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On the implementation of SGFEM simulation of cohesive crack propagation problems

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Abstract. The present work aims to discuss some details regarding the implementation of a cohesive crack propagation system using the Stable Generalized Finite Element Method (SGFEM), a relatively new approach that derives from a simple modification of enrichment functions used in Generalized/eXtended Finite Element Method (G/XFEM). For this, a combination between Heaviside functions that employ a stabilization parameter, presented in Wu and Li [1], and a trigonometric function [2] is used as enrichment. A cohesive crack model is considered. Though nonlinear material models, e. g., damage or plasticity, could be used, the bulk is considered as a linear elastic material for the discussions carried out in this work. Results involving crack direction criteria, as proposed by Wells and Sluys [3] and Moës and Belytschko [2], are also compared. To the best of the authors knowledge, SGFEM has never been applied to simulate cohesive crack propagation problems with the presence of trigonometric enrichment functions. This work is related to a proposal of expansion of the INSANE (INteractive Structural ANalysis Environment) system, an open-source project developed at the Structural Engineering Department of the Federal University of Minas Gerais. This platform has enabled the resources that allowed the implementation discussed in this work.

Keywords: Stable Generalized Finite Element Method; Generalized Finite Element Method; Computational Mechanics; Fracture Mechanics; Object Oriented Programming; Non-linear analysis.

1 Introduction

Non-smooth properties, such as jumps, kinks, corner singularities, high gradients or oscillations, can be found in the solution of problems in many engineering applications. In these situations, conventional finite element methods (FEM) may require a mesh to fit the non-smoothness and/or a great level of refinement in order to produce a reasonable solution. Since those mesh operations, as mesh fitting and refinement, may increase computational cost, it is interesting to work with approaches that might simplify mesh generation. In this context, the Generalized/eXtended Finite Element Method (G/XFEM) [4–7] has become an efficient tool to solve problems in which prior knowledge about the solution characteristic is available. This method allows to locally incorporate the main features of the problem, such as the behavior of weak and strong discontinuities, into the global approximation. Thus, G/XFEM can be understood as an expansion of the standard FEM approximation space by a set of local spaces generated by the so-called enrichment functions [8].

G/XFEM has been able to solve a large variety of problems [9], including fracture mechanics problems. Recently, new strategies that deliver excellent results for complex crack simulations have been developed to deal with that kind of problems. Some examples are nonlocal/gradient approaches, phase-field modelling and peridynamics theory. According to Bento et al. [8], however, all of them rely on length-scale parameters and they also require very fine meshes in the region encompassed by it. G/XFEM, as already mentioned, is able to loose some of FEM mesh requirements, allowing the mesh to be constructed regardless crack surface configuration. Despite this great features, is is noteworthy that G/XFEM presents also some shortcomings. An important concern is the ill-conditioning of system matrix. Other drawback that may be mentioned is G/XFEM performance in the existence of the so-called blending elements [10].

Many techniques have been developed to deal with these issues, including enrichment functions modification. Particularly, the enrichment modification proposed originally by Babuška and Banerjee [11] has caught attention for its performance, while being relatively easy to implement. It consists in subtracting from a enrichment function its FE interpolant and was called by the authors Stable Generalized Finite Element Method (SGFEM). Since its proposal, SGFEM has been also applied to a variety of problems, such as fracture mechanics problems - being the main strategy, as in [8, 12, 13] or associated with the global-local strategy [14, 15] - interfaces [16, 17], dynamic problems [18] and interfacial crack problems in bi-materials [19]. Is is also worthy to mention the work of Wu and Li [1], that handles cohesive crack propagation simulations using a modified version of Heaviside function. Thought its potentialities, works as those of Zhang et al. [20] and Zhang et al. [16] have showed that the simple extension of ideas presented in [11] was not enough to provide a robust G/XFEM. In order to attain this, the proposed SGFEM should deliver same convergence rate as a smooth problem, generate stiffness matrices with the same conditioning rate growth as the one obtained using standard FEM and keeping the previous requirements regardless the relative position between the special feature and the mesh [20]. Some of the cited works were able to deliver such a robust SGFEM using the methodology proposed by Babuška and Banerjee [11], along with some additional modifications, as the use of different Partitions of Unity (PoUs) to generate enrichment spaces, local ortogonalization, generalization of continuous interpolant, among others [8].

