Analysis of the nonlinear vibration of a clamped cylindrical shell considering the influence of the internal fluid and oceanic waves

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Abstract. Cylindrical shells are used in various engineering installations, such as nuclear power plants, tanks, cooling towers and oil platforms, which can fluid-filled. In addition, offshore structures are subject to the incidence of ocean waves, which can change their dynamic response. Due to the slenderness of the cylindrical shells, the interaction between these structures and the fluid presents a complex dynamic behavior, and understanding these phenomena is of great interest for the development of these designs. This article presents an analytical-numerical comparative study to analyze the nonlinear vibrations of a clamped cylindrical shell, considering the effects of the presence of an internal fluid, that is incompressible, non-viscous and irrotational, and its free surface. It is also applied to the outer side walls of the cylindrical shell, a load from the action of ocean waves, which are derived from the Airy theory and are time dependent. To describe the deformation field and curvature changes of the middle surface of the cylindrical shell, the nonlinear Sanders-Koiter theory was used. Chebyshev polynomials are applied to define the modal expansions that describe the displacement field of the structure. Finally, the Rayleigh-Ritz method is used to obtain the nonlinear equations of motion of the system. The free vibrations were compared with a numerical model obtained by the Finite Element Method (FEM), with the aid of the commercial software ANSYS. As a response of the nonlinear analysis, the response in time and phase-plane is obtained for the cylindrical shell with the presence of the internal fluid and the action of ocean waves, in addition to evaluating the influence of two types of waves, for shallow and intermediate waters.

Keywords: Cylindrical shells, Fluid-structure interaction, Nonlinear vibrations, Dynamic response, Analytical-numerical study.

1 Introduction

Cylindrical shells are structures used in various engineering installations, such as nuclear power stations, tanks, cooling towers and oil platforms. In most of their applications, they are filled with fluids. In relation to offshore structures, the incidence of ocean waves is an important factor in evaluating their dynamic response. Due to the slenderness of the cylindrical shells, the fluid-structure interaction presents a complex dynamic behavior, and understanding these phenomena is of great interest for the development of designs in different areas of engineering. A literature review on research involving the various variables around nonlinear vibrations of cylindrical shells between the years 2003 to 2013, is made by Alijani and Amabili [1]. Several aspects of nonlinear vibrations of cylindrical shells are addressed, such as fluid-structure interaction, geometric imperfections, boundary conditions, thermal loads, reduced-dimensional models and perturbation techniques.

The mathematical formulation necessary for the analysis of fluid-structure interaction in offshore structures is presented by Pedroso [2], discussing the wave-induced forces and the importance of the parameters involved in the problem. Haritos [3] exposes some wave theories, such as Airy theories, Stokes and Cnoidal, including their equation, indicating important factors to consider in the study of fluid-structure interaction involving the action of ocean waves, in addition to the prediction of statistical values.

So, the aim work is to perform an analytical-numerical comparative study for analysis of nonlinear vibrations of a cylindrical shell clamped at bottom and free at top, considering the effects of the presence of an internal fluid and its free surface, in addition to the action of oceanic waves applied to the external sides of the structure. Initially, linear vibrations are obtained analytically and numerically, through the FEM commercial software ANSYS, presenting good convergence between the results. Finally, a nonlinear time response analysis is carried out, considering the influence of all variables on the dynamic responses of the system.

2 Problem formulation

2.1 Shells equations

Consider a perfect cylindrical shell, whose geometry is defined by a length L, radius R, and thickness h. The cylindrical shell material is isotropic, homogeneous and elastic with Young's modulus E, Poisson ratio ν and density ρ_s. The displacements on the mid-surface of the shell are: axial $u(x, \theta, t)$, circumferential $v(x, \theta, t)$ and radial $w(x, \theta, t)$, being t the time. Figure 1 illustrates the geometry of the shell and its coordinate system.

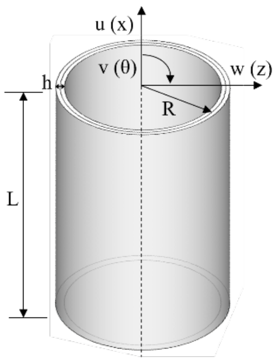


Figure1. Geometry of the cylindrical shell and its coordinate system.

