroelasticity problem involves two coupled differential equations obtained from two laws: the law that describes the relationship between fluid motion and pressure within the porous medium; and the porous matrix structural displacement law, known as Biot's theory.

The solution to this problem requires smoothers that deal efficiently with displacement and pressure coupling, given that the methods that smooth the variables in an uncoupled way, although they are widely used for ease of implementation, are not as efficient.

Oosterlee and Gaspar [1] presented the Vanka smoother, which handles variables in a coupled way. Franco et al. [2] used the color ordering applied to the Vanka smoother using the Finite Difference and the Multigrid method, and comparing it with the Time-Stepping and Waveform Relaxation methodologies.

Luo et al. [3] applied the Uzawa smoother in the 2D poroelasticity equations with Finite Volume discretization, associated with the Multigrid method.

Borregales et al. [4] proposed a new version for the Fixed-stress method, in which it considers the temporal variable as an additional direction for the parallelization of the Biot's problem and presents numerical experiments and convergence analysis to demonstrate the robustness of the method.

The objective of this study is to present a comparison among Vanka, Fixed-stress split and Uzawa smoothers, using Finite Volume discretization and Time-Stepping methodology for time advancement.

In Section 2, we present the mathematical and numerical models of the poroelasticity problem. Section 3 is dedicated to numerical methods, in particular, the Multigrid method and the smoothers proposed for this paper. In Section 4, we present the numerical results obtained from the comparison of smoothers. Finally, conclusions are shown in Section 5.

2 Mathematical and numerical model

2.1 Mathematical model

The poroelasticity problem is formulated by a coupled system of time-dependent partial differential equations. In this work, we consider the classic problem for a saturated, homogeneous and isotropic porous medium and the incompressible fluid, Gaspar [5]. This system of differential equations is composed of two equations: the first equation models the displacement *u* and the second equation models the pressure *p*:

$$\begin{cases} -E\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial p}{\partial x} = \mathcal{U} \\ \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial x}\right) - K\dfrac{\partial^2 p}{\partial x^2} = \mathcal{P} \end{cases}. \tag{1}$$

The constant E represents the Young's modulus and the constant K, called hydraulic conductivity, represents the physical properties related to the porosity and permeability of the medium. The term $\mathcal{U}$ represents the density of the force applied to the body and $\mathcal{P}$ is the injection or extraction force of the fluid in the porous medium.

The variables of interest are displacement and pressure, represented by $u(x,t)$ and $p(x,t)$ which depend on spatial position and time. The time interval $(0,t_f]$ is considered, where t_f is the final time, and the spatial interval by [0,1/2].

This problem presents boundary conditions with left boundary without displacement variation (Neumann's condition) and with free drainage (Dirichlet's condition), thus:

$$\begin{cases} -E\dfrac{\partial u}{\partial x}(0,t) = 0 \\ p(0,t) = 0 \end{cases}. \tag{2}$$

The right boundary is rigid (Dirichlet's condition) and has no pressure variation (Neumann's condition), so:

$$\begin{cases} u\left(\frac{1}{2},t\right) = 0 \\ K\dfrac{\partial p}{\partial x}\left(\frac{1}{2},t\right) = 0 \end{cases}. \tag{3}$$

For comparison purposes, analytical solutions were proposed:

$$u(x,t) = \cos(\pi t)e^{-t} \tag{4}$$

$$p(x,t) = \sin(\pi t)e^{-t}. \tag{5}$$

Therefore, the source terms that satisfy the problem are given by

$$\mathcal{U}(x,t) = (E\pi + 1)\pi\cos(\pi t)e^{-t} \tag{6}$$

$$\mathcal{P}(x,t) = (1 + K\pi)\pi\sin(\pi t)e^{-t}. \tag{7}$$

2.2 Numerical model

The spatial domain is discretized using the Finite Volume Method (FVM) considering a collocated uniform mesh, that is, the pressure p and the displacement u are in the center of the control volume. The representative volumes and measurements of the mesh are described in Figure 1, below:

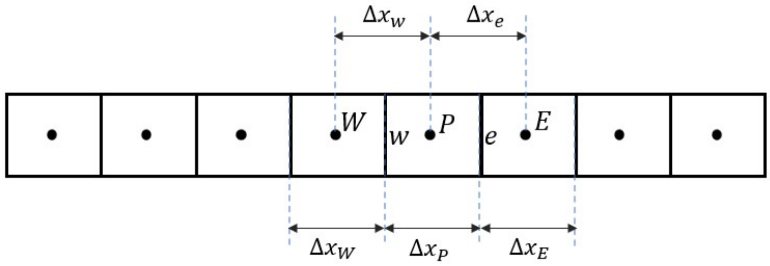

Figure 1. Representation of control volumes

Considering a uniform mesh, in which the size of each control volume is standard, so:

$$\Delta x_W = \Delta x_P = \Delta x_E = \Delta x_w = \Delta x_e. \tag{8}$$

Given a spatial domain L, and dividing into N control volumes, the size of each control volume is given by:

$$h = \frac{L}{N}. \tag{9}$$

The temporal approximation and the spatial/temporal connection are made using the Implicit Euler method. The time step τ is given by:

$$\tau = \frac{t_f - t_0}{N_t}, \tag{10}$$

where t_f is the final time, t_0 is the initial time and N_t is the number of time steps.

Gaspar et al. [6] presented a reformulated version for the poroelasticity system which consists of the additional use of a smoothing term on the left side of the pressure equation. This term does not change the result and improves the stability of the system for the numerical solution. The smoothing term is given by:

$$\frac{-h^2}{4E}\frac{\partial}{\partial t}\left(\frac{\partial^2 p}{\partial x^2}\right). \tag{11}$$

Performing the discretization, we have the following system of equations:

$$\begin{cases} \frac{-E}{(\Delta x)^2}(u_E - 2u_P + u_W) + \frac{1}{2\,\Delta x}(p_E - p_W) = \mathcal{U}_P \\ \frac{1}{\Delta x \Delta t}(u_E - u_W) - \frac{(\Delta x)^2 + 4KE\Delta t}{4\,E\Delta t\,(\Delta x)^2}(p_E - 2p_P + p_W) = \mathcal{P}_P + \\ \quad + \frac{1}{\Delta x \Delta t}({u_E}^0 - {u_W}^0) - \frac{1}{4\,E\Delta t}({p_E}^0 - 2{p_P}^0 + {p_W}^0). \end{cases} \tag{12}$$

Ghost volumes are considered for the boundary conditions. For the left boundary we have: $u_P = u_E$ and $p_P = -p_E$; and for the right boundary, $u_P = -u_W$ and $p_P = p_W$. The process consists of solving a system of equations in the format:

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} U \\ P \end{pmatrix}, \tag{13}$$

where A and B are the matrices of the displacement-related coefficients, and B^T and C are the matrices of the pressure-related coefficients. The elements U and P are vectors with the independent terms of displacement u and pressure p, respectively.

3 Numerical methods

In this section the numerical methods used in this work are presented.

3.1 Multigrid

The Multigrid method is a numerical technique that consists of accelerating the convergence of iterative methods to solve systems of algebraic equations, reducing the solution time. This technique can be applied in any type of discretization, including the Finite Volume Method, and for iterative methods that have the smoothing property, that is, fast reduction of the error oscillatory modes, Briggs [7].

In order to obtain faster convergence, the main idea of this method is to alternate smoothing at different mesh levels and the approximations of these solutions in coarser meshes, reaching the various error frequencies (oscillatory and smooth).

According to Wesseling [8], the sequence in which the different grids are visited determines a Multigrid cycle. The most common are V, W and F-cycle. In addition, two schemes can be used to deal with linearity or non-linearity. For linear problems the Correction Scheme (CS) is used and for nonlinear problems the Full Approximation Scheme (FAS) is used.