This paper aims to discuss some details regarding the implementation of a cohesive crack propagation system using the Stable Generalized Finite Element Method (SGFEM). To the best of authors knowledge, the works published so far with SGFEM dealt with this kind of problem employing only versions of Heaviside functions as enrichment functions. Here, a combination between Heaviside functions that use a stabilization parameter [1] and a trigonometric function [2] (in the context of the enrichment modification presented by Babuška and Banerjee [11]) is proposed as enrichment. A cohesive crack model [3] is considered, and the bulk is taken as linear elastic material for the simulations. This work is related to a proposal of expansion of the INSANE (INteractive Structural ANalysis Environment) system, an open-source project developed at the Structural Engineering Department of the Federal University of Minas Gerais. Based on the formulations presented in section 2, we discuss in section 3 a couple of details regarding the expansion of the previous G/XFEM code in order to enable SGFEM cohesive crack simulations and the use of both enrichment functions as the crack propagates. Finally, in section 4, some results of numerical experiments performed with the L-shaped panel experimentally analyzed by Winkler et al. [21] are presented. Those simulations are compared in order to investigate crack direction criteria - as presented in Wells and Sluys [3] and Moës and Belytschko [2]. Accuracy of both approaches and their suitability to SGFEM cohesive crack propagation are discussed.

2 Model problem

2.1 Governing equations

For conciseness, the equations presented in this section are severely summarized. For more details on their development, the reader is referred to Wang and Waisman [22]. Consider an isotropic solid occupying the domain $\overline{\Omega} = \Omega \cup \partial\Omega$ in $\mathbb{R}^2$, with an external boundary divided into $\partial\Omega = \partial\Omega^u \cup \partial\Omega^\sigma$ with $\partial\Omega^u \cap \partial\Omega^\sigma = \oslash$ (Fig. 1). Indices u and σ indicate regions where Dirichlet and Neumman boundary conditions are applied, respectively.

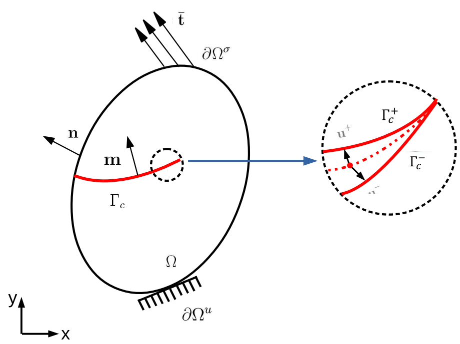

Figure 1. Schematic representation of a solid body in domain Ω crossed by a crack Γ_c.

A crack with surfaces denoted by $\Gamma_c \subset \mathbb{R}^1$ is assumed to gradually propagate through the solid domain $\overline{\Omega}$ in such a way that $\Gamma_c(t) \subseteq \Gamma_c(t+\Delta t)$. The displacement field $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^2$ may be discontinuous across the internal crack boundary Γ_c, yielding a crack opening $[\![\mathbf{u}]\!]$, defined as the difference between displacements $\mathbf{u}^+$ in Γ_c^+ and $\mathbf{u}^-$ in Γ_c^-: $[\![\mathbf{u}]\!] = \mathbf{u}^+ - \mathbf{u}^-$.

Crack boundary Γ_c may be divided into two disjoint parts, a traction-free surface and a fracture process zone Γ_{ZPF}, over which cohesive tractions are active. Restricting our attention to quasi-static problems under small deformation the fracture mechanics problem can be recast as the following principle of virtual work:

$$\int_{\overline{\Omega}} \nabla^s \delta\mathbf{u} : \boldsymbol{\sigma} \;\; d\overline{\Omega} + \int_{\Gamma_{\text{ZPF}}} \delta[\![\mathbf{u}]\!] \cdot \mathbf{t} \;\; d\Gamma = \int_{\partial\Omega^\sigma} \delta\mathbf{u} \cdot \bar{\mathbf{t}} \;\; d\mathbf{s} \tag{1}$$

where $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor, ∇^s, the symmetric part of the gradient operator. In Fig. 1 $\mathbf{n}$ is the outward unit normal vector on the boundary $\partial\Omega$ whereas $\mathbf{m}$ is the unit vector normal to the internal discontinuity Γ_c. $\bar{\mathbf{t}}$ are prescribed tractions on Neumman boundary $\partial\Omega^\sigma$ and $\mathbf{t}$ are the tractions existent in the ZPF.