From nonlinear Sanders-Koiter theory, at a point on the mid-surface of the shell, the deformation fields are given by:

$$\left[\varepsilon_x, \varepsilon_\theta, \gamma_{x\theta}\right] = \left[u_{,x} + \frac{{w_{,x}}^2}{2} + \frac{1}{8}\left(v_{,x} - \frac{u_{,\theta}}{R}\right)^2, \frac{v_{,\theta} + w}{R} + \frac{\left(w_{,\theta} - v\right)^2}{2R^2} + \frac{1}{8}\left(\frac{u_{,\theta}}{R} - v_{,x}\right)^2, \frac{u_{,\theta}}{R} + v_{,x} + w_{,x}\left(\frac{w_{,\theta} - v}{R}\right)\right] \quad (1)$$

The changes in curvature of the mid-surface are expressed by:

$$\left[\kappa_x, \kappa_\theta, \kappa_{x\theta}\right] = \left[-w_{,xx}, \frac{v_{,\theta} - w_{,\theta\theta}}{R^2}, -\frac{2w_{,x\theta}}{R} + \frac{1}{2R}\left(3v_{,x} - \frac{u_{,\theta}}{R}\right)\right] \quad (2)$$

To evaluate the nonlinear dynamic behavior of cylindrical shells, it is necessary to adopt a modal solution for the displacement fields, u, v, and w, which must satisfy the boundary and continuity conditions of the problem. In this work, it will be used the first-order Chebyshev polynomials $T_m(x)$. So, the modal solutions for cylindrical shell displacements are defined by a series of Chebyshev polynomials as defined by:

$$\begin{aligned} &U(x,\theta,t) = \sum_{m=1}^{N_u}\sum_{n=0}^{N} U_{m,n}(t)T_m^*(x)\cos(n\theta),\; V(x,\theta,t) = \sum_{m=1}^{N_v}\sum_{n=1}^{N} V_{m,n}(t)T_m^*(x)\sin(n\theta) \\ &W(x,\theta,t) = \sum_{m=1}^{N_w}\sum_{n=0}^{N} W_{m,n}(t)T_m^*(x)\cos(n\theta) \end{aligned} \quad (3)$$

For the condition of the cylindrical shell clamped-free, the displacement fields must satisfy the following boundary conditions [4]:

$$\begin{aligned} &u = v = w = w_{,x} = 0 \text{ to } x = 0, \\ &N_x = N_{x\theta} + M_{x\theta}/R = M_x = Q_x + M_{x\theta,\theta}/R = 0 \text{ to } x = L \end{aligned} \tag{4}$$

The internal deformation energy of cylindrical shell, for a homogeneous and isotropic material, is given by:

$$U_I = \frac{ER}{4\left(1-\nu^2\right)} \int_0^{2\pi}\int_0^L\int_{-h/2}^{h/2} \left(2\varepsilon_x^2 + 2\varepsilon_\theta^2 + \gamma_{x\theta}^2 + 4\nu\varepsilon_x\varepsilon_\theta - \nu\gamma_{x\theta}^2\right) dx\left(1+z/R\right) d\theta\, dz \tag{5}$$

The kinetic energy for the shell is defined as [5]:

$$T_C = \frac{\rho_S h R}{2} \int_0^{2\pi}\int_0^L \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dx\, d\theta \tag{6}$$

The work of the external loads, applied in the cylindrical shell, is defined by the contribution of the hydrodynamic pressure p_H from the presence of the internal fluid, and from p_W derived from the action of ocean waves:

$$W = R\int_0^{2\pi}\int_0^L p_H\, w\, dx\, d\theta + R\int_0^{\pi}\int_0^L p_W\, w\, dx\, d\theta \tag{7}$$

Finally, according to Popov *et al*. [6], the work of the dissipation forces is obtained by:

$$R_E = \left(\eta_1 \rho h \omega\right)\int_0^L\int_0^{2\pi} \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dx\, R\, d\theta + \left[\eta_2 E h^3 / 24\left(1-\nu^2\right)\right]\int_0^L\int_0^{2\pi} \left(\nabla^2 \dot{w}\right)^2 dx\, R\, d\theta \tag{8}$$

Once the cylindrical shell energy functionals are defined, the Lagrange function is obtained, and then the Rayleigh-Ritz method is applied to obtain the nonlinear equilibrium equations of the problem:

$$\overline{L} = T_C - U_I + W - R_E \rightarrow \frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial \dot{q}_i}\right) - \frac{\partial \overline{L}}{\partial q_i} = 0,\ i = 1, 2, \ldots M_T \tag{9}$$

being M_T the total number of degrees of freedom defined by the sum of the contributions of each displacement field, obtained as a function of the modal amplitudes $U_{m,n}$, $V_{m,n}$ and $W_{m,n}$, and, q the vector of the displacements.