3.2 Smoothers in Multigrid method

The Vanka smoother (Oosterlee and Gaspar [1]) performs block smoothing and is indicated to solve problems modeled by linear systems with the presence of saddle points, a situation that brings numerical instability. It is an iterative process in which the displacement and pressure unknowns are updated as a function of the nodes that are in the neighborhood, in a coupled way, Franco [2].

The Vanka smoother iteration takes place over all mesh volumes following lexicographic ordering and using four color (in order to parallelize). In the one-dimensional case, a 3x3 linear system is solved in three consecutive volumes, with two displacement variables and one pressure variable (u_W, p_P and u_E). In each iteration, the pressure variable is updated once, while the displacement is updated twice.

Fixed-Stress, Kim [9], is a sequential method that initially consists of solving the pressure, keeping displacement constant, then performing the calculation of the displacement, through the pressure already calculated.

This approach leaves the sequential nature of the temporal variable, considering it as a position variable and allowing for parallelization. This method can be interpreted as a Richardson preconditioner with a block triangular operator. For this, the red-black symmetric Gauss-Seidel method is used, Borregales et al. [4].

The discretization given by Eq. (13) can be rewritten, decomposing it into two new systems, as follows:

$$\begin{pmatrix} A & B^T \\ B & C \end{pmatrix} = \begin{pmatrix} A & B^T \\ 0 & -C + \frac{\alpha^2}{K_b} M_P \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -B & \frac{\alpha^2}{K_b} M_P \end{pmatrix}, \tag{14}$$

where the matrix M_P is an upper triangular block preconditioner, a Schur's complement to the matrix A, given by: $-(C + BA^{-1}B^T)$. Thus, the system can be solved in the form of two triangular systems by blocks.

The Uzawa method, Zhi-Hao Cao [10], is a decoupled approach that can be interpreted as the Richardson method for solving the Schur complement equation. In this case, symmetric Gauss-Seidel is used, which can be described as an incomplete LU, Luo et al. [3]. The central idea of the Uzawa smoother is to decompose the coefficient matrix obtained in the discretization into two matrices, as follows:

$$\begin{pmatrix} A & B^T \\ B & C \end{pmatrix} = \begin{pmatrix} M_A & 0 \\ B & -\omega^{-1} I \end{pmatrix} \begin{pmatrix} M_A - A & -B^T \\ 0 & C - \omega^{-1} I \end{pmatrix}, \tag{15}$$

where ω is a positive term and M_A is a smoother for matrix A which makes the process less expensive for each iteration.

4 Numerical results

4.1 Computational details

In this work, according to Franco et al. [2] we use constant E (Young's modulus) with a fixed value of 10^4, a realistic value, and we vary the values of constant K (hydraulic conductivity) between 10^{-12} and 10^0, which can be compared in Franco et al. [2]. For the proposed problem, we used the W-cycle Multigrid method, with full weighting as a restriction operator and linear interpolation for the prolongation operator, one pre- and one post-smoothing, limiting to a maximum of 30 iterations. The stopping criterion was based on the error tolerance, set at 10^{-6}, considering the infinity norm of the error.

The software used for the computational code is MatLab R2020b. The tests were run on a computer with an Intel(R) Core(TM) i7-8550U CPU @ 1.80 GHz, 8 GB RAM, Windows 10 Home Single Language and 64-bit OS.

4.2 Comparison among solvers

For the proposed comparison, we used the convergence factor and the complexity order.

Convergence factor: The convergence factor is evaluated by:

$$\rho = \frac{\|res(it)\|_\infty}{\|res(it-1)\|_\infty}, \tag{15}$$

where *res(it)* is the residual in iteration *it*.

The average convergence factor is given by:

$$\rho_m = \sqrt[n]{\frac{\|res(it)\|_\infty}{\|res(0)\|_\infty}}, \tag{16}$$

where *res(0)* is the residual in the initial estimate.