2.2 Cohesive Law

The cohesive law adopted in this work is based on a simplification of the formulation presented by Wells and Sluys [3], where the normal traction force t_n and the shear traction t_s acting on the discontinuity surface are defined as

$$t_n = f_t \exp\left(-\frac{f_t}{G_f}\kappa\right), \quad t_s = d_{\text{init}}[\![u]\!]_s \tag{2}$$

where f_t is the tensile strength of the material, G_f is the fracture energy and κ is a history parameter, equal to the largest value of normal crack opening $[\![u]\!]_n$ reached. d_{init} is the initial crack shear stiffness (when $\kappa = 0$) and $[\![u]\!]_s$ is the crack sliding displacement. This cohesive law is suitable to mixed mode fracture, as long as $d_{\text{init}} \neq 0$ [3].

2.3 Enrichment functions

In order to present the enrichment functions selected to simulate cohesive crack propagation with the SGFEM here proposed it is necessary to define a standard Heaviside function:

$$\mathcal{H}(x,y) = \begin{cases} 1, & \text{if } \bar{y} \leq 0 \\ 0, & \text{if } \bar{y} > 0 \end{cases} \quad \text{for a local coordinate } \; \bar{y}\text{, normal to the crack segment.} \tag{3}$$

To simulate SGFEM cohesive crack propagation, the authors consider that all nodes of an element that contains the crack tip are enriched with a trigonometric function [2], defined as

$$L_k^{\text{trig}}(\mathbf{x}) = L_k{}^{G_S}(\mathbf{x}) - I_{\omega_j}(L_i{}^{G_S}(\mathbf{x})), \quad \text{where} \quad L_k{}^{G_S}(\mathbf{x}) = R_\gamma \;\; \left(r^{\frac{3}{2}} \sin\frac{\theta}{2} - r_k^{\frac{3}{2}} \sin\frac{\theta_k}{2} \right) \tag{4}$$

with (r, θ) being the local polar coordinates at the point $\mathbf{x}$ regarding a local coordinate system defined at the crack tip, (r_k, θ_k), the local polar coordinates at node $\mathbf{x_k}$, and R_γ representing a rotation matrix. Finally, $I_{\omega_j}(L_k{}^{G_S}(\mathbf{x}))$ is the finite element interpolant, defined as [12]:

$$I_{\omega_j}(L_{ji})(\mathbf{x}) = \sum_{k=1}^{n_e} N_k(\mathbf{x}) L_{ji}(\mathbf{x}_k) \tag{5}$$

where L_{ji} is an enrichment function, vector $\mathbf{x}_k$ has the coordinates of node k of element e, N_k is the piecewise linear FE shape function for node k, and n_e is the number of element nodes. The nodes of remaining elements cut by the crack are enriched with the modified Heaviside functions proposed by Wu and Li [1] and computed from:

$$\mathcal{H}_{mod}(x,y) = \mathcal{H}(x,y) - [\alpha I_{\omega_j}(\mathcal{H}(x,y))(\mathbf{x}) + (1-\alpha)\mathcal{H}(x_j,y_j)] \tag{6}$$

where $\mathcal{H}(x_j, y_j)$ is the Heaviside function from Eq. (3) computed at the node $\mathbf{x_j}$ and $0 \leq \alpha \leq 1$ is a stabilization parameter. In this work, $\alpha = 0.1$.