2.2 Fluid equations

The shell is completely filled with fluid, incompressible, irrotational and non-viscous, considering the free surface of the internal fluid. The fluid is described by a velocity potential $\phi\ (r,\ \theta,\ x,\ t)$ that it must satisfy the following Laplace equation:

$$\phi_{,rr} + \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} + \phi_{,xx} = 0 \tag{10}$$

The fluid applies a hydrodynamic pressure p_H to the inner walls of the shell, that is obtained from the Bernoulli equation and dependent on the velocity potential $\phi\ (r,\ \theta,\ x,\ t)$:

$$p_H = -\rho_F\, \dot{\phi}\Big|_{r=R} \tag{11}$$

Through the principle of superposition of effects, the velocity potential of the fluid is defined by $\phi = \phi^{(1)} + \phi^{(2)}$ [7], and $\phi^{(1)}$ describes the potential of the fluid considering the flexible shell walls and the rigid bottom, and $\phi^{(2)}$ corresponds to the velocity potential of the sloshing to a shell with the rigid walls. The height of the fluid inside the structure is given by H. So, the potential ϕ must satisfy the following boundary conditions:

$$\phi_{,x}^{(1)}\left(x, r, \theta\right) = 0\Big|_{x=0} \tag{12.1}$$

$$\phi_{,r}^{(1)}\left(x, r, \theta\right) = \dot{w}\left(x, \theta, t\right)\Big|_{r=R} \tag{12.2}$$

$$\phi^{(1)}(x,r,\theta)=0\Big|_{x=H} \tag{12.3}$$

$$\phi_{,x}{}^{(2)}(x,r,\theta)=0\Big|_{x=0} \tag{12.4}$$

$$\phi_{,r}{}^{(2)}(x,r,\theta)=0\Big|_{r=R} \tag{12.5}$$

$$\phi_{,tt}{}^{(2)}(x,r,\theta)+\phi_{,t}{}^{(2)}(x,r,\theta)=-g\left(\phi_{,x}{}^{(1)}+\phi_{,x}{}^{(2)}\right)\Big|_{x=H} \tag{12.6}$$

In this work, the following velocity potential $\phi^{(1)}$, proposed by Paak, Païdoussis and Misra [8], satisfies the fluid-structure boundary conditions:

$$\phi^{(1)}(x,r,\theta)=\sum_{n=0}^{N_w}\sum_{\bar{m}=1}^{N_f}\left\{A_{i\bar{m}}(t)\cos(i\,n\theta)\sin(\lambda_i x)\,\mathrm{I}_{\,i\times n}\left(\lambda_i r\right)\right\} \tag{13}$$

being $\lambda_I = i/H\pi$. The amplitude of each mode $A_i(t)$, in eq. (13), is obtained by the condition of impenetrability, expressed by the eq. (12.2), and using the Galerkin method, using as weight function, the trigonometric terms of the eq. (13). So, $A_i(t)$ can be expressed as a function of w as:

$$A_i(t)=\int_0^L\left[\dot{w}\sin(\lambda_i x)\right]dx\Bigg/\int_0^L\left[\phi_{,r}\sin(\lambda_i x)\right]dx\Bigg|_{r=R} \tag{14}$$