In Figure 2, we present the comparison among the average convergence factors of Multigrid methods, when using Fixed-stress, Uzawa and four-color Vanka solvers:

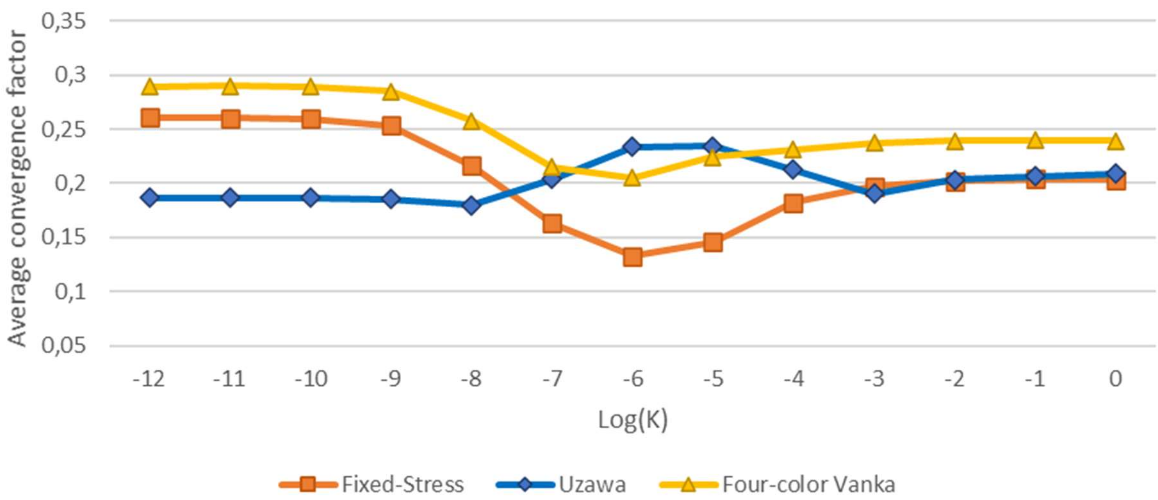

Figure 2. Average convergence factor ρ_m as a function of the *log K* value for the grid with $N = 1024$ points.

We note that the average convergence factor has a maximum value less than 0.3, which means that the problem quickly converges for the three solvers. In addition, the Uzawa solver is better than the others for K values between 10^{-12} and 10^{-8}, and for K between 10^{-3} and 10^0. Fixed-stress gives better results in the K range between 10^{-7} and 10^{-4}.

Complexity order: In Figure 3, we present the analysis of computational time *versus* log(K) for the analyzed solvers.

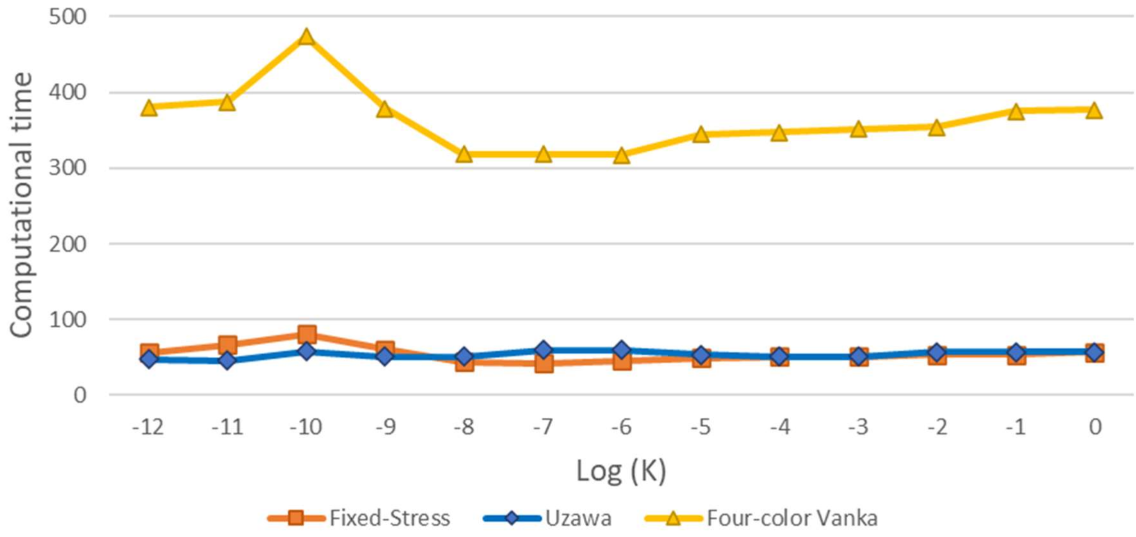

Figure 3. Computational time as a function of log(K).