2.4 SGFEM displacement approximation

Based on developments of section 2.3, one can write an approximation for displacement field as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{j \in S} N_j(\mathbf{x})\tilde{\mathbf{u}}_j + \sum_{j \in S_H} N_j(\mathbf{x})\mathcal{H}_{mod}(\mathbf{x})\mathbf{a}_j + \sum_{j \in S_T} N_j(\mathbf{x}) L_j^{\text{trig}}(\mathbf{x})\mathbf{b}_j \tag{7}$$

where $N_j(\mathbf{x})$ are the standard FE shape functions (here, the bi-linear Q4 functions), S is the set of all nodes in the domain, $\tilde{\mathbf{u}}_j = \{u_j, v_j\}^T$ is the vector of displacements, and $\mathbf{a}_j, \mathbf{b}_j$ are nodal parameters. S_H is the set of nodes whose basis function support is entirely split by the crack whilst S_T contains nodes with the crack tip located in the support of their basis functions.

Considering the definition of $[\![\mathbf{u}]\!]$ and eq. (5), eq. (6), eq. (7), $N_j(\mathbf{x})$ and $I_{\omega_j}(L_{ji})$ continuity over an element, one can conclude that, at the cracked elements:

$$[\![\mathbf{u}^h]\!](\mathbf{x}) = \sum_{j \in S_H} N_j(\mathbf{x})[\![\mathcal{H}(\mathbf{x}) - (1-\alpha)\mathcal{H}(\mathbf{x}_j)]\!]\mathbf{c}_j + \sum_{j \in S_T} N_j(\mathbf{x})[\![L_j^{G_s}(\mathbf{x})]\!]\mathbf{d}_j \tag{8}$$

where $[\![L_{ji}(\mathbf{x})]\!]$, $L_{ji} = \mathcal{H}(\mathbf{x}) - (1-\alpha)\mathcal{H}(\mathbf{x}_j)$ or $L_{ji} = L_j^{G_s}(\mathbf{x})$, is the difference between an enrichment L_{ji} computed at position $\mathbf{x}$ in Γ_c^+ and the same enrichment function computed at position $\mathbf{x}$ in Γ_c^-. In other words, it is the enrichment function jump at the crack.

3 On the SGFEM implementation at the INSANE system

This section aims to discuss only some relevant details regarding the expansion of INSANE G/XFEM system for cohesive crack propagation. The implementations carried out in this work were made in order to enable SGFEM simulations. For more details concerning INSANE G/XFEM cohesive crack propagation system, the reader is referred to Malekan et al. [23].

The expansion of the aforementioned system to enable SGFEM simulations was not difficulty to be made. It will be described in general lines, highlighting one aspect that has caught our attention. Further works may deal with more implementation details. Crack propagation was managed mainly by a class called *DiscontinuityByGfem*, which is contained in the project Model, at INSANE processor [23]. *DiscontinuityByGfem* represents a crack segment. The main modifications were performed in this class, enabling the use of different enrichment functions and loosening the way the considered discontinuity was updated in order to represent propagation. Until this work, only Heaviside functions were used, and this made it necessary to adapt the crack tip new position computation. Other implementations were made in the context of classes *GFemModelDiscreteCrack*, *GFemElementDiscreteCrack* and *GFemPhysicallyNonLinearDiscreteCrack*. The goals of code developments were, respectively, provide a correct initialization of new crack segments, perform the computation of the jump at enrichment functions, and to allow the process zone of fracture term integration for building stiffness matrices in the non-linear process of solution. It as also necessary to create class *CohesiveCrackTipEnrichment*, as a child class of *EnrichmentType* (a generic class that represents a enrichment function), in order to allow the use of eq. (4) as enrichment function. Adaptation of the computation of stable version of this enrichment (eq. (4)) was not needed since Oliveira et al. [13] have developed the generic class *StableEnrichmentType*. This class provides the computations concerning enrichment modification represented by eq. (5) for any enrichment function. Implementation of modified Heaviside enrichment functions (eq. (6)) in INSANE was performed by Oliveira et al. [24]. Finally, SGFEM expansion required some modifications in class *PersistenceAsXml*, in order to adequate the *xml* data entrance file to modifications carried out in INSANE processor. Some examples are the creation of a *tag* that identifies the method chosen for numerical analysis (G/XFEM in its traditional form or SGFEM) and a place where the user may input crack increment considered during the propagation. The crack nucleation procedure [23] and cohesive law originally implemented (section 2.2) were not modified.