The potential velocity of the fluid because of sloshing, $\phi^{(2)}$, is given by [9]:

$$\phi_2(x,r,\theta)=\sum_{i=1}^{Nv}\sum_{k=1}^{N_s}B_{ik}(\mathrm{t})h^2\omega\cosh\left(\frac{\varepsilon_{ik}x}{R}\right)J_n\left(\frac{\varepsilon_{ik}r}{R}\right)\cos(n\theta) \tag{15}$$

being ω the natural frequency of cylindrical shell, J_n the function of first-class Bessel, N_s the number of terms used in the expansion to the potential $\phi^{(2)}$ and $B_{ik}(t)$ the modal amplitudes. The eq. (15) is a solution of the Laplace equation and obeys the boundary condition given in (12.4). Applying the boundary condition (12.5), ε_{ik} are the solutions obtained from the following differential equation:

$$\left[J_n\left(\varepsilon_{ik}r/R\right)\right]_{,r}=0\Big|_{r=R} \tag{16}$$

Obtaining the velocity potentials $\phi^{(1)}$ and $\phi^{(2)}$, the Galerkin method is applied in the eq. (15), using the term as a weight function $J_n(\varepsilon_{ik}\,r/R)$. When this operation is performed, N-th additional equations are generate to the system of equations of the cylindrical shell, which couple the effects of the free surface to the shell-fluid problem.

2.3 Oceanic waves

In this work will be considered the action of ocean waves, using Airy's theory. In a simplified way, the minimum formulation necessary for the analysis of this load in the problem will be presented. The wave profile is obtained as a function of amplitude (A) at a given time (t) and the horizontal position of the wave (x), being [3]:

$$\eta(x,t)=A\cos\left(k_o\,x-\omega_o\,t\right) \tag{17}$$

The number of the wave k_o is related through the period of the wave T_o:

$$k_o=\left(2\pi/T_o\right)^2\Big/g\tanh\left(k_o h_o\right) \tag{18}$$

The natural frequency is obtained as a function of the number of waves k_o and the depth h_o is:

$$\omega_o=\sqrt{k_o g\tanh\left(k_o h_o\right)} \tag{19}$$

The velocity potential of wave is ϕ_3 expressed by [2]:

$$\phi_3(x,z,t)=\frac{Ag}{\omega_o}\frac{\cosh\left[k_o\left(z+h_o\right)\right]}{\cosh\left(k_o h_o\right)}\sin\left(k_o x-\omega_o\,t\right) \tag{20}$$

Wave pressure is obtained from velocity potential ϕ_3, that it must satisfy Bernoulli equation [2]:

$$p_W\left(x,z,t\right)=\rho_f\, g\, A\frac{\cosh[k_o\left(z+h_0\right)]}{\cosh\left(k_o h_0\right)}\cos\left(k_o x-\omega_o\, t\right) \tag{21}$$

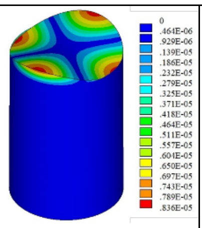
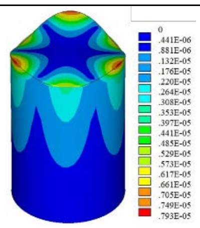
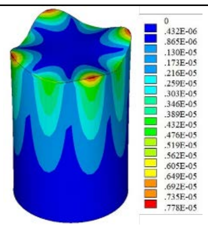
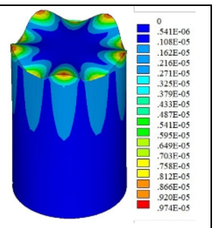
This pressure will be applied to the outer sides of the cylindrical shell.

3 Numerical results

It is considered a homogeneous and isotropic material (reinforced concrete class C70) for the cylindrical shell. So, the physical and geometric parameters of the clamped-free cylindrical shell are: R = 40 m, L = 120 m, h = 0.25 m; E = 42 GPa; ν = 0.2 and ρ_s = 2500 kg/m³. In relation to the internal fluid, its density is ρ_f = 1000 kg/m³ and the cylindrical is completely fluid-filled. Initially, the free vibrations natural frequencies were validated by comparing the proposed analytical model and the numerical results obtained from a FEM analysis.

For the numerical FEM model, a mesh of 5304 finite elements was used (obtained from a mesh previously tested to guarantee the convergence of the results), composed of two types of elements of the ANSYS's library [10]: SHELL281 and FLUID220, referring to the shell and fluid, respectively, suitable chosen to evaluate the effects of bending and membrane of the shell and the fluid-structure interaction. Regarding the proposed analytical model, the number of terms used for modal expansion in axial (N_u), circumferential (N_v) and radial (N_w) directions were equal to 5, for fluid N_f = 5 and sloshing N_s = 1. Table 1 presents the first four vibration modes and the comparison between the frequencies obtained for the analytical and numerical models.