Comparing the computational time, the Fixed-Stress and Uzawa methods take less time to reach the stopping criteria, using the Multigrid cycles with N = 1024 (10 grid levels), than the four-color Vanka method, which presents a computational time at least 5 times greater than those required for the other solvers.

According to Burden and Faires [11], considering the result of computational time t_{cpu}, it is possible to perform a exponential least square by geometric fit to verify the complexity of the algorithm using the equation:

$$t_{cpu} = c.\boldsymbol{N}^p \, , \tag{17}$$

where c is the method-related coefficient, p represents the order of solver complexity associated with the slope of the fit curve, and $\boldsymbol{N}$ is the number of unknows of the problem. In the ideal case, the value of p should be one and the value of c should be close to zero, Trottenberg et al. [12]. For the problem studied in this work, data on parameters c and p for $K = 10^{-12}$, $K = 10^{-6}$ and $K = 10^{-2}$ are provided in Table 1:

Table 1. Geometric fit parameters

Solver	$K = 10^{-12}$		$K = 10^{-6}$		$K = 10^{-2}$	
	c	p	c	p	c	p
Fixed-stress	0.009	1.10	0.008	1.11	0.008	1.12
Uzawa	0.008	1.12	0.008	1.12	0.009	1.06
four-color Vanka	0.020	1.09	0.017	1.12	0.025	1.03

In this table, the Fixed-stress, Uzawa and four-color Vanka methods have equivalent complexity orders.

Discretization error: In Figure 4, the graph for the Fixed-Stress solver analyzed at $K = 1$ is shown, and we can see that the drop in the discretization error is directly proportional to the number of unknowns. Comparing the number of meshes with the error obtained, we can verify the method convergence. All solvers have the same type of error drop. This behavior is qualitatively similar for other values of K, and quantitatively for the other solvers. In all cases, the iterative process was applied until the rounding error was reached.

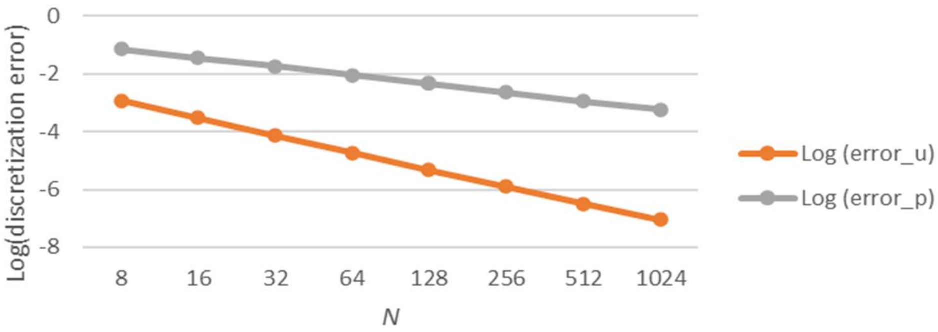

Figure 4. Discretization error using $K = 1$.

In this figure we notice that the error drop for displacement is smaller than the error drop for pressure, Marchi [13]. This is a characteristic of this poroelasticity problem.

5 Conclusions

With the data presented in this paper, we can conclude that Fixed-stress and Uzawa are the best methods, depending on the value of K analyzed. This analysis refers to the convergence factor, computational time and the complexity order of the methods. Further analysis of the variation in the E parameter could be added. We can also conclude that four-color Vanka solver had a similar behavior, but inferior to Fixed-Stress in the convergence factor; and computational time compared to the Uzawa solver. Therefore, putting together a hybrid method with Fixed-stress and Uzawa seems to be promising.

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Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors or has the permission of its owners.

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