One question that arose during code expansion, that, to the the best of the authors knowledge, has little or none discussion in literature, was how to treat the enrichment nodes change in the non-linear process of solution as the crack propagates. While new crack segments are created, elements that contained the crack tip can become fully cracked, or may have a new tip inside them. Thus, changing the enrichment functions at their nodes - since trigonometric functions (eq. (4)) represent the behaviour of solution near to the crack tip - is needed. Modified Heaviside enrichment functions (eq. (6)) should then be considered to simulate the element fully cracked, and a new trigonometric enrichment function should be applied on element's nodes that have a new crack tip. Aiming to perform this change, the method *changeRestraintsOfCohesiveEnrich* (Node *currentNode*) was created in *DiscontinuityByGfem*. This method deactivates the current trigonometric enrichment at a node by changing the restraints of DOFs associated with the aforementioned enrichment function from "false" to "true". With this, DOFs related to the current function represented by eq. (4) will be no longer effectively be considered in matrices, displacements and stress or strain computations. Additionally, in order to assure the coherence in the computation of displacements vector in the non-linear process of solution, this method performs one more task. Considering that, proceeding with the process of crack propagation simulation, the same node will receive an order to incorporate the modified Heaviside enrichment functions (eq. (6)) or a new trigonometric function (eq. (4)), the method initializes new values for states variables, contemplating the DOFs added by the new enrichment function. This aims to assure that these new DOFs will be managed as new nodal parameters, different from those previous nodal parameters associated with the trigonometric function on displacement vector. Doing so, we could overcome some instabilities in numerical analysis.

4 On crack propagation criteria

Once the implementations described in previous section were carried out, some numerical experiments were performed. The one chosen for this paper refers to the L-shaped panel experimentally analyzed for Winkler et al. [21] (Fig. 2).

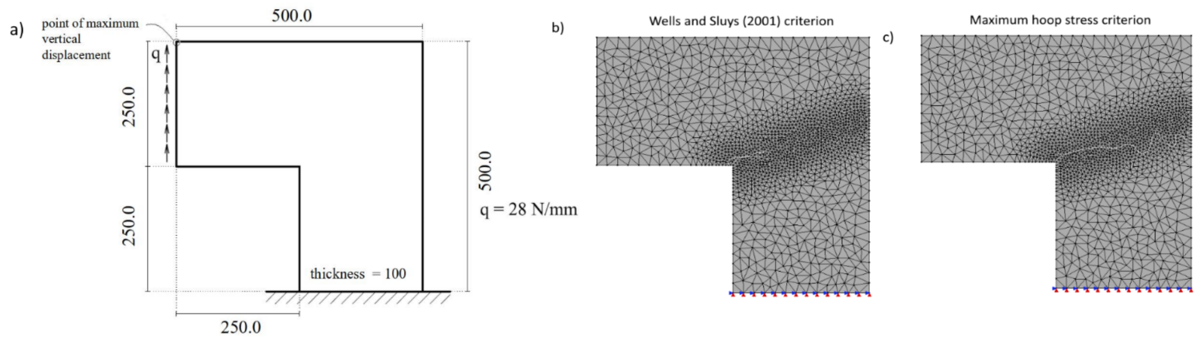

Figure 2. a) Geometry and loading of L-shaped panel. Dimensions in mm. b), c) Crack paths obtained with the proposed SGFEM for both criteria here studied.

Linear elastic material is considered. The following parameters were used: Young's modulus $E = 25850.0\ N/mm^2$, Poisson's ratio $\nu = 0.18$, tensile strength $f_t = 2.7\ N/mm^2$, fracture energy $G_f = 0.065\ N/mm$ and initial crack shear stiffness $d_{\text{init}} = 0.0\ N/mm^3$. For the simulations performed in this paper, the crack increment is equal to $5.0\ mm$. Plane stress condition is assumed. Experiments were made using T3 elements (Fig. 2). Numerical integration was performed with Gaussian quadrature, employing 6 points. Integration on the crack boundary was computed with 4 points. The non-linear analysis is performed with displacement control, with an increment of $0.005\ mm$ in the vertical displacement of the point of maximum vertical displacement (Fig. 2), a tangent constitutive tensor approximation, and a convergence tolerance of 1×10^{-4} in terms of displacement.