Table 1. Four first modes of vibration and comparison of the values of the frequencies obtained.

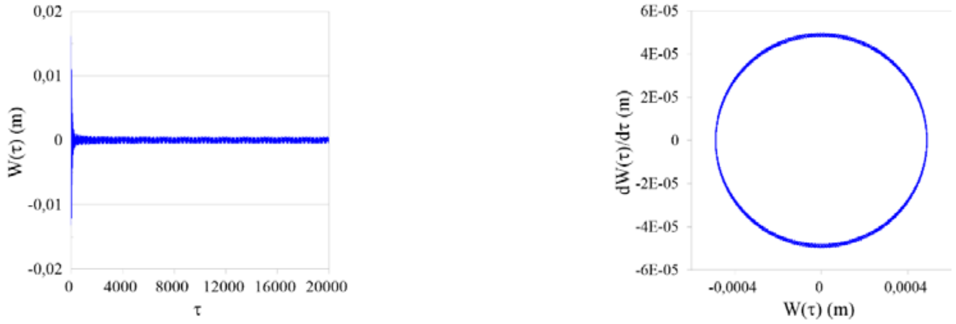
Modal deformed				
$(m, n) = (k, n)$	(1, 2)	(1, 3)	(1, 4)	(1, 5)
ω_{ana} (Hz)	0.13086	0.13166	0.13834	0.17978
ω_{mef} (Hz)	0.13222	0.13249	0.13709	0.17818
E (%)	1.03	0.63	0.91	0.90

It is observed a good convergence of the values of natural frequencies between the analytical and numerical methods, with differences of less than 1.03 %. The lowest natural frequency was obtained for the vibration mode (m, n) = (1, 2), which will be used as a fundamental frequency (ω) in the subsequent nonlinear analysis. It is important to note that the results obtained consider the effects of free surface on the structural system. Other important dynamical aspect is the density of natural frequencies that this cylindrical shell present around the vibration mode (m, n) = (1, 2) that can lead to nonlinear resonances and modal coupling.

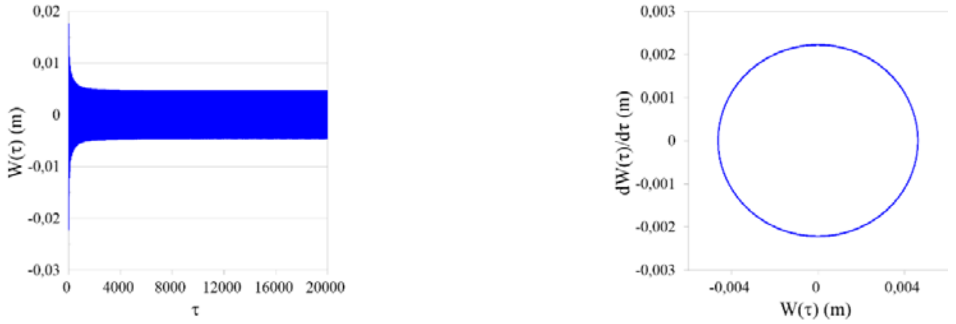
Then, a nonlinear analysis will be conducted for the forced vibration problem, evaluating the incidence of oceanic waves applied to the outer side walls of the cylindrical shell. Two types of waves will be considered using Airy's theory, classified as shallow (long waves) and intermediate waters. This classification is made according to the variation of the parameter $k_o h_o$, expressed in the eqs. (18) to (21), in addition to the relationship between water height (h_o) and wavelength (L_o). The numerical ranges of these interfaces are detailed in [2]. In this work, a water height was considered focusing along the entire length of the shell ($h_o = L$ = 120 m) and the periods (T_o) for each type of wave were: 90 s and 18 s for shallow and intermediate waters, respectively. The coordinate system (z) in which the system is inserted will be at a point (0, 0), and the wave amplitude (A) equal to 2 m. With this data, it is possible to calculate the incident wave pressure, eq. (21).