Experiments performed in this section aim to discuss the suitability of crack propagation criterion as proposed by Wells and Sluys [3] for the simulation of SGFEM cohesive crack propagation using the combination of enrichment functions presented in section 2.3. Until this work, this criterion - which consists in presuming that the crack propagates in an orthogonal direction to the principal direction obtained through a non-local stress [3] tensor - was the only direction criterion available in INSANE G/XFEM cohesive crack propagation system as conceived by Malekan et al. [23]. Here, it was implemented the maximum hoop stress criterion, whose formulation may be found in, for example, Moës and Belytschko [2]. Crack paths of both criteria are shown in Fig. 2.

It can be seen from Fig. 2 and from results analysis that the crack direction criterion proposed by Wells and Sluys [3] fails to represent the proper crack path behaviour after step 32, once the compression in the model becomes significantly dominant. The path, then, takes an impossible direction. This directly impacts the results obtained for equilibrium paths of this model, which are not shown here. In the other hand, the SGFEM model with the maximum hoop stress criterion for crack direction is able to represent crack pattern more accurately. The equilibrium path obtained with this strategy is show in fig Fig. 3.

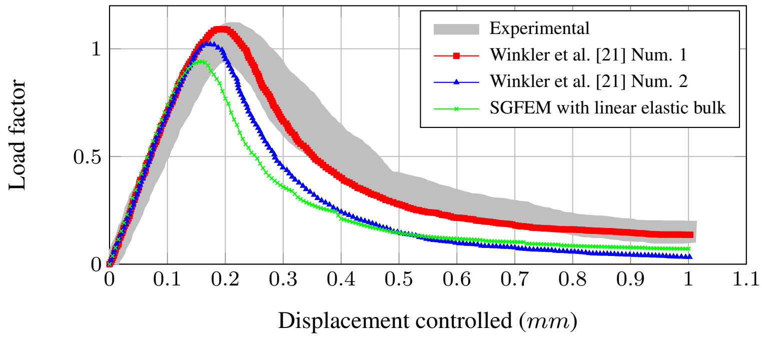

Figure 3. Equilibrium paths obtained for L-shaped panel of Fig. 2. Reference results are taken from Winkler et al. [21].

We can see from Fig. 3 that equilibrium path obtained with SGFEM employing the maximum hoop stress criterion and linear elastic material is quite reasonable, although its peak is a little under than the ones of reference results and experimental behaviour. This may be due to the more fragile nature of discrete crack models, since stiffness loss is more intense. Future works may consider the influence of the choice of cohesive law to prevent this issue. From those results, as well as others that are not presented in this work, maximum hoop stress criterion was taken in INSANE G/XFEM cohesive crack propagation system as the crack direction criterion.

5 Conclusions

In this paper, a SGFEM is proposed for cohesive crack propagation based on the combination between modified Heaviside enrichment functions (eq. (6)) and a trigonometric function (eq. (4)). To the best of authors knowledge, this is the first attempt to propose a SGFEM for this kind of problem combining enrichment functions of such nature. The implementation process in the context of the expansion of INSANE G/XFEM cohesive crack propagation system is discussed. Particularly, we discuss some details regarding the nodes enrichment change as the crack propagates. The SGFEM here proposed is shown to have reasonable accuracy in providing an equilibrium path for the L-shaped panel of Winkler et al. [21] when employing maximum hoop stress criterion for crack direction. Crack direction criterion proposed by Wells and Sluys [3] fails to represent the proper crack behaviour. Numerical results of other benchmark problems, as well as a study of scaled condition numbers, for both G/XFEM in its traditional form and the SGFEM here proposed are under investigation. Further works may also deal with numerical integration, discontinuity mapping and coupling with non-linear material models.

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