The values used for the viscous damping coefficients (η_1) and viscoelastic (η_2) of the material were equal to 0.005 and 0.0001, respectively and the non-dimensional time $\tau = \omega\, t$ was considered in the numerical results. Finally, the obtained EDO system is time-integrated using the fourth-order Runge-Kutta method to obtain the time responses. For a sufficient long time, in which it can disregard the transient region, the phase-plane in the permanent response were obtained to demonstrated the encountered stable periodic orbits. The value adopted for the initial conditions of modal amplitudes was equal to 1×10^{-4} m (near to the rest position).

A convergence study was carried out to calibrate the number of terms needed to obtain satisfactory results in the nonlinear analysis. In all cases, the number of circumferential waves was expanded by a sum of N=3 in eq. (3). The following modal expansion was considered, N_u=N_v=N_w=2 (in eq. (3)), N_f = 5, N_s= 1, giving a discrete system with 26 degrees of freedom, with good convergence of the values of modal amplitudes. All time responses and phase-planes of Fig. 2 were obtained for a point at the top of the cylindrical shell ($x = L$, $\theta = 2\pi$), which represent the maximum values of displacements. This figure shows stables time responses and phase-planes of the permanent response, considering the two classifications for ocean waves, differing from each other in the magnitude of the displacements and velocities due to the strong variation of the pressure of the oceanic wave.



(a) shallow water (T_o = 90 s)



(b) Intermediates water (T_o = 18s)

Figure 2. Time responses and phase-planes of the cylindrical shell filled by fluid and subjected to the action of the oceanic waves, considering the two types of waves.

Figure 3 presents the variation of the maximum radial displacements in the permanent response of the cylindrical shell as a function of increasing the wave amplitude (A) for constant periods, considering the two types of waves (shallow and intermediate waters). The blue lines represent the periods T_o equal to 18, 20 and 22 s corresponding to intermediate waters, while the black lines illustrate the periods T_o equal to 90, 94 and 98 s, equivalent to shallow waters. It is observed that radial displacements vary linearly with the increase of wave amplitude for a constant period. In addition, the modal amplitudes for shallow water present lower values compared to the incidence of waves in intermediate waters and they are not influenced by the oceanic waves period.

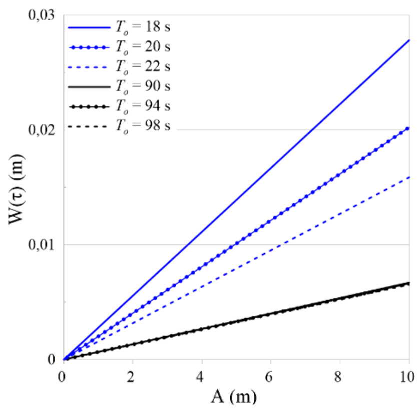


Figure 3. Variation of $W(\tau)$ as a function of the increment of A (T_o fixed), considering the two types of waves.

4 Conclusions

Through this work it was possible to carry out an analytical-numerical comparative study to analyze the nonlinear vibrations of a cylindrical shell clamped at bottom and free at top, considering the effects of the presence of an internal fluid and its free surface, in addition to the action of ocean waves applied to the outer side walls of the structure. Regarding the analytical model, the nonlinear Sanders-Koiter theory was used and the displacement fields of the clamped-free cylindrical shell were obtained from the Chebyshev polynomials. To define the load to be applied to the structure, the Airy theory was adopted, which provides the necessary formulation to describe the behavior of ocean waves. Two types of waves were considered, classified as shallow and intermediate waters, defined from certain parameters such as the depth of the water depth, length, amplitude and period of the wave. Initially, linear vibrations are obtained analytically and numerically, with good convergence between the results. Finally, a nonlinear analysis is carried out, evaluating through the time responses, phase-planes and, the dynamic behavior of the of the cylindrical shell with the influence of all the variables under analysis.

Therefore, it was possible to observe a good agreement of the results obtained by the FEM in comparison with the analytical method adopted, to perform the study of free vibrations of cylindrical shells set with the presence of internal fluid. For nonlinear analysis, the developed model proved to be efficient for the evaluation of forced vibrations from the incidence of ocean waves. It is important to point out that the work is ongoing, and several other analyses are under development, such as the study of deepwater waves and evaluation of the overall stability of the system, with the achievement of frequency-amplitude relationships and resonance curves.

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